# Conformal Einstein's equations

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# Asymptotically simple solutions

## Asymptotically de Sitter spacetimes

Future asymptotically simple solutions to Einstein equations:

$$g_{\mu\nu} = \Omega^{-2}\hat{g}_{\mu\nu}, \qquad G_{\mu\nu}[g] = \Lambda g_{\mu\nu}, \quad \Lambda > 0$$

and $\Omega = 0$ at a Cauchy surface of $\hat{g}_{\mu\nu}$ (Penrose compactification).

## Goal

Classify such solutions, application to stability of de Sitter and similar spacetimes (long time behaviour of solutions).

Einstein's equations written in terms of $\hat{g}_{\mu\nu}$ are singular at $\Omega = 0$ surface and we cannot impose easily initial conditions nor say something about dynamics.

# Conformal method of Friedrich

## Friedrich's approach

Find set of equations which

1. are more general i.e. every solution to Einstein's equation is also a solution to these equations,
2. transform nicely under conformal transformations,
3. the system is hyperbolic (after imposing suitable gauge),
4. the scale factor $\Omega$ and properties of being conformal to Einsteinian metric propagate by hyperbolic equation too.

Stability follows from stability of hyperbolic equations.
Friedrich's solution invented for $3+1$.

# Anderson's proposition (dimension $d$ even)

Vanishing of Fefferman-Graham obstruction tensor $H_{\mu\nu}$ follows from Einstein's equations

$$G_{\mu\nu} = \Lambda g_{\mu\nu} \implies H_{\mu\nu} = 0$$

**Anderson's version, any even dimension $d \geq 4$**

Use equation $H_{\mu\nu} = 0$. Nice conformal transformations and

$$H^{\mu}_{\mu} = 0, \qquad \nabla^{\mu} H_{\mu\nu} = 0, \quad \text{Lagrangean formulation}$$

Complicated high order tensor.

**Question**

Is this equation well-posed (after fixing gauge)?

In $d = 4$ proved by Guenther '70. Proofs in higher dimensions nontrivial (Anderson, Anderson-Chruściel).

# Choquet-Bruhat's method for Einstein's equations

1. Gauge freedom (diffeomorphisms): gauge fixing needed.
2. Constraints on the initial data surface $G_{\mu\nu}n^\mu = 0$.

We decompose Ricci tensor into gauge fixed part $E_{\mu\nu}$ and the rest

$$R_{\mu\nu} = E_{\mu\nu} + \frac{1}{2}\left(\nabla_\mu F_\nu + \nabla_\nu F_\mu\right), \quad E_{\mu\nu} = -\frac{1}{2}\Box g_{\mu\nu} + \ldots,$$

$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and harmonic gauge $F_\mu = g^{\xi\chi}\left(\partial_\xi g_{\chi\mu} - \frac{1}{2}\partial_\mu g_{\xi\chi}\right) = \Box x_\mu$.
Bianchi identities

$$0 = \nabla^\mu G_{\mu\nu} = \nabla^\mu\left(E_{\mu\nu} - \frac{1}{2}g_{\mu\nu}E\right) + \left(\frac{1}{2}\Box + \ldots\right)F_\nu$$

so $E_{\mu\nu} = 0$ implies $F_\nu = 0$ if it holds on $\Sigma$ (this due to constraints).

**Quasi-linear wave equation is well-posed**

Equation of the form $\Box_{g(u)}u + F(D^1 u) = 0$ is well-posed (existence and uniqueness of the local development, propagation with a speed of light, continuity in some finite time).

# Anderson's proposition

Similar Bianchi identity $\nabla^\mu H_{\mu\nu} = 0$

1. Gauge freedom: diffeomorphisms and conformal transformations

$$F_\mu = \Box x_\mu = 0, \quad R = 0 \quad \text{(gauge fixing, always possible)}$$

2. Constraints $H_{\mu\nu} n^\mu|_\Sigma = 0$ for initial data $D^{d-1} g_{\mu\nu}|_\Sigma$

The gauge fixed equation is now

$$\Box_g^{\frac{d}{2}} g_{\alpha\beta} + F(D^{d-1} g) = 0$$

**Higher order equations**

Multiple characteristics of the principal symbol $(p_\mu p^\mu)^{d/2}$ of the equation $\Box_{g(u)}^{d/2} u + F(D^{d-1} u) = 0$. Not necessary well-posed (one needs to control many lower order terms), different then Euclidean signature where it is automatically elliptic. Example

Similar phenomenon for GJMS operator: $P\phi = \Box_g^{d/2} \phi + \ldots$.

