A twistor sigma model for Plebanski generating functions and gravity scattering

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Work with: Tim Adamo & Atul Sharma arxiv: 2103.16984, . . .

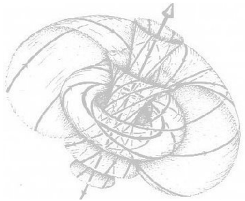
Revisiting M & Wolf CMP, 288, '09, CMP, and M & Skinner CMP 294, '10 in light of more recent developments.

Intro

Plebanski generating functions are functions Ω on hyper-Kahler manifolds M^{4k}.

- The 'first kind' is a Kahler scalar for a choice of Kahler structure.
- Generating functions of BPS & DT/Gromov-Witten invariants etc. are Plebanski functions of 'second kind'.
- We show here that it generates the gravity MHV amplitude.

- Nonlinear graviton encodes M^{4k} into a deformed twistor space $\mathscr{PT}$.
- M^{4k} = moduli space of holomorphic curves in $\mathscr{PT}$.
- We will see that Ω = 'action' of curve for a new sigma model in $\mathscr{PT}$.
- Generalizations control full tree S-matrix.



Gravity amplitudes at MHV (maximal helicity violating)

In spinors, null momenta $k_{i\alpha\dot\alpha} = \kappa_{i\alpha}\kappa_{i\dot\alpha}$, $\alpha = 0, 1$, $\dot\alpha = \dot 0, \dot 1$.
We have skew pairings

$$\langle 1\,2\rangle := \kappa_{1\alpha}\kappa_2^{\alpha}\,,\ [1\,2] := \kappa_{1\dot\alpha}\kappa_2^{\dot\alpha}\,,\ \langle 1|2|3] = \kappa_{1\alpha}k_2^{\alpha\dot\alpha}\kappa_{3\dot\alpha}\,.$$

In 2008, had MHV amplitude formula after BGK

$$\mathcal{M} = \frac{\delta^4\left(\sum_i k_i\right)}{\Pi_{i=1}^n [i\, i+1]}\frac{[12]^7}{[1\,n][n\,2]}\prod_{k=3}^{n}\frac{\langle k|k_{k+1}+\cdots+k_n|1]}{[k1]}+\text{Perms}_{3,\ldots,n-1}$$

Since then Hodges 2012 formula

$$\mathcal{M} = \langle 12\rangle^6 \det{}'\mathbb{H}\; \delta^4\left(\textstyle\sum_i k_i\right)$$

$$\mathbb{H}_{ij} = \begin{cases} \frac{[ij]}{\langle ij\rangle} & i \neq j \\ -\sum_k \frac{[ik]}{\langle ik\rangle} & i = j\,. \end{cases}$$

$\mathbb{H}$ is Laplace matrix for a matrix-tree theorem $\rightsquigarrow$[Feng,He 2012]
Sum of tree diagrams [Bern,Dixon,Perelstein,Rosowski '98, Nguyen, Spradlin, Volovich, Wen '10]

Outline

1. Generating functional for the gravity MHV amplitude from the Plebanski scalar via Plebanski action.
2. Twistor space and nonlinear graviton.
3. A twistor sigma model for the Plebanski scalar and the tree formulae.
4. Extension to cosmological constant and Swann Hyper-Kahler structure.
5. Extension to full gravity tree S-matrix.

