Variational approach to conformal curves

Josef Šilhan (joint with V. Žádník)

Masaryk University, Brno, Czech Republic

"SCREAM" opening workshop, Aug 2021, Ilawa

Plan

— Motivation: curves in the Euclidean/Riemannian space

— Conformal curves

— Variational approach: the simplest conformal invariant α

— The second variation of α

Motivation: curves in the Euclidean/Riemannian space

Conformal curves

Variational approach to conformal curves

The second variation of α

— How to deal with a curve $\Gamma \subseteq E_n$ or a Riemannian manifold M^n:

- *arc length* parametrization $s : \Gamma \to \mathbb{R}$ for which the corresponding tangent vector U^a, $U^a \nabla_a s = 1$ has unit length,
- canonical way how to differentiate vectors along Γ using $\frac{d}{ds}(\,) = U^a \nabla_a(\,) = (\,)'$ where $\nabla_a = \frac{\partial}{\partial x^a}$ (or Levi-Civita connect.)
- the canonically associated ortonormal *Frenet frame* $(e_1, e_2, \ldots, e_n)$ along Γ where $e_1 = U^a \rightsquigarrow$ curvatures/torsions $\kappa_1, \ldots, \kappa_{n-1}$

$$
\begin{aligned}
e_1' &= \kappa_1 e_2, \\
e_i' &= -\kappa_{i-1} e_{i-1} + \kappa_{i+1} e_{i+1}, \quad 2 \leq i \leq n-1, \\
e_n' &= -\kappa_{n-1} e_n.
\end{aligned}
$$

- arc-length parametrized *geodesics*: $e_1' = U' = 0$, i.e. all $\kappa_i = 0$.
- Variational approach: minimize $\int_{t_1}^{t_2} U \cdot U$ (with fixed endpoints) yields the same equation.

Motivation: curves in Euclidean/Riemannian space

— How to deal with a curve $\Gamma \subseteq E_n$ or a Riemannian manifold M^n:

- *arc length* parametrization $s : \Gamma \to \mathbb{R}$ for which the corresponding tangent vector U^a, $U^a \nabla_a s = 1$ has unit length,
- canonical way how to differentiate vectors along Γ using $\frac{d}{ds}(\,) = U^a \nabla_a(\,) = (\,)'$ where $\nabla_a = \frac{\partial}{\partial x^a}$ (or Levi-Civita connect.)
- the canonically associated ortonormal *Frenet frame* $(e_1, e_2, \ldots, e_n)$ along Γ where $e_1 = U^a \rightsquigarrow$ curvatures/torsions $\kappa_1, \ldots, \kappa_{n-1}$

$$\begin{aligned} e_1' &= \kappa_1 e_2, \\ e_i' &= -\kappa_{i-1} e_{i-1} + \kappa_{i+1} e_{i+1}, \quad 2 \le i \le n-1, \\ e_n' &= -\kappa_{n-1} e_n. \end{aligned}$$

- arc-length parametrized *geodesics*: $e_1' = U' = 0$, i.e. all $\kappa_i = 0$.
- Variational approach: minimize $\int_{t_1}^{t_2} U \cdot U$ (with fixed endpoints) yields the same equation.

Motivation: curves in Euclidean/Riemannian space

— How to deal with a curve $\Gamma \subseteq E_n$ or a Riemannian manifold M^n:

- *arc length* parametrization $s : \Gamma \to \mathbb{R}$ for which the corresponding tangent vector U^a, $U^a \nabla_a s = 1$ has unit length,
- canonical way how to differentiate vectors along Γ using $\frac{d}{ds}(\,) = U^a \nabla_a (\,) = (\,)'$ where $\nabla_a = \frac{\partial}{\partial x^a}$ (or Levi-Civita connect.)
- the canonically associated ortonormal *Frenet frame* $(e_1, e_2, \ldots, e_n)$ along Γ where $e_1 = U^a \rightsquigarrow$ curvatures/torsions $\kappa_1, \ldots, \kappa_{n-1}$

$$
\begin{aligned}
e_1' &= \kappa_1 e_2, \\
e_i' &= -\kappa_{i-1} e_{i-1} + \kappa_{i+1} e_{i+1}, \quad 2 \leq i \leq n-1, \\
e_n' &= -\kappa_{n-1} e_n.
\end{aligned}
$$

- arc-length parametrized *geodesics*: $e_1' = U' = 0$, i.e. all $\kappa_i = 0$.
- Variational approach: minimize $\int_{t_1}^{t_2} U \cdot U$ (with fixed endpoints) yields the same equation.

