Simple nonholonomic systems on the plane

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Introduction

The purpose of this lecture is to give an introduction for studies of geometries of mechanical systems obeying nonholonomic constraints. I will not talk about the *dynamics* of such systems. It turns out that in the nonholonomic regime, already the *kinematics* is quite interesting. Even in the case of systems consisting of a few points moving on the plane, the ZOO of geometric structures apearing on their configuration spaces very quickly becomes fascinating. In particular, several simple Lie groups find their realizations as symmetries of such systems.

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Configuration space and the movement

- In classical mechanics one usually models the movement of a mechanical system using an n-dimensional manifold M, which is interpreted as the *configuration space of the system*. Its points $q \in M$ correspond to all *positions* that the system may assume during its evolution. The number n corresponds to the number of degrees of freedom of the system.
- A *movement* of the system from a given position q_i at time t_i to a position q_f at time t_f is modelled in terms of a (piecewise) smooth curve $]t_i, t_f[\ni t \to q(t) \subset M$. The derivative $v = \frac{\mathrm{d}q}{\mathrm{d}t}$ represents the *velocity* of the system at time t in the point $q = q(t)$ on the curve.
- All possible velocities at $q \in M$, as tangent to all possible curves at q, form the *tangent space* T_qM to M at q. It is an n-dimensional *vector space*.

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Unconstrained velocity space

- Another manifold frequently used in classical mechanics to describe mechanical systems is the tangent bundle TM to M, whose points are pairs (q, v), where $q \in M$ and $v \in T_qM$. The tangent bundle TM represents all possible positions (q) and velocities (v) of the system. It can be visualised as an n dimensional manifold M of positions of the system, with an n-dimensional vector spaces of possible velocities T_qM attached to every point $q \in M$.

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Constraints

We have to decide if the system we want to describe does, or does not, obey any *constraints*. There are two different classes of constraints that we have to consider:

- First, there could be some imposed restrictions on positions, which means that there are some relations between the *positions*, $F(q) = 0$. We assume that locally such constraints define a submanifold of M, which becomes a new, lower dimensional, configuration space of the constrained system. Restricting the movement to this submanifold, say N, merely diminishes the number n of degrees of freedom, and the system can now be described in terms of the new configuration space N and its tangent bundle TN as before.

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- The second class of constraints we want to discuss is much more interesting. These are the constraints that impose relations on points in the tangent bundle TM to M. In physical terms these are the constraints that make restrictions on *velocities*. They can be schematically described by relations of the form $H(q, v) = 0$. Since the velocities v's are related to the positions q's by taking derivatives, it may happen that the relations $H(q, v) = 0$ can be *integrated* to $F(q) = 0$.

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Nonholonomic constraints

- In other words it may happen that, roughly, the velocity/position constraints $H(q, v) = 0$ are related to the *differential* of $F(q)$, in such a way that $H(q, v) = 0$ if and only if $F(q) = 0$. These kinds of velocity/position constraints, which we will call *integrable* ones, are therefore equivalent to the constraints on positions $F(q) = 0$, which were discussed before.
- We will exclude such velocity/position constraints from our consideration from now on, and we will focus on the velocity/position constraints $H(q, v) = 0$ which are *not* integrable. Such constraints are called *nonholonomic*.

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Velocity distribution

- We note that at each point q of the configuration space M, the nonholonomic relations $H(q, v) = 0$, define subsets

$$D_q = \{ v \in T_q M \mid H(q, v) = 0 \} \subset T_q M$$

  in the tangent space $T_q M$. In general these sets are *nonlinear* subsets in $T_q M$.

- We will focus on the situations when these sets D_q are *vector subspaces*. This corresponds to the *linear* constraints on velocities.
- Furthermore, we will only deal with the *regular* systems, for which vector subspaces D_q will be such that their dimension k is constant along M.

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Velocity distribution

- The configuration space M of such systems is equipped with a smooth assignment

$$M \ni q \overset{D}{\to} D_q \subset T_q M$$

of k-planes D_q to each point q of M.

- Such an assignment is called *rank k distribution* on M.
- Since this distribution is the distribution of all possible velocities of the mechanical sytsem, it is called *velocity distribution*.
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Frobenius theorem

- We recall that a rank k distribution D is *integrable* if and only if there is a foliation of M by submanifolds tangent to the k-planes D_q of the distribution D.
- The *Frobenius theorem* states that M is foliated by such submanifolds if and only if the space $D_{-2} = [D, D]$, consisting of all commutators of vector fields from D, is equal to D.
- Thus, the mechanical system with configuration space M and velocity distribution D is nonholonomic if and only if D is nontrivially contained in D_{-2}, $[D, D] \supsetneq D$.

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Skate blade on an ice rink

- We now show how the velocity distribution D looks like in the case of a mechanical system, which for obvious reasons, we call a *skate on an ice rink*.
- We idealize the skate blade as an interval of a fixed length on the Cartesian plane. We assume that the blade moves *without skidding*, which means that the velocity of the mid point of the blade is always parallel to the line defined by the direction of the blade.

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- We now show how the velocity distribution D looks like in the case of a mechanical system, which for obvious reasons, we call a *skate on an ice rink*.
- We idealize the skate blade as an interval of a fixed length on the Cartesian plane. We assume that the blade moves *without skidding*, which means that the velocity of the mid point of the blade is always parallel to the line defined by the direction of the blade.

