Differential invariants of Kundt spacetimes

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A Lorentzian metric g on an n-dimensional manifold M is a Kundt metric if there exists a vector field ℓ such that

$$\|\ell\|_g^2 = 0, \quad \nabla_\ell^g \ell = 0, \quad \mathrm{Tr}(\nabla^g \ell) = 0, \quad \|\nabla^g \ell^{\mathsf{sym}}\|_g^2 = 0, \quad \|\nabla^g \ell^{\mathsf{alt}}\|_g^2 = 0,$$

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where ∇^g is the Levi-Civita connection given by g. We call g a degenerate Kundt metric if, in addition,

- The Riemann tensor Riem is aligned and of algebraically special type II, and
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For any Kundt metric, there exist local coordinates $u, x^1, ..., x^{n-2}, v$ in which g takes the form

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In these coordinates, $\ell = \partial_v$, and g is a degenerate Kundt metric if and only if $(W_i)_{vv} = 0$ and $H_{vvv} = 0$.

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One of the standard ways is to use polynomial curvature invariants, i.e. total contractions of the curvature tensor and its covariant derivatives. However, not all spacetimes are separated by such invariants. In particular, in dimension $n = 4$ the degenerate Kundt spacetimes are exactly those that can not separated by polynomial curvature invariants (Coley, Hervik, Pelavas 2009).

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Therefore we need to use other invariants!

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Theorem

The transformations preserving the form of (1) *are given by*

$$(u, x^i, v) \mapsto \left(C(u), A^i(u,x), \frac{v}{C'(u)} + B(u,x)\right), \quad \det[A^i_{x^j}] \neq 0, C'(u) \neq 0.$$

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The Lie algebra $\mathfrak{g}$ *corresponding to this Lie pseudogroup consists of the vector fields*

$$c(u)\partial_u + a^i(u,x)\partial_{x^i} + \left(b(u,x) - c'(u)v\right)\partial_v.$$

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The vector fields of $\mathfrak{g}$ can be lifted to $F \times M$ by requiring the lifts to preserve the horizontal symmetric form

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The lift $\hat{X}$ of the vector field $X = c(u)\partial_u + a^i(u,x)\partial_{x^i} + \big(b(u,x) - c'(u)v\big)\partial_v$ is found by setting $\hat{X} = X + A_{ij}\partial_{h_{ij}} + B_i\partial_{W_i} + C\partial_H$, and determining the coefficients from the equation $L_{\hat{X}}G = 0$:

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$$\begin{aligned}\hat{X} =& c\partial_u + a^i\partial_{x^i} + (b - c'v)\partial_v - (a^l_i h_{lj}\partial_{h_{ij}} + a^l_i h_{li}\partial_{h_{ii}}) \\ & - (c'W_i + a^j_i W_j + b_i + 2a^j_u h_{ij})\partial_{W_i} - (2c'H - c''v + b_u + a^j_u W_j)\partial_H.\end{aligned}$$

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We are not working with arbitrary sections, but with sections satisfying certain differential equations.

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  For $k > 3$, we define $\tilde{\mathcal{E}}^k \subset J^k\pi$ as the prolongation of $\tilde{\mathcal{E}}^3$.

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Any vector field X in $\mathfrak{g}$ prolongs to a vector field $\hat{X}^{(k)}$ on $J^k\pi$ in the standard way, and $\hat{X}^{(k)}$ is tangent to $\mathcal{E}^k$ and $\tilde{\mathcal{E}}^k$ for each k.

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We would like to distinguish sections of π satisfying $\mathcal{E}$ or $\tilde{\mathcal{E}}$ under the equivalence relation given by the Lie algebra $\mathfrak{g}$. Both of these are of infinite dimension. However, each manifold $\mathcal{E}^k, \tilde{\mathcal{E}}^k \subset J^k\pi$ is of finite dimension, and so are $\mathfrak{g}$-orbits on these.

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Theorem (Lie-Tresse-...-Kruglikov-Lychagin)

The algebra of rational differential invariants separates orbits in general position in $\mathcal{E}^\infty$ and $\tilde{\mathcal{E}}^\infty$. It is generated by a finite number of differential invariants and invariant derivations.