Plebanski gravity action

Expanding about the SD sector, Abou-Zeid, Hull hep-th/0511189

- Use Plebanski-Palatini formulation with variables on M^4:
  - $\mathbf{e}^{\alpha\dot{\alpha}} =$ tetrad of 1-forms s.t.

$$\mathrm{d}s^2 = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\,\mathbf{e}^{\alpha\dot{\alpha}}\mathbf{e}^{\beta\dot{\beta}}\,, \qquad \mathbf{e}^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix} T+iZ & X+iY \\ X-iY & T-iZ \end{pmatrix}$$

  - $\Gamma_{\alpha\beta} = \Gamma_{(\alpha\beta)}$ the ASD spin connection 1-forms.
- Action uses ASD two-forms $\Sigma^{\alpha\beta} = \mathbf{e}^{(\alpha}_{\dot{\alpha}} \wedge \mathbf{e}^{\beta)\dot{\alpha}}$

$$S = \int_M R d^4x = \int_M \Sigma^{\alpha\beta}\left(d\Gamma_{\alpha\beta} + \kappa^2\,\Gamma_\alpha^\gamma \wedge \Gamma_{\beta\gamma}\right)\,,$$

- Field equations:

$$d\Sigma^{\alpha\beta} = 2\kappa^2\Gamma_\gamma^{(\alpha} \wedge \Sigma^{\beta)\gamma}\,, \qquad d\Gamma_{\alpha\beta} + \kappa^2\,\Gamma_\alpha^\gamma \wedge \Gamma_{\beta\gamma} = \Psi_{\alpha\beta\gamma\delta}\Sigma^{\gamma\delta}\,.$$

- $\Rightarrow \kappa^2\Gamma_{\alpha\beta} =$ ASD spin connection 1-form, $\qquad \mathrm{Ricci} = 0$.

The SD sector and MHV amplitudes

SD sector: Set $\kappa = 0$, $S_{SD} = \int_M \Sigma^{\alpha\beta} \mathrm{d}\Gamma_{\alpha\beta}$, $\rightsquigarrow$ field equs

$$\mathrm{d}\Sigma^{\alpha\beta} = 0 \Rightarrow \quad \text{metric is SD, and}$$

$$\mathrm{d}\Gamma_{\alpha\beta} \wedge \mathbf{e}^{\alpha\dot\alpha} = 0 \,, \Rightarrow \mathrm{d}\Gamma_{\alpha\beta} = \psi_{\alpha\beta\gamma\delta}\Sigma^{\gamma\delta}$$

and $\psi_{\alpha\beta\gamma\delta}$ is linearized ASD Weyl spinor on SD background.

- All $+$ amplitude $= 0 \leftrightarrow$ consistency of SD sector.
- One $-$, rest $+$ amplitude $= 0 \leftrightarrow$ integrability of SD sector.

MHV interactions:

$$\mathcal{M}(1^-, 2^-, e^+) = \int_M \kappa^2 \Sigma^{\alpha\beta} \wedge \Gamma_{1\alpha\gamma} \wedge \Gamma^{\gamma}_{2\beta} \,.$$

MHV amplitude $\leftrightarrow$, probability of helicity flip of $-$ helicity particle on SD background given by $\Sigma^{\alpha\beta}$.

Plebanski scalar as MHV generating function

Eliminating gauge choice in '08 paper with Skinner.

- An ASD linear field of momentum $k_{\alpha\dot\alpha} = \kappa_\alpha \tilde\kappa_{\dot\alpha}$ is

$$\Gamma_{\alpha\beta} = \mathbf{e}^{\gamma\dot\gamma} b_{\dot\gamma}\kappa_\gamma\kappa_\alpha\kappa_\beta e^{ik\cdot x} \quad \text{with} \quad [b,\kappa] = 1.$$

  At MHV have two of these with momenta k_1, k_2.
- Spin frame aligned along $\kappa_{1\alpha}, \kappa_{2\alpha} \rightsquigarrow$ 'complex' coords:

$$z^{\dot\alpha} = x^{1\dot\alpha}\,, \qquad \tilde z^{\dot\alpha} = x^{2\dot\alpha}\,,$$

- Plebanski: general SD metric is determined by $\Omega(z^{\dot\alpha}, \tilde z^{\dot\alpha})$ subject to Monge-Ampere:

$$\Sigma^{11} = d^2 z, \quad \Sigma^{22} = d^2\tilde z, \quad \Sigma^{12} = \partial\tilde\partial\Omega\,, \qquad \det\partial\tilde\partial\Omega = 1.$$

- Then can integrate by parts twice to obtain

$$\mathcal{M}(1^-, 2^-, \Omega) = \langle\kappa_1\,\kappa_2\rangle^4 \int_M d^2 z\, d^2\tilde z\, \Omega\, \mathrm{e}^{[\tilde\kappa_1 z] + [\tilde\kappa_2 \tilde z]}.$$

How can we generate Ω from twistor space?

