On variational approach to conformal geodesics

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Overview

In the Riemannian geometry geodesics are distinguished curves characterized (locally) by the length minimizing property. They are solutions to a system of second order ODEs, derived from a first order Lagrangian by methods of the calculus of variations.

In the conformal geometry there is also a distinguished class of curves, already known to Yano, called conformal circles (or conformal geodesics). They are solutions to a third order system of ODEs which makes a variational approach problematic. In this talk I'd like to show how one can deal with the difficulties.

- Joint work with Maciej Dunajski

Conformal geodesics

Let $(M,[g])$ be a conformal n-dim manifold. A curve $\gamma\colon [t_0,t_1]\to M$ is a conformal geodesic if it satisfies the third order ODE (Bailey-Eastwood formulation):

$$E \equiv \nabla_U A - \frac{3g(U,A)}{|U|^2}A + \frac{3|A|^2}{2|U|^2}U - |U|^2 P^\sharp(U) + 2P(U,U)U = 0.$$

where ∇ is the Levi-Civita connection for g,

$$P = \frac{1}{n-2}\Big(Ric - \frac{1}{2(n-1)}Sg\Big),$$

is the Schouten tensor, $g(P^\sharp(U),V) = P(U,V)$, and

$$U = \dot\gamma, \qquad A = \nabla_U U.$$

The equation is conformally invariant.

Variational formulation

Goal: characterize conformal geodesics as critical points of a functional.

Ingredients:

- functional $\mathcal{I}$, usually integral of a Lagrangian L along a curve

$$\mathcal{I}(\gamma) = \int_{t_0}^{t_1} L(t, \gamma(t), \dot{\gamma}(t), \ldots)$$

- a class of variations γ_s, i.e. a class of 1-parameter families of curves satisfying certain conditions: e.g. curves joining fixed points, or submanifolds; or e.g. more complicated non-holonomic constraints in the sub-Riemannian geometry

General approach

Compute

$$\delta\mathcal{I}(\gamma) = \frac{d}{ds}\mathcal{I}(\gamma_s)|_{s=0}$$

Integrating by parts as many times as needed one gets

$$\delta\mathcal{I}(\gamma) = \int_{t_0}^{t_1} F(t, \gamma(t), \dot{\gamma}(t), \ldots)V + BT$$

where BT are boundary terms and $V = \frac{d}{ds}\gamma_s|_{s=0}$ is a vector field along γ and F depends on derivatives of γ up to the order $2k$ provided that L is a Lagrangian of order k (non-degenerate case).

If γ is a critical point of $\mathcal{I}$ then $\delta\mathcal{I} = 0$ for all admissible V. The fundamental lemma of the calculus of variations implies that $F = 0$ (provided that one considers arbitrary variations with $BT = 0$).

Generalization

Stop integration by parts earlier and aim for $\delta\mathcal{I}$ in the form

$$\delta\mathcal{I}(\gamma) = \int_{t_0}^{t_1} F(t, \gamma(t), \dot{\gamma}(t), \ldots) D(V) + BT$$

where D is a differential operator along γ acting on V.

Then adjust a class of variations such that a variant of the fundamental lemma of the calculus of variations can be applied for $W = D(V)$ instead of V and conclude that γ is a critical point of $\mathcal{I}$ for this specific class of variations iff $F = 0$ which now is of order lower than $2k$ (precisely $2k - s$, where s is the order of D).

One may expect: larger class of variations $\longrightarrow$ lower order of ODEs.

In the case of the conformal geodesics all ingredients (differential operator, boundary terms, class of variations etc.) have to be conformally invariant.

Conformally invariant differential operator

The following first-order differential operator

$$D(V) = \nabla_U V + |U|^{-2}(g(A, V)U - g(U, V)A - g(A, U)V)$$

is conformally invariant along a given curve γ.

It depends on the second jet of curve γ.

Invariant Lagrangian

First candidate by Bailey-Eastwood

$$L_{BE} = \frac{1}{2}\frac{|A|^2}{|U|^2} - \frac{g(U,A)^2}{|U|^4} + P(U,U).$$

L_{BE} is conformally invariant up to a differential of a function, which is sufficient if one considers variations vanishing at the endpoints.

L_{BE} can be extended to a conformally invariant third order Lagrangian

$$L = L_{BE} + 2\frac{g(U,\nabla_U A)}{|U|^2} + 2\frac{g(A,A)}{|U|^2} - 4\frac{g(U,A)^2}{|U|^4}$$

In fact the additional term is a differential, so L and L_{BE} differ only by a boundary term and L and L_{BE} give exactly the same fourth order Euler-Lagrange equations.

Invariant Lagrangian

L can be put in the following compact form

$$L = \frac{g(E, U)}{|U|^2}.$$

It proves that L is conformally invariant.

Variation of the functional

Theorem

The first variation of the functional $\mathcal{I}$ is given by

$$\delta\mathcal{I} = \int_{t_0}^{t_1} |U|^{-2}(g(K,V) - g(E - 2LU, D(V)))dt + BT(V)|_{t_0}^{t_1},$$

where K is a vector field along γ given, in terms of the Weyl tensor W, by

$$K^e = g^{ec}(W_{bca}{}^{d} U^a U^b A_d - 2|U|^2 \nabla_{[c} P_{a]b} U^a U^b),$$

and

$$BT(V) = |U|^{-2}(g(U, D^2(V)) - g(E - 2LU, V)),$$

Note that $E - 2LU$ vanishes if and only if $E = 0$.

Class of variations

In the flat case $K = 0$ and we consider variations satisfying

$$BT(V)|_{t_0}^{t_1} = 0$$

which for γ being a conformal geodesic gives a second order condition $g(D^2(V), U) = 0$ at the endpoints.

In the general case we need the following

$$BT(V)|_{t_0}^{t_1} = -\int_{t_0}^{t_1} |U|^{-2} g(K, V) dt.$$

Note that the right hand side is a well defined linear functional acting on variations V.

Equations

Theorem

A curve γ is a critical point of $\mathcal{I}$ in the class of variations satisfying

$$BT(V)|_{t_0}^{t_1} = -\int_{t_0}^{t_1} |U|^{-2} g(K, V) dt.$$

if and only if γ is a conformal geodesic.

Thank you for your attention!