# Fefferman-Graham ambient metric construction

Ambient metric on $\mathbb{R}_+ \times M \times \mathbb{R}$ with coordinates $t, x^\mu, \rho$ indices $I = 0, \mu, \infty$.

$$\mathbf{g}_{IJ} dx^I dx^J = 2\rho dt^2 + 2t dt d\rho + t^2 \tilde{g}_{\mu\nu}(x^\mu, \rho) dx^\mu dx^\nu$$

Conformal Killing vector $\mathbf{T} = t\partial_0$, where $\tilde{g}_{\mu\nu}$ is a $\rho$-dependent metric on $M$.

**Graham-Jenne-Mason-Sparling (GJMS) equation**

$$P\phi = 0 \iff \Box\boldsymbol{\phi} = O(\rho^{d/2}), \;\; \mathcal{L}_{\mathbf{T}}\boldsymbol{\phi} = 0$$

where $\boldsymbol{\phi} = \sum_{n=0}^{d/2-1} \boldsymbol{\phi}^{[n]} \rho^n + \ldots$ and $\phi = \boldsymbol{\phi}^{[0]}_{t=1}$.

Equivalent formula for $\tilde{\phi} = \boldsymbol{\phi}|_{t=1}$, **(recursive)**

$$[\Box_{\tilde{g}} \tilde{\phi}]^{[n]} + (d - 2 - 2n)(n+1)\tilde{\phi}^{[n+1]} = 0.$$

It allows use to determine $\tilde{\phi}^{[n]}$ for $1 \leq n \leq d/2 - 1$ and plug recursively to obtain $P\phi = 0$.

# GJMS operators as evolution system

Equivalent formula for $\tilde{\phi} = \phi|_{t=1}$

$$[\Box_{\tilde{g}}\tilde{\phi}]^{[n]} + (d-2-2n)\,(n+1)\tilde{\phi}^{[n+1]} = 0.$$

Instead of eliminating higher orders, let us keep them as independent variables $\tilde{\phi}^{[n]}$, $n \leq d/2 - 1$.

## Evolution equation

$$\begin{bmatrix} \Box & 0 & \cdots & 0 \\ * & \Box & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Box \end{bmatrix} \begin{bmatrix} \tilde{\phi}^{[0]} \\ \tilde{\phi}^{[1]} \\ \vdots \\ \tilde{\phi}^{[\frac{d}{2}-1]} \end{bmatrix} + \begin{bmatrix} c_0\tilde{\phi}^{[1]} \\ c_1\tilde{\phi}^{[2]} \\ \vdots \\ 0 \end{bmatrix} = 0$$

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## Evolution equation

$$\begin{bmatrix} \Box & 0 & \cdots & 0 \\ * & \Box & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Box \end{bmatrix} \begin{bmatrix} u^{[0]} \\ u^{[1]} \\ \vdots \\ u^{\left[\frac{d}{2}-1\right]} \end{bmatrix} + L \begin{bmatrix} c_0 u^{[1]} \\ c_1 u^{[2]} \\ \vdots \\ 0 \end{bmatrix} + \ldots = 0$$

We introduce $\tilde{\phi}^{[k]} = L^k u^{[k]}$, $L = \sqrt{1+\Delta}$.

Well-posed system, but in skewed Sobolev spaces (Leray hyperbolic).
Recursive $\Longrightarrow$ Solution also to $P\phi = 0$.

# Fefferman-Graham ambient metric

The obstruction tensor is related to Einstein's equations in higher dimension. Vanishing of the obstruction tensor

1. **Geometric:** Conformal Killing $\boldsymbol{\nabla}_I \mathbf{T}_J = \mathbf{g}_{IJ}$, introduce $\boldsymbol{\rho} = \frac{1}{2}\mathbf{T}_I\mathbf{T}^J$

   $\mathbf{g}_{IJ}|_{\{\boldsymbol{\rho}=0\}}$ tautological bundle of the conformal structure for $h_{\mu\nu}$

   Condition on the Einstein tensor

   $$\mathbf{G}_{IJ}\mathbf{X}^I\mathbf{X}^J = O(\boldsymbol{\rho}^{d/2}), \quad \forall \mathbf{X}^I \text{ tangent to } \{\boldsymbol{\rho}=0\}$$

2. **Gauge:** $\mathbf{T} = t\partial_0$ and $\partial_\infty$ null geodesic (locally)

   $$\mathbf{g}_{IJ}dx^I dx^J = 2\rho dt^2 + 2tdtd\rho + t^2\tilde{g}_{\mu\nu}(x^\mu,\rho)dx^\mu dx^\nu, \quad \boldsymbol{\rho} = \rho t^2$$