Flat twistor space $\mathbb{T} = \mathbb{C}^4$ or $\mathbb{PT}' = \mathbb{CP}^3 - \mathbb{CP}^1$ with hgs coords:

$$Z = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}, \qquad Z \sim aZ\,, a \neq 0\,, \quad \lambda_\alpha \neq 0.$$

Theorem (Penrose, 1976)

$$\left\{\begin{matrix}\textit{Deformations of complex}\\ \textit{structure: } \mathbb{PT}' \rightsquigarrow \mathscr{PT}\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Self-dual deformations}\\ \textit{of conformal structure}\\ (\mathbb{M}, \eta) \rightsquigarrow (M, [g]).\end{matrix}\right\}$$

Main ideas: Deform d-bar op by $\bar\partial_0 \to \bar\partial_0 + V$.
The $\mathbb{CP}^1$s in $\mathscr{PT}$ survive deformation. Define space-time by

$$M = \{\text{moduli space of degree-1 } \mathbb{CP}^1\text{s} \subset \mathscr{PT}\}.$$

$x, y \in M$ connected by a light ray $\Leftrightarrow$ Incidence $\mathbb{CP}^1_x \cap \mathbb{CP}^1_y \neq \emptyset$.
$\rightsquigarrow$ SD conformal structure, $[g]$, Weyl$^- = 0$ on M.

For Einstein $g \in [g]$, $\mathscr{PT}$ must have a holomorphic Poisson structure $\{\,,\,\}$, bivector of weight -2.

Hamiltonians for quaternion/hyperkahler spaces

after M. & Wolf 2009

Flat twistor space $\mathbb{T} = \mathbb{C}^4$ or $\mathbb{PT}' = \mathbb{CP}^3 - \mathbb{CP}^1$ with hgs coords:

$$Z^I = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}, \qquad Z^I \sim aZ^I \,, a \neq 0\,, \quad \lambda_\alpha \neq 0.$$

- Introduce Poisson structure: $\{,\} = I^{IJ}\partial_{Z^I} \wedge \partial_{Z^J}$.
- I^{IJ} rank 4 for quaternion-Kahler, rank 2 for hyperkahler.
- $h \in \Omega^{0,1}(2)$ gives *Hamiltonian deformation* wrt Poisson bracket
$$\bar\partial_h f = \bar\partial_0 f + \{h, f\}\,, \qquad .$$
- Integrability $\Leftrightarrow$ $\bar\partial_0 h + \{h, h\} = 0$.
- Points $x \in M^4 \leftrightarrow$ holomorphic maps $F(x,\sigma)^I : \mathbb{CP}^1_\sigma \to \mathbb{PT}'$

$$\bar\partial_\sigma F^I = I^{IJ} \left.\frac{\partial h}{\partial Z^J}\right|_{Z^I = F^I}$$

Plebanski scalar from sigma model action on $\mathbb{PT}$

Hyperkahler case, I^{IJ} rank 2: $\{,\} := I^{IJ}\partial_{Z^I}\partial_{Z^J} = \varepsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\mu^{\dot\alpha}}\wedge\frac{\partial}{\partial\mu^{\dot\beta}}$.