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Then $I_g^1, ..., I_g^n$ can be used as coordinates on M, and we can write g in these coordinates:

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On this Zariski open set, $\hat{d}I^1, ..., \hat{d}I^n$ form an invariant horizontal coframe.

Invariants for general Kundt metrics

Generic Kundt metrics can be distinguished by polynomial curvature invariants. In particular, we may take the invariants

$$I^i = \mathrm{Tr}(\mathrm{Ric}^i)$$

for $i = 1, ..., n$, where $\mathrm{Ric}\colon TM \to TM$ is the Ricci operator. We have

$$\hat{d}I^1 \wedge \cdots \wedge \hat{d}I^n \neq 0$$

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On this Zariski open set, $\hat{d}I^1, ..., \hat{d}I^n$ form an invariant horizontal coframe. There is also a dual frame of invariant derivations $\hat{\partial}_i = AD_u + B_i^j D_{x^j} + CD_v$, satisfying $\hat{d}I^i(\hat{\partial}_j) = \delta^i_j$.

Invariants for general Kundt metrics

We can express the horizontal symmetric form

$$G = du\left(dv + H\,du + W_i\,dx^i\right) + h_{ij}\,dx^i dx^j.$$

as

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where $K_{ij} = G(\hat{\partial}_i, \hat{\partial}_j)$ are rational differential invariants of order 3.

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Theorem

The algebra of rational differential invariants is generated by I^i, K_{ij} and the invariant derivations $\hat{\partial}_i$.

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For $n-1$ horizontally independent invariants $J^1, \ldots, J^{n-1}$ from the above set, let $\nabla_1, \ldots, \nabla_{n-1}$ be the G-duals to $\hat{d}J^1, \ldots, \hat{d}J^{n-1}$.

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Define the nth derivation by

$$G(\nabla_1, \nabla_n) = 1, \qquad G(\nabla_i, \nabla_n) = 0 \text{ for } i = 2, ..., n.$$

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Theorem

The algebra of differential invariants is generated by the differential invariants L_{ij}, c^k_{ij} and the invariant derivations ∇_i.

Other choices of generators of invariants

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The above approach is very flexible regarding the choice of n invariants or n invariant derivations. All we require is that $\hat{d}I^1 \wedge \cdots \wedge \hat{d}I^n$ is nonvanishing, or that $\nabla_1, ..., \nabla_n$ are independent, on generic points.

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We will show another choice of generators for $n = 3$.

Three-dimensional Kundt spacetimes

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Theorem

The derivations

$$\nabla_1 = \frac{W_v}{W_{vv}} D_v, \qquad \nabla_2 = \frac{2}{W_v} D_x + \frac{h_x W_v - 2hW_{xv}}{hW_v W_{vv}} D_v,$$
$$\nabla_3 = \frac{1}{W_v} \left(H_{vv} D_x - W_{vv} D_u + (W_{uv} - H_{xv}) D_v \right)$$

are invariant, and they are independent on a Zariski open subset of $\mathcal{E}^2$.

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The derivations satisfy $[\hat{X}^{(\infty)}, \nabla_i] = 0$ for each $X \in \mathfrak{g}$, and they were found by solving this system of PDEs.

Three-dimensional Kundt spacetimes

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If we let α^j denote the elements of the dual horizontal coframe ($\alpha^j(\nabla_i) = \delta_i^j$), then the horizontal symmetric 2-form G written in terms of this coframe will have coefficients given by $G(\nabla_i, \nabla_j)$.

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where $I_1 = \frac{W_v^2}{h}$ is the first-order differential invariant from the previous slide and

$$J_1 = \frac{HW_{vv}^2 + (-H_{vv}W + H_{xv} - W_{uv})W_{vv} + H_{vv}^2 h}{h},$$
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Theorem

For $n = 3$ the algebra of differential invariants on $\mathcal{E}$ is generated by the differential invariants I_1, J_1, J_2 and the invariant derivations $\nabla_1, \nabla_2, \nabla_3$.