Skate blade

- To parametrize the configuration space of the blade we attach coordinates (x, y) to its middle point. Then the position of the blade on the plane is totally determined by numbers x, y and an angle α, which the blade direction forms with the Ox axis. Thus the configuration space of the skate blade is $M = \mathbb{R}^2 \times \mathbb{S}^1$, and the movement of the *velocity unconstrained* blade is described in terms of a curve

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Velocity distribution for the skate blade

- The velocity constraint of nonskidding in this language says that $(\dot{x}, \dot{y})$ is parallel to $(\cos\alpha, \sin\alpha)$, or what is the same, that
$$\dot{x}\sin\alpha - \dot{y}\cos\alpha = 0.$$
- This last condition says that the velocity $v = (\dot{x}, \dot{y}, \dot{\alpha})$ of the blade must satisfy the following *linear* relation $(\sin\alpha)\cdot v_x - (\cos\alpha)\cdot v_y + 0\cdot v_\alpha = 0$. Since we have only *one scalar* constraint on the $v's$, then two out of the three velocity components of the system are free. Thus, at every point $q = (x, y, \alpha)$ the nonskiding condition distinguishes a 2-plane of possible velocities D_q. This can be easily seen to be spanned by ∂_α and $(\cos\alpha)\partial_x + (\sin\alpha)\partial_y$.

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- This defines a rank 2 distribution

$$D = Span((\cos\alpha)\partial_x + (\sin\alpha)\partial_y, \partial_\alpha)$$

  on M, to which every movement of the skate blade obeying the nonskidding constraint must be tangent.
- We use the Frobenius theorem to show that our skate blade mechanical system is *nonholonomic*. Indeed, taking the two vector fields $X_1 = \partial_\alpha$ and $X_2 = (\cos\alpha)\partial_x + (\sin\alpha)\partial_y$ belonging to the velocity distribution D of this system, we see that $[X_1, X_2] = -(\sin\alpha)\partial_x + (\cos\alpha)\partial_y$. And this does not belongs to D for all values of (x, y, α). Thus $D_{-2} \supsetneq D$, which according to the Frobenius theorem implies that the *skate blade mechanical system is nonholonomic*.

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Achievability

- Suppose now that we have a mechanical system with an n-dimensional configuration space M and linear velocity constraints. Then we have a rank $k < n$ velocity distribution D on M, and all movements of the system obeying these constraints are described by curves $q = q(t) \subset M$, which are always tangent to D. Such curves are called *horizontal with respect to* D, or *horizontal* for short.
- Now we encounter the *problem of reaching a given configuration by the velocity constrained system*. We formulate it as follows: determine if two points q_1 (i.e. the starting configuration) and q_2 (i.e. the final configuration) are *horizontally path connected* on M.

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- For example, if the velocity distribution D is integrable on M, i.e. if $D_{-2} = D$, then two points q_1 and q_2, which lie on two different leaves of the foliation defined by D are *not* horizontally path connected. Simply, horizontal movements of points which lie on a given leaf, being tangent to the leaf, will stay at this leaf. In other words, the integrability of the velocity distribution D is an obstruction to horizontal path connectivity of M: two different leaves are never horizontally path connected.
- It follows from this example that, to consider linear velocity constrained systems that can reach any configuration point starting from any other configuration, the system must be nonholonomic, or what is the same, its velocity distribution D should be such that $D_{-2} \supsetneq D$. The question arises if the necessity of $D_{-2} \supsetneq D$ is *sufficient* for such reachability. The answer to this question is given by the Chow-Raszewski theorem.

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Bracket generating distributions

- Its formulation requires an introduction of the following sequence of distributions on the configuration space M.
- We start with $D_{-1} = D$, and define a sequence of nested distributions

$$D_{-1} \subset D_{-2} \subset \cdots \subset D_{-s} \subset \cdots \subset TM,$$

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If this happens the distribution D is called *bracket generating*, or *maximally nonintegrable*, the integer

$$r = s_0 + 1$$

is called the *step*, and the sequence of integers

$$(rank(D_{-1}), rank(D_{-2}), \ldots, rank(D_{-(r-1)}), rank(D_{-r})) =: \vec{N}$$

is called the *growth vector* of the distribution D. In particular the growth vector, as carrying information about (a) the rank of D (the first entry in $\vec{N}$), (b) the dimension of M (the last entry in $\vec{N}$), and (c) the step (the number of components of $\vec{N}$), gives the simplest *numerical invariants* of the distribution.

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- Note that if D is integrable, we have $D_{-s} = D_{-1} = D$ for all s, and in such situation the above sequence has only one element D_{-1}. Often however, there exists $s_0 > 0$ such that

$$D_{-1} \subsetneq D_{-2} \subsetneq \cdots \subsetneq D_{-s_0} \subsetneq D_{-(s_0+1)} = TM.$$

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- Note that the above definitions are *pointwise*, and therefore the growth vector is a vector valued *function* on M. In what follows we will however consider only situations when the growth vector is *locally constant* over some open set $U \in M$,
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  - $D_{-1} = Span((\cos\alpha)\partial_x + (\sin\alpha)\partial_y, \partial_\alpha)$,
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- The converse to this theorem is true in the case of *analytic* distributions but it fails in general, even if the distribution is smooth (see e.g. R. Montgomery's book, Section 2.1). Anyhow, in the piecewise smooth category, this theorem gives a *sufficient* condition for a mechanical system with linear velocity constraints to have the ability to move from any given configuration to any other one. For this it is enough that the velocity distribution of the system is bracket generating.
- It turns out that there is also another, much stronger, theorem giving sufficient conditions for a system to reach any configuration. It combines results of Nagano and Sussman and states the following:

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- **Nagano-Sussman theorem.** *Let* $\mathcal{F} = \{X_i\}$ *be a family of vector fields* X_i *on a manifold* M*. Suppose that a finite number of brackets of the* X_i*s and a finite number of iterations of these brackets generate* T_qM *at every* $q \in M$ *(we say that the family* $\mathcal{F}$ *is bracket generating). Then the orbit of this family of vector fields at each point is all of* M*.*
- Here the term *orbit of a family at a point* $q \in M$ means all points in M that can be connected with q by piecewise smooth segments of integral curves of vector fields X_i from the family $\mathcal{F}$. The fact that the orbit of the family $\mathcal{F}$ through every point is all of M means that every point $q \in M$ can be reached by such broken integral curves of vector fields X_i regardless of the starting point $q_0 \in M$.