- λ_α holomorphic so have fibration $\mathbb{PT}' \to \mathbb{CP}^1_\lambda$ by $Z^I \to \lambda_\alpha$.
- Curves are $\mu^{\dot\alpha} = F^{\dot\alpha}(x,\lambda,\bar\lambda)\,, \qquad \bar\partial_\lambda F^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\left.\frac{\partial h}{\partial\mu^{\dot\beta}}\right|_{\mu=F}$.
- In Plebanski coords $x = (z^{\dot\alpha}, \tilde z^{\dot\alpha})$ write

$$F^{\dot\alpha}(x,\lambda,\bar\lambda) = \lambda_2 z^{\dot\alpha} + \lambda_1\tilde z^{\dot\alpha} + \lambda_1\lambda_2\, M^{\dot\alpha}(z,\tilde z,\lambda,\bar\lambda)\,.$$

- M satisfies: $\quad \bar\partial_\lambda M^{\dot\alpha} = \frac{\varepsilon^{\dot\alpha\dot\beta}}{\lambda_1\lambda_2}\left.\frac{\partial h}{\partial\mu^{\dot\beta}}\right|_{\mu=F}$.
- Action: $\quad S_{\mathbb{PT}}[M] = \int D\lambda\left([M\bar\partial M] + \frac{2}{\lambda_1^2\lambda_2^2}h|_{\mu=F}\right)$

Key proposition: Plebanski scalar = value of on-shell action

$$\Omega(z,\tilde z) = S_{\mathbb{PT}}[M]\,.$$

MHV generating function and tree formula

- MHV generating function becomes

$$\mathcal{M}(1,2,h) = \langle \kappa_1\, \kappa_2 \rangle^4 \int_M d^2 z d^2 \tilde{z}\, \mathrm{e}^{[\kappa_1 z] + [\kappa_2 \tilde{z}]}\, S_{\mathbb{PT}}[M,h]$$

- Perturbatively expand h in momentum eigenstates

$$h = \sum_{i=3}^{n} h_i\,, \qquad h_i = \int \frac{ds}{s^3} \bar{\delta}^2(s\lambda_\alpha - \kappa_{i\alpha})\, \mathrm{e}^{is[\mu\, \kappa_i]}\,.$$

- On-shell action has tree expansion (ignoring $O(h_i^2)$)

$$S_{\mathbb{PT}}[M,h] = \langle V_{h_3} \dots V_{h_n} \rangle_{\text{tree}}\,, \qquad V_{h_i} = \int_{\mathbb{CP}^1} h_i D\lambda$$

- V_{h_i} are vertex operators, and propagators

$$\frac{[\partial_\mu h_i\, \partial_\mu h_j]}{\langle i\, j \rangle} = \frac{[i\, j]}{\langle i\, j \rangle} h_i h_j$$

- Gives tree-diagram formalism of Bern et. al. 1998.

Matrix-tree theorem and Hodges formula

Feng-He 2012

- Matrix-tree thm $\Rightarrow\ \langle V_{h_3}\dots V_{h_n}\rangle_{\rm tree} = \det{}'\mathbb{H}$
- Sum of tree diagrams $=$ reduced determinant of Laplace matrix:

$$\mathbb{H}_{ij} = \begin{cases} \frac{[ij]}{\langle ij\rangle} & i \neq j \\ -\sum_k \frac{[ik]}{\langle ik\rangle} & i = j\,. \end{cases}$$

- Here $\mathbb{H}_{ij}$ for $i \neq j$ gives propagator from vertex i to vertex j.
- Diagonal entries fixed by vanishing row sum.
- Integrating out $(z^{\dot\alpha}, \tilde z^{\dot\alpha})$ gives

$$\mathcal{M} = \langle 12\rangle^6 \det{}'\mathbb{H}\ \delta^4(\textstyle\sum_i k_i)$$

$\rightsquigarrow$ Hodges formula.

Generalized Plebanski scalar

More -ve helicity particles & cosmological constant from higher degree curves

Cosmological constant: non-degenerate Poisson structure I^{IJ}.

