Conformal groups of compact Lorentzian manifolds

Karin Melnick
Department of Mathematics
University of Maryland, USA
joint work with Charles Frances

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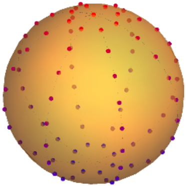
Outline

$\mathbf{S}^n$ with the usual metric g_{+1}

$$\mathrm{Isom}(S^n) \cong O(n+1)$$

$$\mathrm{Conf}(S^n) \cong PO(1, n+1)$$



Def: A group ... $(M, g))$ is essential if H is not isometric for any $g' \in [g]$.

Lichnerowicz Conjecture = Ferrand-Obata Theorem

Thm (Ferrand '71; Obata '71) Given a compact Riemannian manifold (M^n, g), $n \geq 3$, if $\mathrm{Conf}^\circ(M, [g])$ is essential, then $(M, g) \underset{\text{conf diff.}}{\cong} (S^n, [g_H])$.

Obata '71

Ferrand '71

Schoen '95 (also for strictly ps-convex CR structures)

Frances '07 (for all rk 1 parabolic geometries)

all use "source-sink dynamics" of conformal Riemannian transformations"

Higher rank

Q (D'Ambra + Gromov '90): higher signature analogue?

model space

Riemannian $\quad \mathbb{S}^n \quad PO(1,n+1)$

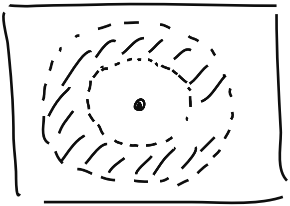
(p,q)-pseudo Riemannian $\quad \mathbb{S}^{p,q} \quad PO(p+1,q+1)$ higher $rk_{\mathbb{R}}$ once $\min(p,q) \geq 1$

$\rightsquigarrow$ not just source-sink

Examples

Hopf manifold

$$\mathrm{Min}^{1,2} \xrightarrow{\text{conf.}} \mathbb{S}^{1,2}$$



$$\mathrm{Min}^{1,2} \setminus \{0\} \Big/ \{2^k \mathrm{Id}_3 : k \in \mathbb{Z}\}.$$

$$\simeq S^1 \times S^2 = M$$

$$\mathrm{Conf}(M) \simeq O(1,2) \times S^1$$

essential.

But not $\underset{\text{conf}}{\simeq} \mathbb{S}^{1,2}$.

Lorentzian Lichnerowicz Conjecture

A lot more examples: Frances '05 $\forall g \geq 1$ Lorentzian metric on $\Sigma_g \times S^1$ with essential flow (in fact, ∞-many distinct) $\longrightarrow$ all conf. flat

LLC: Given compact Lorentzian manifold (M^n, g), $n \geq 3$, if $\mathrm{Conf}(M, [g])$ essential, then (M, g) is conformally flat.

For $p, q \geq 2$, polynomial deformation of $\mathrm{Min}^{p,q}$ is not flat but $\mathbb{R}^{p+q} \setminus \{0\} / \langle \tau^k : k \in \mathbb{Z} \rangle \cong S^1 \times S^{n-1}$ admits essential conf. flow

$n = p + q$

Our result

Thm (Frances + Me. '21) Let (M,g) be a compact, 3-dimensional, real-analytic Lorentzian manifold. If $\mathrm{Conf}^0(M,[g])$ is essential, then (M,g) is conformally flat.

Outline

Ferrand-Obata Theorem and the Lorentzian Lichnerowicz Conjecture

- Conformal groups of Riemannian manifolds
- Conformal transformations of the sphere
- Higher signature?

About the proof

- Step 1: Essential conformal vector field
- Step 2: Bound on isotropy
- Step 3: Proofs for each dimension

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2. $$\mathfrak{z}_X = \{\text{local conformal vector fields commuting with } X\}$$

   Lie algebra is well-defined by Amores/Gromov, using C^ω. dim $\mathfrak{z}_X \leq 4$—more precisely the isotropy at any $x \in M$ has dimension ≤ 1.