3. **Normalization:** $\mathbf{R}_{\infty\infty}^{[d/2-2]} = 0$. It is non-dynamical condition on $\operatorname{tr}\tilde{g}^{[d/2]}$. Then $(\mathbf{R}_{0I} = 0)$

   $$\mathbf{R}_{\mu\nu} = O(\rho^{d/2}), \quad \mathbf{R}_{\mu\infty} = O(\rho^{d/2-1}), \quad \mathbf{R}_{\infty\infty} = O(\rho^{d/2-1}).$$

# Gauge fixing

## Tension between FG gauge and the harmonic gauge

1. **Excessive gauge fixing.** The propagation of the gauge in Choquet-Bruhat method uses Bianchi identity. Here Bianchi identity already used to recover

$$\mathbf{R}_{\mu\infty} = O(\rho^{d/2-1}) \text{ and } \mathbf{R}_{\infty\infty} = O(\rho^{d/2-1})$$

   from $\mathbf{R}_{\mu\nu} = O(\rho^{d/2})$. This last condition allows us to recursively determine $\tilde{g}^{[k]}_{\mu\nu}$ for $k = 0, \ldots, d/2-1$ and $\operatorname{tr}\tilde{g}^{[d/2]}$. We construct gauge fixing functions from $\mathbf{R}_{\mu\infty}$ and $\mathbf{R}_{\infty\infty}$.

2. **Nondynamical fields.** Trace $\operatorname{tr}\tilde{g}^{[d/2]}_{\mu\nu}$ is nondynamical (it appears only without derivatives). It can be cancelled from equations.

# Gauge fixing

Introduce,

$$\tilde{S}_{\mu\nu} := \mathbf{R}_{\mu\nu}|_{t=1}, \quad \tilde{S}_{\mu\infty} := \mathbf{R}_{\mu\infty}|_{t=1}, \quad \tilde{S}_{\infty\infty} := \mathbf{R}_{\infty\infty}|_{t=1}.$$

Define gauge fixing functions ($\partial_\infty^{-1}$ formal integration in $\rho$)

$$\tilde{\gamma} = -\frac{1}{2}\tilde{g}^{[0]\xi\chi}\tilde{g}^{[1]}_{\xi\chi} + \partial_\infty^{-1}\tilde{S}_{\infty\infty},$$
$$\tilde{G}_\mu = \tilde{F}^{[0]}_\mu + 2\partial_\infty^{-1}\tilde{S}_{\mu\infty} - \partial_\mu\partial_\infty^{-1}\tilde{\gamma},$$

and the gauge fixed tensor

$$\tilde{E}_{\mu\nu} = \tilde{S}_{\mu\nu} - \frac{1}{2}(\tilde{\nabla}_\mu\tilde{G}_\nu + \tilde{\nabla}_\nu\tilde{G}_\mu) - \tilde{g}_{\mu\nu}\tilde{\gamma}.$$

**Remark**

If the metric is Fefferman-Graham then zero order terms of gauge functions:

$$\tilde{\gamma}^{[0]} \propto R, \qquad \tilde{G}^{[0]}_\mu = F_\mu = \Box x_\mu.$$

# The AFG equation is well-posed

1. The equation $\tilde{E}_{\mu\nu} = \tilde{S}_{\mu\nu} - \frac{1}{2}(\tilde{\nabla}_\mu \tilde{G}_\nu + \tilde{\nabla}_\nu \tilde{G}_\mu) - \tilde{g}_{\mu\nu}\tilde{\gamma} = O(\rho^{d/2})$

$$\tilde{E}^{[n]}_{\mu\nu} = -\frac{1}{2}\left[\Box_{\tilde{g}}\tilde{g}_{\mu\nu}\right]^{[n]} + \ldots + c_n \tilde{g}^{[n+1]}_{\mu\nu},$$

   where $c_{d/2-1} = 0$ (recursive and generalized hyperbolic system for $\tilde{g}^{[k]}_{\mu\nu}$ for $k = 0, \ldots, d/2-1$).

2. Bianchi identity gives hyperbolic equations for the gauge

$$-\frac{1}{2}\Box_{\tilde{g}}\tilde{\gamma} + \ldots = O(\rho^{d/2-1}), \quad -\frac{1}{2}\Box_{\tilde{g}}\tilde{G}_\mu + \ldots = O(\rho^{d/2}).$$

   Vanishing of the initial condition for this system follows from vanishing of $R$ and $F_\mu$ to sufficient order on $\Sigma$ and constraints on $\Sigma$.

3. Nondynamical $\tilde{\gamma}^{[d/2-1]} \propto \tilde{S}^{[d/2-2]}_{\infty\infty}$.

Well-posedness in this gauge follows from standard gluing technique the same way as in case of the Einstein's equations.