- Take ASD gravitons: $\tilde{h}_r(Z_r) \in H^1(\mathbb{PT}, \mathcal{O}(-6))$, $r = 1, \ldots, k$, insert at $Z_r \in \mathbb{T}$, $\sigma_r \in \mathbb{CP}^1$ by:

$$Z(\sigma) = \sum_{r=1}^{k} \frac{Z_r}{\sigma - \sigma_r} + M(x, \sigma) : \mathbb{CP}^1 \to \mathbb{PT}\,.$$

- Given Z_r, σ_r, $\exists!$ holomorphic curve with $Z(\sigma) \in \sqrt{\Omega^{1,0}_{\mathbb{CP}^1}}$.
- Set $\langle Z_1, Z_2 \rangle = I_{IJ} Z_1^I Z_2^J)$ and define action by

$$S[Z(\sigma), Z_r, \sigma_r, h] = \int_{\mathbb{CP}^1} d\sigma \left(\langle M\,, \bar{\partial} M \rangle + 2h(Z) \right)$$

$$= \int_{\mathbb{CP}^1} d\sigma \left(\langle Z\,, \bar{\partial} Z \rangle + 2h(Z) \right) + \sum_{r=1}^{k} \langle Z_r, Z(\sigma_r) \rangle\,.$$

Define: Generalized Plebanski scalar = on shell action:

$$\Omega(Z_r, \sigma_r) := S[Z(\sigma), Z_r, \sigma_r, h] \in C^\infty(\mathbb{PT}^k \times \mathcal{M}_{0,k})$$

Swann bundle and Przanowski scalar

For $k = 2$, define $\Omega = \Omega(Z_1, Z_2)]$.

- Euclidean signature:
  $\mathscr{T}$ = total space of spin bundle
  = *Swann bundle* (up to $\mathbb{Z}_2$) over M.
- Swann defines a hyperkahler structure on $\mathscr{T}$.
- **Claim:** $\Omega(Z, \hat{Z})$ is Kahler scalar on twistor space for Swann (hyper-) Kahler structure in standard complex structure.
- Przanowski scalar is a function on M that defines quaternion Kahler structure as a Hermitian.
- *Omega* testricts to give 'Przanowski scalar' on holomorphic hypersurfaces in $\mathscr{T}$.

General amplitudes with cosmological constant

Scattering of k ASD particles on nonlinear background h is:

$$\mathcal{M}(1^-,\ldots,k^-,h)=\int_{(\mathbb{CP}^1\times\mathbb{PT})^k/\mathrm{GL}_2}\Omega(Z_r,\sigma_r)\,\det{}'\tilde{\mathbb{H}}\prod_{r=1}^{k}\tilde{h}_r D^3 Z_r d\sigma_r\,.$$

- here $\det{}'\tilde{\mathbb{H}}$ is reduced determinant of 'conjugate' $\mathbb{H}$ matrix

$$\tilde{\mathbb{H}}_{rs}=\begin{cases}\frac{\langle Z_r Z_s\rangle}{\sigma_r-\sigma_s} & r\neq s\\ -\sum_q \frac{\langle Z_r Z_q\rangle}{\sigma_r-\sigma_q}\,, & r=s\,.\end{cases}$$

- Expanding $h=\sum_{i=k+1}^{n} h_i$, tree expansion of Ω gives

$$\mathcal{M}(1,..n)=\int_{\frac{(\mathbb{CP}^1\times\mathbb{PT})^k}{\mathrm{GL}_2}}\langle h_{k+1}\ldots h_n\rangle_{\text{tree}}\det{}'\tilde{\mathbb{H}}\prod_{r=1}^{k}\tilde{h}_r D^3 Z_r d\sigma_r\,,$$

$$=\int_{\frac{(\mathbb{CP}^1)^n\times\mathbb{PT}^k}{\mathrm{GL}_2}}\det{}'\mathbb{H}\,\det{}'\tilde{\mathbb{H}}\prod_{i=k+1}^{n} h_i(Z(\sigma_i))d\sigma_i\prod_{r=1}^{k}\tilde{h}_r D^3 Z_r d\sigma_r$$

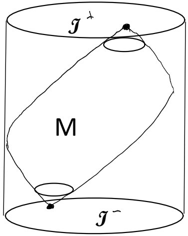
Our final formula for the gravity tree S-matrix, $\Lambda\neq 0$.