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Contact distributions

- An important class of nonholonomic distributions is given by *contact* distributions. These are rank $k = 2m$ distributions D on a $(2m+1)$-dimensional manifold, which *annihilate* a single 1-form λ on M such that its corresponding 2-form $\omega = \mathrm{d}\lambda$ is not degenerate on D. More formally, given a 1-form λ such that

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on M, a *contact distribution* is

$$D = Span\{X \in TM \;:\; \lambda(X) = 0\}.$$

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- An example of a contact distribution is given by the velocity distribution D of the skate blade considered earlier. Indeed, take the 1-form $\lambda = -(\sin\alpha)\mathrm{d}x + (\cos\alpha)\mathrm{d}y$ on our skate blade configuration space M. Clearly we have $\lambda(\partial_\alpha) = \lambda((\cos\alpha)\partial_x + (\sin\alpha)\partial_y) = 0$, i.e. the velocity distribution D of the skate blade is an annihilator of λ. Also $\lambda \wedge \mathrm{d}\lambda = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\alpha \neq 0$, meaning that D is contact.
- We mentioned above that the growth vector provides the simplest *invariants* of a distribution. They are used to distinguish if two distributions are (locally) equivalent. The precise definition of a (local) *equivalence* is as follows.

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- An example of a contact distribution is given by the velocity distribution D of the skate blade considered earlier. Indeed, take the 1-form $\lambda = -(\sin\alpha)\mathrm{d}x + (\cos\alpha)\mathrm{d}y$ on our skate blade configuration space M. Clearly we have $\lambda(\partial_\alpha) = \lambda((\cos\alpha)\partial_x + (\sin\alpha)\partial_y) = 0$, i.e. the velocity distribution D of the skate blade is an annihilator of λ. Also $\lambda \wedge \mathrm{d}\lambda = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\alpha \neq 0$, meaning that D is contact.
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Geometry of maximally nonintegrable distributions

- Let D_1 and D_2 be two rank k distributions living on two, not neccessarily different, n-dimensional manifolds M_1 and M_2. We say that the two distributions D_1 and D_2 are (locally) *equivalent* if and only if there exists a (local) diffeomorphism

$$\phi : M_1 \to M_2$$

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These (local) *self*equivalences are called (local) *symmetries* of the distribution D. Since they can be composed and inverted as maps, they form a *group*. The set of all (local) symmetries is *the (local) symmetry group* $Aut(D)$ of the distribution D.

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- Given two infinitesimal symmetries Y_1 and Y_2 of D, their commutator $[Y_1, Y_2]$ is also an infinitesimal symmetry of D, and the set of all infinitesimal symmetries of D naturally has the structure of a *Lie algebra*. This Lie algebra is called *the symmetry algebra* $\mathfrak{aut}(D)$ of D.
- The local Lie group $Aut(D)$ and the Lie algebra $\mathfrak{aut}(D)$ are closely related. In particular, for every value of the real parameter t, the flow $\phi_t(Y)$ of an infinitesimal symmetry $Y \in \mathfrak{aut}(D)$ is a local diffeomorphism of M. It forms a 1-parameter subgroup in the local symmetry group $Aut(D)$.

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Example of a $\vec{N} = (2,3,5)$ distribution

- We illustrate the notion of an infinitesimal symmetry of a distribution by the following example in 5 dimensions.
- Consider $\mathbb{R}^5$ with local coordinates (x, y, p, q, z). Let $D_{CE} = Span(X_1, X_2)$ be a rank 2-distribution spanned over the smooth functions on $\mathbb{R}^5$ by the following two vector fields:

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$$X_1 = \partial_q, \quad X_2 = \partial_x + p\partial_y + q\partial_p + \tfrac{1}{2}q^2\partial_z.$$

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this is clearly a regular bracket generating distribution, with a growth vector $\vec{N} = (2,3,5)$.

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Example of a $\vec{N} = (2, 3, 5)$ distribution

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Distribution with a remarkable symmetry

Cartan-Engel theorem. *The symmetry algebra of the distribution* $\mathcal{D}_{CE}$ *is a 14-dimensional split real form of the exceptional simple Lie algebra* $\mathfrak{aut}(\mathcal{D}_{CE}) = \mathfrak{g}_2$. *It can be spanned over the reals by the following vector fields* Y_μ, $\mu = 1, 2, \ldots, 14$ *on* $\mathbb{R}^5$*:*

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\begin{aligned}
Y_1 =& (12p^2 - 18qy)\partial_x + (8p^3 - 18pqy + 18yz)\partial_y + (18pz - 9q^2y)\partial_p + (18qz - 6pq^2)\partial_q + \\
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- Continuing the example of the skate blade kinematics, we recall that:
  - the skate blade configuration space M is locally $\mathbb{R}^2 \times \mathbb{S}^1$ with coordinated (x, y, α); the velocity distribution D is *contact*, and is defined as the annihilator of the contact form $\lambda = -\sin\alpha \mathrm{d}x + \cos\alpha \mathrm{d}y$;
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Para-CR structures

- This leads to the following Definition: A $(2m+1)$-dimensional manifold M with a contact distribution D is called a *para-CR structure* if D splits onto $D = D_1 \oplus D_2$, where the distributions D_1 and D_2, each of the same rank m, are integrable.
- Note that although D_1 and D_2 are integrable, the distribution D is *not*. As we remember it is an annihilitar of a nondegenerate (contact) 1-form λ such that $\lambda \wedge \underbrace{\mathrm{d}\lambda \wedge \mathrm{d}\lambda \wedge \cdots \wedge \mathrm{d}\lambda}_{m \text{ times}} \neq 0$.
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- **Theorem**. *The symmetry algebra* $\mathfrak{aut}(D_1 \oplus D_2)$ *of the para-CR structure* $D = D_1 \oplus D_2$ *on the configuration space* $M = \{(x, y, \alpha) \mid (x, y) \in \mathbb{R}^2, \alpha \in \mathbb{S}^1\}$ *of the skate blade is isomorphic to* $\mathfrak{sl}(3, \mathbb{R})$. *It is spanned over the reals by the following vector fields* Y_μ, $\mu = 1, 2, \ldots, 8$ *on* M*:*