Three-dimensional Kundt spacetimes

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We also have the second-order differential invariants

$$\nabla_3(I_1) = 2\,\frac{H_{vv}W_{xv} - H_{xv}W_{vv}}{h} - \frac{W_v(H_{vv}h_x - W_{vv}h_u)}{h^2},$$

$$\begin{aligned} J_3 = {} & \frac{W_{vv}^2(h_u^2 - 2hh_{uu})}{h^3} - \frac{2W_{vv}(H_vW_{vv} - H_{vv}W_v)h_u}{h^2} \\ & - \frac{((H_vW - H_x + W_u)W_{vv}^2 - W_v(H_{vv}W - H_{xv} + W_{uv})W_{vv} + 2H_{vv}^2hW_v)h_x}{h^3} \\ & + \frac{(-2H_xW_v + 2H_vW_x + 2H_{xv}W - 2H_{xx} + 2W_{ux})W_{vv}^2}{h^2} \\ & + \frac{((-2H_{vv}W + 2H_{xv} - 2W_{uv})W_{xv} - 4H_{xv}H_{vv}h)W_{vv} + 4H_{vv}^2hW_{xv}}{h^2} \end{aligned}$$

which, together with I_1, J_1, J_2, constitute a transcendence basis for the field of second-order differential invariants on $\mathcal{E}^2$.

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Let us now consider degenerate kundt metrics. We still have the invariant $I_1 = \frac{W_v^2}{h}$.
We define the following functions:

$$I_{2a} = H_{vv}, \quad I_{2b} = \frac{W_v h_x - 2hW_{xv}}{h^2}, \quad K_{2a} = \frac{H_{xv} - W_{uv}}{W}, \quad K_{2b} = \frac{W_v h_u - 2hW_{uv}}{Wh}.$$

The functions I_{2a} and I_{2b} are second-order differential invariants on $\tilde{\mathcal{E}}^2$.

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The functions I_{2a} and I_{2b} are second-order differential invariants on $\tilde{\mathcal{E}}^2$. We also define

$$Q = \frac{(2I_{2a}K_{2b} + I_{2b}K_{2a} - I_{2a}I_{2b})W}{I_1},$$
$$R = \frac{I_{2b}HW_v^2}{I_1} - \frac{(I_{2b}I_{2a}^2 - 2K_{2a}(I_{2b} - 2K_{2b})I_{2a} + I_{2b}K_{2a}^2)W^2}{4I_{2a}^2}.$$

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A fourth second-order differential invariant is given by

$$\begin{aligned} I_{2c} =& \frac{1}{Q^2}\Bigg(\frac{(I_1^2 h_u(WW_v + h_u) - (h_x(H_vW - H_x + W_u)I_1^2 - W_v^4 I_{2b}H))I_{2a}I_{2b}}{W_v^2} \\ & -2\left(W_v H_x + (h_u - W_x)H_v + H_{xx} - W_{ux} + h_{uu}\right) I_1 I_{2a} I_{2b} \\ & -W^2 I_1 (K_{2b}(I_{2b} - K_{2b})I_{2a} - 2I_{2b}K_{2a}^2)\Bigg). \end{aligned}$$

Three-dimensional degenerate Kundt spacetimes

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We have $\hat{d}I_1 \wedge \hat{d}I_{2a} \wedge \hat{d}I_{2c} \neq 0$ on a Zariski open set in $\tilde{\mathcal{E}}^3$.

Three-dimensional degenerate Kundt spacetimes

We have $\hat{d}I_1 \wedge \hat{d}I_{2a} \wedge \hat{d}I_{2c} \neq 0$ on a Zariski open set in $\tilde{\mathcal{E}}^3$. It is possible to express G in terms of $\hat{d}I_1, \hat{d}I_{2a}, \hat{d}I_{2c}$ as before, and in this way find a generating set of invariants.

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Alternatively, we may use the following invariant derivations:

$$\nabla_1 = \frac{I_1}{I_{2a}I_{2b}} \cdot \frac{Q}{W_v} D_v, \qquad \nabla_2 = \frac{1}{W_v}\left(D_x - \frac{K_{2a}}{I_{2a}} W D_v\right),$$
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Theorem

The algebra of differential invariants on $\tilde{\mathcal{E}}$ is generated by the differential invariants I_1, I_{2a}, I_{2c} and the invariant derivations $\nabla_1, \nabla_2, \nabla_3$.

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