Checks on Einstein gravity formula

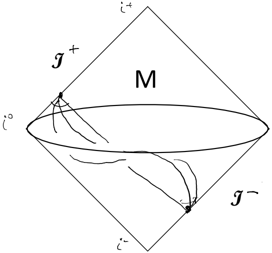
$$\mathcal{M}(1,..,n) = \int_{\frac{(\mathbb{CP}^1)^n \times \mathbb{PT}^k}{\mathrm{GL}_2}} \det{}'\mathbb{H} \prod_{i=k+1}^{n} h_i(Z(\sigma_i))d\sigma_i \det{}'\tilde{\mathbb{H}} \prod_{r=1}^{k} \tilde{h}_r D^3 Z_r d\sigma_r$$

$$\mathbb{H}_{ij} = \begin{cases} \frac{1}{\sigma_i - \sigma_j}\{,\}_{ij}\,, & i \neq j, \quad i = k+1, \dots n \\ -\sum_{k\neq i} \mathbb{H}_{ik}\,, & i = j\,. \end{cases}$$

- $k = 2$, $\Lambda \neq 0$, $\Leftrightarrow$ formula from space-time action [M, Adamo '13].
- $\Lambda = 0$, reduces to Cachazo-Skinner formula, [CS, 2012, CMS 2012.)
- $\Lambda \neq 0$, $k > 2$, $\Leftrightarrow$ Adamo 2015 formula (conjectural).



Asymptotically de Sitter



Asymptotically Flat

Contrast with Skinner $N = 8$ twistor-string

The Skinner twistor-string for $N = 8$ supergravity also generates Cachazo-Skinner formula.

- Skinner model has target ambitwistor space, new model has target twistor space.
- $2\times$ bosonic fields $+$ as many fermions and gaugings.
- Skinner formulae are full quantum correlators on worldsheet, new model just uses trees.
- Skinner model has target ambitwistor space, space of null geodesics in M^4.
- New model has target twistor space, encodes 'infinity twistor' Poisson structure for null geodesics at $\mathscr{I}$. More 'palatial'!

Conclusions & discussion

- Einstein Gravity tree amplitudes generated by on-shell action of sigma model for curves in $\mathbb{PT}$.
- Integral of new on-shell sigma model action at degree 1 $\rightsquigarrow$ on-shell Einstein-Hilbert action.

Further developments:

- at MHV directly translates to cuts of $\mathscr{I}$, with $h = \int^u \sigma du$; degree of curve $= k - 1$ at N^{k-2}MHV so cuts have higher degree over celestial sphere beyond MHV.
- New sigma model has dimensionful coupling; can quantize curves to get α-deformed MHV:

$$\delta^4\left(\sum_{r=1}^{n} k_r\right) \langle 1\,2\rangle^{2n} \prod_{i=3}^{n} \frac{\exp\left[-\frac{\mathrm{i}\,\alpha}{8\pi}\sum_{j\neq i}\frac{[i\,j]}{\langle i\,j\rangle}\frac{\langle 1\,i\rangle^2\langle 2\,j\rangle^2}{\langle 1\,2\rangle^2}\right]}{\langle 1\,i\rangle^2\langle 2\,i\rangle^2}\,.$$

- But new model is incomplete, $\det{}'\tilde{\mathbb{H}}$ inserted by hand.
- Connections to Atul's twistor action for gravity?

Thank You!