Einstein-Weyl-like conditions, ruled affine spheres, and causal structures in dimension 3

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Einstein-Weyl structures

A *Weyl structure* $(\nabla, [g])$ on a manifold is defined by the property

$$\nabla g = 2\phi \otimes g,$$

for a 1-forms ϕ where (g, ϕ) is equivalence class under the transformation

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In 3D, EW str are in 1-1 correspondence with point equiv classes of 3rd order ODEs with vanishing Wünschmann and Cartan invariants.

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Question : What if we start with a causal structure (null cones not necessarily quadratic) endowed with a 2D family of “null surfaces”

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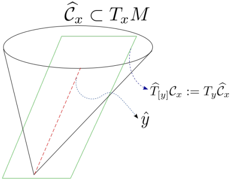

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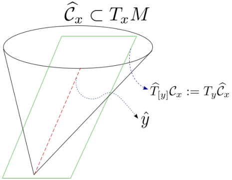

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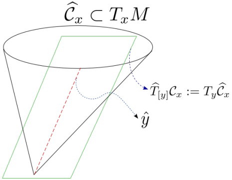

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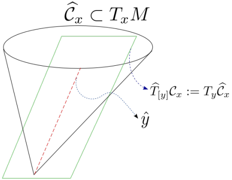

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Theorem (Chern, Godlinski-Nurowski, Holland-Sparling): 3D causal strs correspond to parabolic geometry $(\mathscr{G}^{10} \to \mathscr{C}^4, \omega)$ of type $(SO(3,2), P_{12})$ where $\mathscr{C}^4$ is the sky bundle equipped with an Engel

Augmentation and reduction of causal structures

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$$\{(x;y) \in TM \mid y^3 = h(x\,;t), y^2 = t, y^1 = 1\},$$

where $(x^1, x^2, x^3; t)$ coordinates on $\mathscr{C}$ then $Y_0 (= f_{qqqq})$ has > 2300 terms and

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Theorem (Cartan, Nurowski): 3D augmented causal structures are Cartan geom $(\mathscr{G}^7, \mathscr{C}^4, \omega)$ of type $(R^3 \rtimes \mathrm{CO}(2,1), B)$ where $B \subset \mathrm{CO}(2,1)$ are upper triangular matrices. Fund. inv are I_0, C_0, S_0, R_0.

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Proposition 3D Finsler-Weyl structures define Cartan geometry $(\mathscr{F}^7, \Sigma^5, \omega)$ of type $(\mathbb{R}^3 \rtimes \mathrm{CO}(2,1), \mathrm{CO}(1,1))$. The fundamental invariants

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Proposition :
Int. FW str. with Σ_x ruled affine spheres $\Longleftrightarrow$ 3rd order ODEs with

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The 4D space of paths of these proper FEW metrics are equipped with half-flat ε-Kähler metrics of type III whose Ricci curvature defines ε-Kähler-Einstein metrics of type II with non-zero scalar curvature. They depend on 3 functions of 2 variables.

Special class of ε-Kähler metrics and an ansatz

Here we assume $\varepsilon = 1$. Other cases are treated similarly.

Theorem (- 2021): Let $g = \theta^1\bar{\theta}^2 + \theta^2\bar{\theta}^1$ be a self-dual para-Kähler such that there is a coframe reduction wrt which $Ric = \theta^1\bar{\theta}^2 - \theta^2\bar{\theta}^1$ such that $\mathrm{Ker}\{\theta^1, \bar{\theta}^1\}$ is principal null plane of multiplicity III. Then Ric defines a para-Kähler-Einstein metric of type II.

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All such para-Kähler metrics arise from proper integrable FEW structures of constant negative flag curvature whose indicatrices are ruled affine spheres.

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Theorem (- 2021): Let $g = \theta^1\bar{\theta}^2 + \theta^2\bar{\theta}^1$ be a self-dual para-Kähler such that there is a coframe reduction wrt which $Ric = \theta^1\bar{\theta}^2 - \theta^2\bar{\theta}^1$ such that $\mathrm{Ker}\{\theta^1, \bar{\theta}^1\}$ is principal null plane of multiplicity III. Then Ric defines a para-Kähler-Einstein metric of type II.
All such para-Kähler metrics arise from proper integrable FEW structures of constant negative flag curvature whose indicatrices are ruled affine spheres.

Third ODEs defining such FEW structure satisfy certain PDEs e.g.

$$\frac{\mathrm{D}^2}{\mathrm{d}x^2} f_{qq} - \frac{\mathrm{D}}{\mathrm{d}x} f_{pq} + f_{yq} = 0, \qquad f_{qqq} = 0, \qquad R_1 = 0$$

which can be explicitly solved!

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This suggests that one should be able to integrate the structure equations such ε-Kähler metrics.

Future directions

- Cartan invariants for 4D causal and Finsler-Weyl structures arising from a pair of 2nd order PDEs and the analogue of FEW condition.
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Thank you for your attention