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Y_1 &= x^2\partial_x + yx\partial_y - \cos\alpha(x\sin\alpha - y\cos\alpha)\partial_\alpha\,,\\
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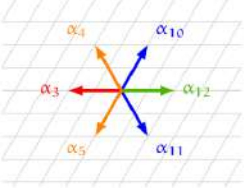

FIGURE 1. The root diagram for $\mathfrak{sl}(3,\mathbb{R})$. The orange roots and the red root, together with the two elements in the Cartan subalgebra, form a 5-dimensional (parabolic) subalgebra $\mathfrak{p}$ in $\mathfrak{sl}(3,\mathbb{R})$. The 3-dimensional $\mathbf{SL}(3,\mathbb{R})$ homogeneus space $M = \mathbf{SL}(3,\mathbb{R})/P$, with P being a subgroup of $\mathbf{SL}(3,\mathbb{R})$ having Lie algebra $\mathfrak{p}$, is naturally equipped with the $\mathbf{SL}(3,\mathbb{R})$ *homogeneous* para-CR structure, which at every point of M is in the tangent space identified with $\mathfrak{sl}(3,\mathbb{R})/\mathfrak{p}$. In this space the blue roots represent the contact distribution D. Thhe split in D is represented by the directions spanned by α_{10} and α_{11} respectively. The $D_{-1} = \mathrm{Span}(\alpha_{10},\alpha_{11})$, and $D_{-2} = \mathrm{Span}(\alpha_{10},\alpha_{11},\alpha_{12})$. This 3-dimensional para-CR structure is the global version of the para-CR structure of the skate blade.

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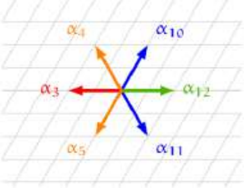

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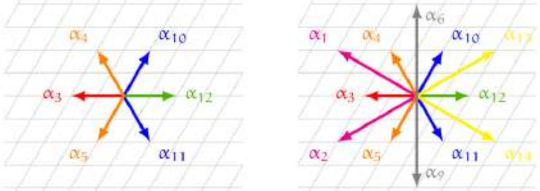

FIGURE . Comparizon of the root diagrams for $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{g}_2$. The addition of yellow, gray, and magenta roots, extends $\mathfrak{sl}(3,\mathbb{R})$ to be a subalgebra in $\mathfrak{g}_2$. The $\vec{N} = (2,3)$ distribution defining the para-CR structure in $M = \mathbf{SL}(3,\mathbb{R})/P$ of the skate blade, is somehow related to the $\vec{N} = (2,3,5)$ distribution on the G_2-homogeneous $\vec{N} = (2,3,5)$ distribution D defined on $M^5 = G_2/P_1$. This rank 2 distribution in *five dimension* is visible in the tangent spaces $\mathfrak{g}_2/\mathfrak{p}_1$ as $D = D_{-1} = \mathrm{Span}(\alpha_{10}, \alpha_{11})$, $D_{-2} = \mathrm{Span}(\alpha_{10}, \alpha_{11}, \alpha_{12})$ and $D_{-3} = \mathrm{Span}(\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14})$.

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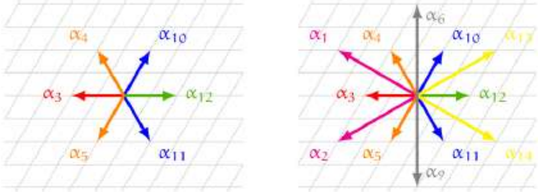

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Is there something more exciting?

- Repeted question: Can we realize G_2 as a symetry of a mechanical nonholonomic system on the plane? By this I mean that the velocity constraints should be imposed on points/lines contained in the plane.
- To realize such a system, its configuration space should be minimum 5-dimensional: the maximal dimension of the proper subgroups H in G_2 is *nine*. So the minimal dimension of a homogeneous space $M^n = G_2/H$ is $n = 14 - 9 = 5$.
- Since a point on the plane $\vec{r}$ has 2 coordinates, $\vec{r} = (x, y)$, we need a minimum number of *three* moving points $\vec{r}_i = (x_i, y_i)$, $i = 1, 2, 3$.

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- Consider a mechanical system of three ants on the floor, which move according to two independent rules:

Rule A - which forces the velocity of any given ant to always point at a neighboring ant, and

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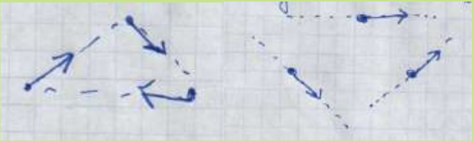

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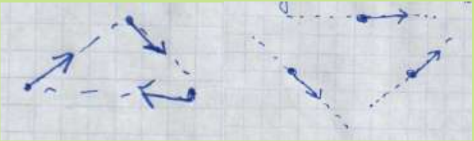

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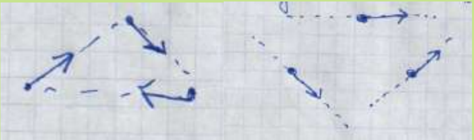

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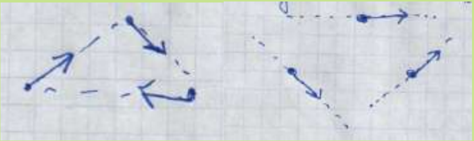

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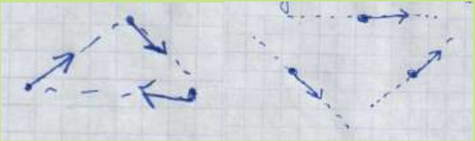