Motivation: curves in Euclidean/Riemannian space

— How to deal with a curve $\Gamma \subseteq E_n$ or a Riemannian manifold M^n:

- *arc length* parametrization $s : \Gamma \to \mathbb{R}$ for which the corresponding tangent vector U^a, $U^a \nabla_a s = 1$ has unit length,
- canonical way how to differentiate vectors along Γ using $\frac{d}{ds}(\,) = U^a \nabla_a(\,) = (\,)'$ where $\nabla_a = \frac{\partial}{\partial x^a}$ (or Levi-Civita connect.)
- the canonically associated ortonormal *Frenet frame* $(e_1, e_2, \dots, e_n)$ along Γ where $e_1 = U^a \rightsquigarrow$ curvatures/torsions $\kappa_1, \dots, \kappa_{n-1}$

$$\begin{aligned} e_1' &= \kappa_1 e_2, \\ e_i' &= -\kappa_{i-1} e_{i-1} + \kappa_{i+1} e_{i+1}, \quad 2 \leq i \leq n-1, \\ e_n' &= -\kappa_{n-1} e_n. \end{aligned}$$

- arc-length parametrized *geodesics*: $e_1' = U' = 0$, i.e. all $\kappa_i = 0$.
- Variational approach: minimize $\int_{t_1}^{t_2} U \cdot U$ (with fixed endpoints) yields the same equation.

— **Conformal geometry** $(M, [g_{ab}])$ on a smooth manifold M, $n = \dim M$ is the class of metrics $[g_{ab}] = \{e^{2\Upsilon} g_{ab} \mid \Upsilon \in C^\infty(M)\}$. This leads in particular to following data:

- density bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$ such that $\mathcal{E}[-n] = \Lambda^n T^* M$,
- the conformal metric $\boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2] \rightsquigarrow$ raising and lowering of abstract indices,
- if ∇_a and $\widehat{\nabla}_a$ are Levi-Civita connections of metrics g_{ab} and $\hat{g}_{ab} = e^{2\Upsilon} g_{ab}$, respectively, then

$$\widehat{\nabla}_a \mu^b = \nabla_a \mu^b + \Upsilon_a \mu^b - \mu_a \Upsilon^b + \mu^c \Upsilon_c \delta_a{}^b, \quad \mu^a \in \mathcal{E}^a$$

where $\Upsilon_a = \nabla_a \Upsilon \in \mathcal{E}_a$.

— **Conformal geometry** $(M, [g_{ab}])$ on a smooth manifold M, $n = \dim M$ is the class of metrics $[g_{ab}] = \{e^{2\Upsilon} g_{ab} \mid \Upsilon \in C^\infty(M)\}$. This leads in particular to following data:

- density bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$ such that $\mathcal{E}[-n] = \Lambda^n T^* M$,
- the conformal metric $\boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2] \rightsquigarrow$ raising and lowering of abstract indices,
- if ∇_a and $\widehat{\nabla}_a$ are Levi-Civita connections of metrics g_{ab} and $\hat{g}_{ab} = e^{2\Upsilon} g_{ab}$, respectively, then

$$\widehat{\nabla}_a \mu^b = \nabla_a \mu^b + \Upsilon_a \mu^b - \mu_a \Upsilon^b + \mu^c \Upsilon_c \delta_a{}^b, \quad \mu^a \in \mathcal{E}^a$$

where $\Upsilon_a = \nabla_a \Upsilon \in \mathcal{E}_a$.

— **Conformal geometry** $(M, [g_{ab}])$ on a smooth manifold M, $n = \dim M$ is the class of metrics $[g_{ab}] = \{e^{2\Upsilon} g_{ab} \mid \Upsilon \in C^\infty(M)\}$. This leads in particular to following data:

- density bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$ such that $\mathcal{E}[-n] = \Lambda^n T^* M$,
- the conformal metric $\boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2] \rightsquigarrow$ raising and lowering of abstract indices,
- if ∇_a and $\widehat{\nabla}_a$ are Levi-Civita connections of metrics g_{ab} and $\hat{g}_{ab} = e^{2\Upsilon} g_{ab}$, respectively, then

$$\widehat{\nabla}_a \mu^b = \nabla_a \mu^b + \Upsilon_a \mu^b - \mu_a \Upsilon^b + \mu^c \Upsilon_c \delta_a{}^b, \quad \mu^a \in \mathcal{E}^a$$

where $\Upsilon_a = \nabla_a \Upsilon \in \mathcal{E}_a$.