# Propagation of almost Einstein structure

**Infinite order ambient Ricci flat extensions (Fefferman-Graham)**

If $H_{\mu\nu} = 0$ then the infinite order Ricci flat extensions

$$\mathbf{R}_{IJ} = O(\rho^\infty)$$

are in $1-1$ correspondence with symetric 2-tensors $k_{\mu\nu} := \mathrm{tf}\, \tilde{g}_{\mu\nu}^{[d/2]}$ such that

$$k_\mu^\mu = 0, \quad \nabla^\mu k_{\mu\nu} = D_\nu$$

for some given $D_\nu$.

- On Lorentzian manifold such extensions always exist (for example it can be propagated from a Cauchy surface by some hyperbolic equation).
- We can assume Ricci flat extension.

# Propagation of almost Einstein structure

**Conformally almost Einstein (Gover, Graham-Willse)**

Existence of the covariantly constant covector

$$\boldsymbol{\nabla}_I \mathbf{I}_J = O(\rho^{d/2-1}) \quad \text{then } I_I = \mathbf{I}_I^{[0]}|_{t=1}$$

We have $\Omega = I_0$ and $\mathbf{I}_I \mathbf{I}^I \propto \Lambda + O(\rho^{d/2-1})$.

For the solution to Einstein equation we can choose extension

$$\tilde{g}_{\mu\nu} = (1 + \lambda\rho)^2 h_{\mu\nu}, \quad \lambda \propto \Lambda$$

then $\mathbf{I}_I = \partial_I \boldsymbol{\sigma}$ where $\boldsymbol{\sigma} = t(1 - \lambda\rho)$ and $\boldsymbol{\nabla}_I \mathbf{I}_J = 0$.

# Propagation of almost Einstein structure

**Conformally almost Einstein (Gover, Graham-Willse)**

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$$\boldsymbol{\nabla}_I \mathbf{I}_J = O(\rho^{d/2-1}) \quad \text{then } I_I = \mathbf{I}_I^{[0]}|_{t=1}$$

We have $\Omega = I_0$ and $\mathbf{I}_I \mathbf{I}^I \propto \Lambda + O(\rho^{d/2-1})$.

Propagation ($\mathbf{I}_I = \partial_I \boldsymbol{\sigma}$) depends on the extension

$$\Box \boldsymbol{\sigma} = O(\rho^{d/2+1}), \quad \mathcal{L}_{\mathbf{T}} \boldsymbol{\sigma} = \boldsymbol{\sigma}.$$

- If $\Box \boldsymbol{\sigma} = O(\rho^{d/2+1})$ and $\mathbf{R}_{IJ} = O(\rho^\infty)$ then (recursive)

$$(\Box + \ldots)\, \boldsymbol{\nabla}_I \mathbf{I}_J = O(\rho^{d/2-1})$$

  If the initial data vanish then $\boldsymbol{\nabla}_I \mathbf{I}_J = O(\rho^{d/2-1})$ everywhere.
  Reduces to standard condition $D^{d-1}\, \mathrm{tf}(\nabla_\mu \nabla_\nu \Omega - P_{\mu\nu}\Omega)|_\Sigma = 0$.

- Additional properties $\boldsymbol{\nabla}^I \mathbf{I}_I = O(\rho^{d/2+1})$, $\boldsymbol{\nabla}_{[I} \mathbf{I}_{J]} = 0$.

# Summary

1. The AFG equation (vanishing of the Fefferman-Graham obstruction tensor) is a well-posed system (in $\Box x_\mu = 0$ and $R = 0$ gauge).
2. The almost Einstein condition propagates by hyperbolic equation too, thus we have stability of future or past asymptotically simple solutions (Anderson, Anderson-Chruściel).
3. Application to other equations constructed by ambient metric like conformal powers of d'Alembertians (GJMS), $Q$-curvature etc.

Thank you!

# Non-strictly hyperbolic problems

## Example of ill-posed problem

On $\mathbb{R} \times S^1$

$$\Box^3 \phi + \partial_2 (\partial_1 + \partial_2)^3 \phi = 0, \qquad \Box = \partial_1^2 - \partial_2^2$$

Example mode solutions $\phi_k(x^1, x^2) = e^{i(\omega(k)x^1 + kx^2)}$ with $\omega(k) = \frac{-1-i\sqrt{3}}{2} k^{1/3}$.

$$\partial_1^n \phi|_\Omega = \sum_{k=0}^{\infty} i^n \omega(k)^n e^{-k^{1/4}} e^{ikx^2}, \quad n = 0 \ldots 5$$

does not admit Cauchy development.