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Here and in the following $i, j = 1, 2, 3$ and $i + j$ is counted modulo 3.
- This rule in coordinates $\vec{r}_i = (x_i, y_i)$ can be written as:
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In case of **rule A** the distribution $\mathcal{D}$ of admissible velocities on M is given by the *annihilator* of the following three 1-forms:

$$\omega_i = (y_{i+1} - y_i)dx_i - (x_{i+1} - x_i)dy_i, \quad i = 1, 2, 3,$$

or which is the same, is spanned by the three vector fileds

(2.1) $$Z_i = (x_{i+1} - x_i)\partial_{x_i} + (y_{i+1} - y_i)\partial_{y_i}, \quad i = 1, 2, 3$$

on M,

$$\mathcal{D} = \mathrm{Span}(Z_1, Z_2, Z_3).$$

Taking the commutators of the vector fields Z_1, Z_2, Z_3 spanning the distribution $\mathcal{D}$ we get three new vector fields

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$$Z_1 \wedge Z_2 \wedge Z_3 \wedge Z_{12} \wedge Z_{31} \wedge Z_{23} = \Big(\sum_{i=1}^{3}(y_i x_{i+1} - x_i y_{i+1})\Big)^3 \partial_{x_1} \wedge \partial_{y_1} \wedge \partial_{x_2} \wedge \partial_{y_2} \wedge \partial_{x_3} \wedge \partial_{y_3},$$

so it follows that the *six* vector fields $Z_1, Z_2, Z_3, Z_{12}, Z_{31}, Z_{23}$ are *linearly independent* at each point m of the configuration space M everywhere, *except* the points on the singular locus, where coordinates of m satisfy

(2.2) $$32\,A = \sum_{i=1}^{3}(y_i x_{i+1} - x_i y_{i+1}) = 0.$$

Since the number A defined above is the *area* of the triangle having the three ants as its vertices, we see that the velocity distribution $\mathcal{D}$ of the three ants moving under **rule A** has a *growth vector* $(3, 6)$ everywhere, except the configuration points corresponding to the three ants staying on a line.

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Now, applying **rule B** to the movement of the three ants, we find that their velocity distribution $\mathcal{D}$ is given by the annihilator of the three Pfaffian forms

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And now the story is *different* than in the case of **rule A**. Calculating $Z_1 \wedge Z_2 \wedge Z_3 \wedge Z_{12} \wedge Z_{31} \wedge Z_{23}$, one gets

$$Z_1 \wedge Z_2 \wedge Z_3 \wedge Z_{12} \wedge Z_{31} \wedge Z_{23} = 0.$$

So the rank of the derived distribution $\mathcal{D}^1 = [\mathcal{D}, \mathcal{D}] + \mathcal{D}$ is *smaller* than 6. The velocity distribution $\mathcal{D}$ for the rule B is *not* bracket generating! Actually one easilly finds that there is *precisely* one linear relation between the vector fields $(Z_1, Z_2, Z_2, Z_{12}, Z_{31}, Z_{23})$, namely

$$Z_1 + Z_2 + Z_3 + Z_{12} + Z_{31} + Z_{23} = 0. \tag{3.3}$$

This shows that the velocity distribution $\mathcal{D}$ for the ants moving under **rule B** has the growth vector $(3, 5)$. The first derived distribution $\mathcal{D}^1$ has rank 5 and is *integrable*! The 6-dimensional configuration space M of ants being in a motion obeying **rule B** is foliated by 5-dimensional leaves. Once ants are in the configuration belonging to a given 5-dimensional leaf in M they can *not* leave this leaf by moving according **rule B**!

Now the question arises about the function that enumerates the leaves of the foliation of the distribution $\mathcal{D}^1$. What is the feature of motion of the ants whose preservation forces the ants to stay on a given leaf?

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Simple nonholonomic systems on the plane. Part 2

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Iława 21.08.2021

This is a joint work with

Andrei Agrachov

Movement of three ants on the plane

- Consider a mechanical system of three ants on the floor, which move according to two independent rules:

Rule A - which forces the velocity of any given ant to always point at a neighboring ant, and

Rule B - which forces the velocity of every ant to be parallel to the line defined by the two other ants.

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- We can parametrize the configuration space M by coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ of the three points $\vec{r}_i = (x_i, y_i)$ in a chosen Cartesian coordinate system (x, y) on the plane. In this parametrization the movement of the system of ants is described in terms of a curve $q(t) = (x_1(t), y_1(t), x_2(t), y_2(t), x_3(t), y_3(t))$, and its velocity at time t is given by $\dot{q}(t) = (\dot{x}_1(t), \dot{y}_1(t), \dot{x}_2(t), \dot{y}_2(t), \dot{x}_3(t), \dot{y}_3(t))$.
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- Equivalently, we can say that the curves $q(t)$ drawn in M by the ants obeying **rule A** must be *horizontal* with respect to the velocity distribution D, which annihilates the following three 1-forms on M:

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Movement of three ants on the plane: Rule A

- Equivalently, we can say that the curves $q(t)$ drawn in M by the ants obeying **rule A** must be *horizontal* with respect to the velocity distribution D, which annihilates the following three 1-forms on M:

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- Now, calculating $Z_1 \wedge Z_2 \wedge Z_3 \wedge Z_{12} \wedge Z_{31} \wedge Z_{23}$, we get

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  so it follows that the *six* vector fields $Z_1, Z_2, Z_3, Z_{12}, Z_{31}, Z_{23}$ are *linearly independent* at each point q of the configuration space M everywhere, *except* the points on the singular locus, where coordinates of q satisfy

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- Since the number A defined above is the *area* of the triangle having the three ants as its vertices, we see that the velocity distribution D of the three ants moving under **rule A** has a *growth vector* $(3, 6)$ everywhere, except the configuration points corresponding to the three ants staying on a line.