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   $\dim \mathfrak{z}_X \leq 4$—more precisely the isotropy at any $x \in M$ has dimension ≤ 1.
3. $\dim \mathfrak{z}_X = 4, 3, 2, 1$

Zeghib's classification

Step 1: Assume $Conf^\circ(M,[g])$ is essential.

Consider unbounded 1-param. subgp $\{\varphi^t\} \leq Conf^\circ(M,[g])$.

If $\{\varphi^t\} \leq Isom(M,g')$ for $g' \in [g]$.

Zeghib proved that then, up to finite cover

a) $M \cong (\mathbb{T}^3, g_{Min})$ | b) $M \cong (PSL_2(\mathbb{R}), g)/\Gamma$, Cartan-Killing metric

$\Gamma < PSL_2(\mathbb{R})$ cocompact lattice

In either case, lift of $Conf^\circ(M)$ to $\tilde{M} = Min^3$ or $PSL_2(\mathbb{R})$ comprises homotheties (i.e. constant conformal factor).

M compact $\Rightarrow$ no nontrivial homotheties

$\therefore$ $Conf^\circ(M,[g])$ isometric for g'.

Stable linear derivative

Given $\{\varphi^t\} \leq \mathrm{Conf}^{loc}(M,[g])$ fixing $p \in M$,

if $D_p\varphi^t = \left\{ \begin{pmatrix} (\mu\lambda)^t & & \\ & \mu^t & \\ & & (\mu/\lambda)^t \end{pmatrix} \right\}$ with $0<\mu<1$, $\mu \leq 1/\lambda \leq 1$

"stable linear derivative"

then Cotton York $C_p \in \Lambda^2 T_p^*M \otimes T_p^*M$ vanishes at p.

conformally init.

Linearization theorem

Thm (FM '10). Given $X \in \mathcal{X}^{conf}(M)$, (M,g) C^ω Lorentzian, $X(p) = 0$

$\{\varphi^t_X\}$ is linearizable at p

OR (M,g) is conf. flat.

Stable flow

$\varphi^t(p) = p \quad \forall t$

If $\{D_p \varphi^t\}$ is "stable linear"

and $\{\varphi^t\}$ linzble near p

$\Rightarrow$ Cotton vanishes on nbhd of p

$\overset{C^\omega}{\Longrightarrow}$ (M,g) conf flat.

If isotropy in $3\times$ dim ≥ 2, get a stable linear flow in derivative.

Table

Table 3D Lich Münster 6/28

$\underline{\dim \mathfrak{z}_x = 4}$: $(M,[g])$ locally homogeneous
construct conformal invariants F_g
for which $F_g^k \cdot g$ is invariant metric

$\underline{\dim \mathfrak{z}_x = 3}$

$\underline{\mathbb{R}^3}$

$\underline{\mathfrak{heis}}$: classification of left-invt Lorentz metrics (Rahmani + Rahmani) rules out 3D orbit

If all orbits 2D, local coordinates $\rightsquigarrow$ conformally flat metric

$\underline{\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}}$: $\exists$ closed $\mathbb{T}^2$-orbit with a complete (G, X)-structure (G is 4D)
flow corresponding to X precompact #

$\underline{\dim \mathfrak{z}_x = 2}$ $\mathfrak{z}_x \cong \mathbb{R}^2$ globalizes on M to $Z_x \simeq S^1 \times \mathbb{R}$
orbits foliate an open dense subset Ω
☆ leaves extend uniquely over 1D orbits on $\partial\Omega$
1D orbits covered by finitely many leaves #

$\underline{\dim \mathfrak{z}_x = 1}$ $\mathfrak{z}_x = \mathbb{R}X$, has fixed $\overline{\text{points}}$
linear unipotent contradicts $\overline{\text{O}}$ locally connected
linear hyperbolic – find $a \geq b > 0$