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- Rank 3 distributions have *differential invariants*. We recall, that two distributions D_1 and D_2 on respective manifolds M_1 and M_2 are (locally) equivalent, if and only if there exists a (local) diffeomorphism $\phi : M_1 \to M_2$ realizing $\phi_* D_1 = D_2$. In particular the statement about rank 3 distributions having invariants, means that there are locally *non*equivalent rank 3 distributions on 6-dimensional manifolds. Among them the $(3, 6)$ distributions are *generic*, and the growth vector $(3, 6)$ distinguishes them locally from, for example, distributions with growth vector $(3, 5)$; these later distributions are rank 3 distributions D in dimension 6 such that in the sequence $D_{-1} = D$, $D_{-(i+1)} = [D, D_{-i}]$, with $i = 1, ...,$, the distribution D_{-2} is *integrable* and has rank 5. More importantly, there are locally nonequivalent $(3, 6)$ distributions.

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- One way of characterizing distributions locally is to determine their Lie algebra of symmetries. Given a manifold M and distribution D, the Lie algebra of symmetries of D consists of vector fields Y on M such that $[Y, D] \subset D$. It is known that for rank 3 distributions with the growth vector $(3, 6)$ the maximal algebra of symmetries is attained for the distribution locally given in Cartesian coordinates (q^i, p_j) in $\mathbb{R}^6$ as the annihilator of three 1-forms $\lambda_i = \mathrm{d}p_i + \epsilon_{ijk} q^j \mathrm{d}q^k$, $i = 1, 2, 3$, where ϵ_{ijk} is the totally skew-symmetric levi-Civita symbol in $\mathbb{R}^3$. This distribution has its Lie algebra of symmetries isomorphic to the 21-dimensional Lie algebra $\mathfrak{spin}(4, 3)$.

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Symmetry algebra of the ants' system moving under **rule A**

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- **Theorem** The Lie algebra of *all* symmetries of the velocity distribution D of the system of three ants moving according **rule A** is isomorphic to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. In coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbb{R}^6$, the 8 independent local symmetries of $D = Span(Z_1, Z_2, Z_3)$ are:

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- We close this theorem with a remark that the vector space over the real numbers spanned by the symmetry vector fields $X_1, X_2, X_3, X_4, X_5 - X_6$ form a Lie algebra isomorphic to the semidirect sum of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathbb{R}^2$, i.e. the Lie algebra of *the area preserving group of motions on the plane* $\mathbb{R}^2$. Here the vector fields X_1 and X_2 on $\mathbb{R}^6$ correspond to translations in the plane in respective directions ∂_x and ∂_y. The vector fields X_3, X_4 and $X_5 - X_6$ correspond to the linear transformations of the plane with unit determinant. In particular we have the following identifications of the respective $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra elements:
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X_8 &= x_1^2\partial_{x_1} + x_2^2\partial_{x_2} + x_3^2\partial_{x_3} + x_1y_1\partial_{y_1} + x_2y_2\partial_{y_2} + x_3y_3\partial_{y_3}.
\end{aligned}$$

- Thus although the symmetry of this $(3,6)$ distribution is far from being maximal among all $(3,6)$ distributions, the ants distribution D, considered here, can be locally identified with one of the *homogeneous models* of $(3,6)$ distributions; a model that lives on the homogeneous manifold $PGL(3,\mathbb{R})/\mathbb{T}^2$, where $\mathbb{T}^2$ is the maximal torus in $PGL(3,\mathbb{R})$.

- We close this theorem with a remark that the vector space over the real numbers spanned by the symmetry vector fields $X_1, X_2, X_3, X_4, X_5 - X_6$ form a Lie algebra isomorphic to the semidirect sum of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathbb{R}^2$, i.e. the Lie algebra of *the area preserving group of motions on the plane* $\mathbb{R}^2$. Here the vector fields X_1 and X_2 on $\mathbb{R}^6$ correspond to translations in the plane in respective directions ∂_x and ∂_y. The vector fields X_3, X_4 and $X_5 - X_6$ correspond to the linear transformations of the plane with unit determinant. In particular we have the following identifications of the respective $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra elements:

$$X_3 \sim \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ X_4 \sim \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \text{ and } X_5 - X_6 \sim \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Symmetry algebra of the ants' system moving under **rule A**

- **Theorem** The Lie algebra of *all* symmetries of the velocity distribution D of the system of three ants moving according **rule A** is isomorphic to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. In coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbb{R}^6$, the 8 independent local symmetries of $D = Span(Z_1, Z_2, Z_3)$ are:

$$\begin{aligned}
X_1 &= \partial_{x_1} + \partial_{x_2} + \partial_{x_3},\\
X_2 &= \partial_{y_1} + \partial_{y_2} + \partial_{y_3},\\
X_3 &= y_1\partial_{x_1} + y_2\partial_{x_2} + y_3\partial_{x_3},\\
X_4 &= x_1\partial_{y_1} + x_2\partial_{y_2} + x_3\partial_{y_3},\\
X_5 &= x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3},\\
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- Thus although the symmetry of this $(3,6)$ distribution is far from being maximal among all $(3,6)$ distributions, the ants distribution D, considered here, can be locally identified with one of the *homogeneous models* of $(3,6)$ distributions; a model that lives on the homogeneous manifold $PGL(3,\mathbb{R})/\mathbb{T}^2$, where $\mathbb{T}^2$ is the maximal torus in $PGL(3,\mathbb{R})$.

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Symmetry algebra of the ants' system moving under **rule A**

- **Theorem** The Lie algebra of *all* symmetries of the velocity distribution D of the system of three ants moving according **rule A** is isomorphic to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. In coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbb{R}^6$, the 8 independent local symmetries of $D = Span(Z_1, Z_2, Z_3)$ are:

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X_1 &= \partial_{x_1} + \partial_{x_2} + \partial_{x_3},\\
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X_3 &= y_1\partial_{x_1} + y_2\partial_{x_2} + y_3\partial_{x_3},\\
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Symmetry algebra of the ants' system moving under **rule A**

- **Theorem** The Lie algebra of *all* symmetries of the velocity distribution D of the system of three ants moving according **rule A** is isomorphic to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. In coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbb{R}^6$, the 8 independent local symmetries of $D = Span(Z_1, Z_2, Z_3)$ are:

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\begin{aligned}
X_1 &= \partial_{x_1} + \partial_{x_2} + \partial_{x_3},\\
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X_8 &= x_1^2\partial_{x_1} + x_2^2\partial_{x_2} + x_3^2\partial_{x_3} + x_1y_1\partial_{y_1} + x_2y_2\partial_{y_2} + x_3y_3\partial_{y_3}.
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Symmetry algebra of the ants' system moving under **rule A**

- **Theorem** The Lie algebra of *all* symmetries of the velocity distribution D of the system of three ants moving according **rule A** is isomorphic to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. In coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbb{R}^6$, the 8 independent local symmetries of $D = Span(Z_1, Z_2, Z_3)$ are:

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Symmetry algebra of the ants' system moving under **rule A**

- **Theorem** The Lie algebra of *all* symmetries of the velocity distribution D of the system of three ants moving according **rule A** is isomorphic to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. In coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbb{R}^6$, the 8 independent local symmetries of $D = Span(Z_1, Z_2, Z_3)$ are:

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- Thus although the symmetry of this $(3,6)$ distribution is far from being maximal among all $(3,6)$ distributions, the ants distribution D, considered here, can be locally identified with one of the *homogeneous models* of $(3,6)$ distributions; a model that lives on the homogeneous manifold $PGL(3,\mathbb{R})/\mathbb{T}^2$, where $\mathbb{T}^2$ is the maximal torus in $PGL(3,\mathbb{R})$.
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- **Theorem** The Lie algebra of *all* symmetries of the velocity distribution D of the system of three ants moving according **rule A** is isomorphic to the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. In coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbb{R}^6$, the 8 independent local symmetries of $D = Span(Z_1, Z_2, Z_3)$ are:

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Movement of three ants on the plane: Rule B

- We have three points in the plane $\vec{r}_i$, $i = 1, 2, 3$, and we want that

$$\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} \quad || \quad (\vec{r}_{i+1} - \vec{r}_{i+2}).$$

Here again $i, j = 1, 2, 3$, and $i + j$ is counted modulo 3.

- We again parametrize the configuration space M by coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ of the three points $\vec{r}_i = (x_i, y_i)$ in a chosen Cartesian coordinate system (x, y) on the plane. In this parametrization the movement of the system of ants is described in terms of a curve $q(t) = (x_1(t), y_1(t), x_2(t), y_2(t), x_3(t), y_3(t))$, and its velocity at time t is given by $\dot{q}(t) = (\dot{x}_1(t), \dot{y}_1(t), \dot{x}_2(t), \dot{y}_2(t), \dot{x}_3(t), \dot{y}_3(t))$.
- In these coordinates the above **rule B** becomes:

$$(y_{i+1} - y_{i+2})\dot{x}_i - (x_{i+1} - x_{i+2})\dot{y}_i = 0, \quad i = 1, 2, 3.$$

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Movement of three ants on the plane: Rule B

- We have three points in the plane $\vec{r}_i$, $i = 1, 2, 3$, and we want that

$$\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} \quad || \quad (\vec{r}_{i+1} - \vec{r}_{i+2}).$$

Here again $i, j = 1, 2, 3$, and $i + j$ is counted modulo 3.

- We again parametrize the configuration space M by coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ of the three points $\vec{r}_i = (x_i, y_i)$ in a chosen Cartesian coordinate system (x, y) on the plane. In this parametrization the movement of the system of ants is described in terms of a curve $q(t) = (x_1(t), y_1(t), x_2(t), y_2(t), x_3(t), y_3(t))$, and its velocity at time t is given by $\dot{q}(t) = (\dot{x}_1(t), \dot{y}_1(t), \dot{x}_2(t), \dot{y}_2(t), \dot{x}_3(t), \dot{y}_3(t))$.
- In these coordinates the above **rule B** becomes:

$$(y_{i+1} - y_{i+2})\dot{x}_i - (x_{i+1} - x_{i+2})\dot{y}_i = 0, \quad i = 1, 2, 3.$$

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- Equivalently, we can say that the curves $q(t)$ drawn in M by the ants obeying **rule B** must be *horizontal* with respect to the velocity distribution D, which annihilates the following three 1-forms on M:

  $$\omega_i = (y_{i+1} - y_{i+2})\mathrm{d}x_i - (x_{i+1} - x_{i+2})\mathrm{d}y_i, \quad i = 1, 2, 3.$$

  In other ords, the velocities of these curves should be spanned by the three vector fileds

  $$Z_i = (x_{i+1} - x_{i+2})\partial_{x_i} + (y_{i+1} - y_{i+2})\partial_{y_i}, \quad i = 1, 2, 3$$

  on M. So the velocity distribution of the ants is now given by $D = Span(Z_1, Z_2, Z_3)$.

- Taking the commutators of the vector fields Z_1, Z_2, Z_3 spanning the distribution D we get three new vector fields

$$Z_{i,i+1} = [Z_i, Z_{i+1}] = (x_i - x_{i+2})\partial_{x_i} + (x_{i+2} - x_{i+1})\partial_{x_{i=1}} + (y_i - y_{i+2})\partial_{y_i} + (y_{i+2} - y_{i+1})\partial_{y_{i=1}}, \quad i = 1, 2, 3.$$

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$$Z_1 \wedge Z_2 \wedge Z_3 \wedge Z_{12} \wedge Z_{31} \wedge Z_{23} = 0.$$
- So the rank of the derived distribution $D_{-2} = [D, D]$ is *smaller* than 6. The velocity distribution D for the rule B is *not* bracket generating!
- Actually one easilly finds that there is *precisely* one linear relation between the vector fields $(Z_1, Z_2, Z_2, Z_{12}, Z_{31}, Z_{23})$, namely
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The invariant characterizing a leaf

- Now the question arises about the function that enumerates the leaves of the foliation of the distribution $[D, D]$. What is the feature of motion of the ants whose preservation forces the ants to stay on a given leaf?
- There is a quick algebraic answer to this question:
- Note that $\mathrm{d}(\omega_1 + \omega_2 + \omega_3) = 0$.
- This means that that there exists a function F such that $\mathrm{d}F = \omega_1 + \omega_2 + \omega_3$.
- One can directly check that

$$F = 32\,A,$$

where A is the *area* of the triangle defined by the ants at every moment. Since all three vector fields Z_i annihilate ω_i, and thus they ennihilate the 1-form $\omega_1 + \omega_2 + \omega_3 = 32\mathrm{d}A$, and in turn they annihilate the one form $\mathrm{d}A$, then they *are tangent* to the 5-dimensional submanifolds $A = \mathit{const}$ in M.

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- Now the question arises about the function that enumerates the leaves of the foliation of the distribution $[D, D]$. What is the feature of motion of the ants whose preservation forces the ants to stay on a given leaf?
- There is a quick algebraic answer to this question:
- Note that $\mathrm{d}(\omega_1 + \omega_2 + \omega_3) = 0$.
- This means that that there exists a function F such that $\mathrm{d}F = \omega_1 + \omega_2 + \omega_3$.
- One can directly check that

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- **Proposition**: The triangle with vertices formed by three ants moving according **rule B** has in every moment of time the same area.
- Apart from the algebraic proof of this proposition given above, it can be also seen by a 'pure thought' observing that any movement of the three ants obeying **rule B** is a superposition of three primitive moves: an ant $\#i$ moves, and ants $\#(i+1)$ and $\#(i+2)$ rest, for each $i = 1, 2, 3$. In each of the three primitive situations, since the vertex $\#i$ of the triangle moves in a line parallel to the corresponding base $\#(i+1) - \#(i+2)$ of the triangle, the area of the triangle formed by the ants $\#1$, $\#2$ and $\#3$ is obviously unchanged. Since the general movement according to **rule B** is a linear combination of the three primitive movements preserving the area, it also preserves the area.

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- So we see that the movement of the ants according to **rule B** stratifies the configuration space: once in an initial position the ants defined a triangle Δ of area A, they move on a 5-dimensional submanifold M_A of M whose configuration points correspond to triangles Δ' having the same area A as Δ.
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$(3,5)$ distribution defines $(2,3,5)$ one

- The 3-distribution D on each leaf M_A is actually a *square* of a rank 2-distribution $\mathscr{D}$. By this I mean that there is a rank 2-distribution $\mathscr{D}$ such that its first derived distribution $\mathscr{D}_{-2} = [\mathscr{D}, \mathscr{D}]$ equals D.
- Its is easy to check that $\mathscr{D} = Span(Z_1 - Z_2, Z_3 - Z_1)$ indeed makes the job. Since $[Z_1 - Z_2, Z_3 - Z_1] = -Z_{12} - Z_{31} - Z_{23}$, then using the dependence relation $Z_1 + Z_2 + Z_3 + Z_{12}, +Z_{31} + Z_{23} = 0$ we get $[Z_1 - Z_2, Z_3 - Z_1] = Z_1 + Z_2 + Z_3$, and consequently $[Z_1 - Z_2, Z_1 - Z_3] \wedge (Z_1 - Z_2) \wedge (Z_3 - Z_1) = 3Z_3 \wedge Z_2 \wedge Z_1$.
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- The 3-distribution D on each leaf M_A is actually a *square* of a rank 2-distribution $\mathscr{D}$. By this I mean that there is a rank 2-distribution $\mathscr{D}$ such that its first derived distribution $\mathscr{D}_{-2} = [\mathscr{D}, \mathscr{D}]$ equals D.
- Its is easy to check that $\mathscr{D} = Span(Z_1 - Z_2, Z_3 - Z_1)$ indeed makes the job. Since $[Z_1 - Z_2, Z_3 - Z_1] = -Z_{12} - Z_{31} - Z_{23}$, then using the dependence relation $Z_1 + Z_2 + Z_3 + Z_{12}, +Z_{31} + Z_{23} = 0$ we get $[Z_1 - Z_2, Z_3 - Z_1] = Z_1 + Z_2 + Z_3$, and consequently $[Z_1 - Z_2, Z_1 - Z_3] \wedge (Z_1 - Z_2) \wedge (Z_3 - Z_1) = 3Z_3 \wedge Z_2 \wedge Z_1$.
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- Thus we have just established the following proposition.
- **Proposition**: The 6-dimensional configuration space M of three ants moving on the plane according to **rule B** is foliated by 5-dimensional submanifolds M_A consisting of configuration points defining triangles of equal area A on the plane. The ants obeying **rule B** must stay on a given leaf M_A of the foliation during their motion. Their velocity distribution D of rank 3, defines a rank 2 distribution $\mathscr{D}$, which is the 'square root' of D, $D = [\mathscr{D}, \mathscr{D}]$. The rank 2 distribution $\mathscr{D}$ has the growth vector $(2, 3, 5)$ on each leaf M_A.

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- We recall that rank 2 distributions with growth vector $(2,3,5)$ on 5-dimensional manifolds have local differential invariants. In particular their symmetry algebra can be as large as 14-dimensional Lie algebra $\mathfrak{g}_2$ of the split real form of the simple exceptional complex Lie group G_2. This happens for the rank 2 distribution given on a 5-dimensional quadric $p_i q^i = 1$ in $\mathbb{R}^6$, with coordinates (q^i, p_i), as the annihilator of three 1-forms $\lambda_i = \mathrm{d}p_i + \epsilon_{ijk} q^j \mathrm{d}q^k$, $i = 1, 2, 3$.

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The theorem about ants' **rule B**

- **Theorem**:
  - The Lie algebra of *all* symmetries of the velocity distribution $\mathscr{D}$ of the system of three ants moving on the plane according to **rule B** is isomorphic to the Lie algebra of *area preserving affine transformations* of the plane.
  - The distribution is one of the homogeneous models of $(2,3,5)$ distributions. It can be locally realized on the 5-manifold being the group of *area preserving affine transformations* of the plane.
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