How to deal with conformal curves

— Considering a curve $\Gamma \subseteq M$ on a conformal manifold M, we would like to find a conformal Frenet construction. We face two obvious problems:

- **Problem 1.**: Is there a **conformal arc length** parametrization?
  Answer: generically yes but actually more than that.
- **Problem 2.**: Is there an **invariant differentiation** along Γ, i.e. independent on the choice of $g \in [g]$?
  Answer: yes but complicated (Fialkow) $\rightsquigarrow$ we replace the tangent Frenet frame by tractors (more conceptual).

— Our setup: choosing an arbitrary parametrization t, we have $U^a \in \mathcal{E}^a$ and $u \in \mathcal{E}[1]$ along Γ as follows:

$$t : \Gamma \to \mathbb{R}, \quad U^a \nabla_a t = 1, \quad u := \sqrt{U^a U_a} \in \mathcal{E}[1].$$

How to deal with conformal curves

— Considering a curve $\Gamma \subseteq M$ on a conformal manifold M, we would like to find a conformal Frenet construction. We face two obvious problems:

- **Problem 1.**: Is there a **conformal arc length** parametrization?
  Answer: generically yes but actually more than that.
- **Problem 2.**: Is there an **invariant differentiation** along Γ, i.e. independent on the choice of $g \in [g]$?
  Answer: yes but complicated (Fialkow) $\rightsquigarrow$ we replace the tangent Frenet frame by tractors (more conceptual).

— Our setup: choosing an arbitrary parametrization t, we have $U^a \in \mathcal{E}^a$ and $u \in \mathcal{E}[1]$ along Γ as follows:

$$t : \Gamma \to \mathbb{R}, \quad U^a \nabla_a t = 1, \quad u := \sqrt{U^a U_a} \in \mathcal{E}[1].$$

How to deal with conformal curves

— Considering a curve $\Gamma \subseteq M$ on a conformal manifold M, we would like to find a conformal Frenet construction. We face two obvious problems:

- **Problem 1.**: Is there a **conformal arc length** parametrization?
  Answer: generically yes but actually more than that.
- **Problem 2.**: Is there an **invariant differentiation** along Γ, i.e. independent on the choice of $g \in [g]$?
  Answer: yes but complicated (Fialkow) $\rightsquigarrow$ we replace the tangent Frenet frame by tractors (more conceptual).

— Our setup: choosing an arbitrary parametrization t, we have $U^a \in \mathcal{E}^a$ and $u \in \mathcal{E}[1]$ along Γ as follows:

$$t : \Gamma \to \mathbb{R}, \quad U^a \nabla_a t = 1, \quad u := \sqrt{U^a U_a} \in \mathcal{E}[1].$$

How to deal with conformal curves

— Considering a curve $\Gamma \subseteq M$ on a conformal manifold M, we would like to find a conformal Frenet construction. We face two obvious problems:

- **Problem 1.**: Is there a **conformal arc length** parametrization? Answer: generically yes but actually more than that.
- **Problem 2.**: Is there an **invariant differentiation** along Γ, i.e. independent on the choice of $g \in [g]$? Answer: yes but complicated (Fialkow) $\rightsquigarrow$ we replace the tangent Frenet frame by tractors (more conceptual).

— Our setup: choosing an arbitrary parametrization t, we have $U^a \in \mathcal{E}^a$ and $u \in \mathcal{E}[1]$ along Γ as follows:

$$t : \Gamma \to \mathbb{R}, \quad U^a \nabla_a t = 1, \quad u := \sqrt{U^a U_a} \in \mathcal{E}[1].$$

How to deal with conformal curves: literature

— Fialkow : "The Conformal Theory of Curves" (TAMS, 1942)
$\rightsquigarrow$ the classical (although technical) presentation of conformal invariants of curves.

— Bailey and Eastwood: "Conformal circles and parametrizations of curves in conformal manifolds" (PAMS), 108(I):215–221, 1990
$\rightsquigarrow$ the first attempt to variational study of curves: a (conformally noninvariant) "BE-functional".

— Bailey, Eastwood and Gover: "Thomas's Structure Bundle for Conformal, Projective and Related Structures" (Rocky Mountain J. Math., 1994)
$\rightsquigarrow$ introduces tractors along curves (and our main motivation).

— Musso: "The Conformal Arclength Functional" (Math. Nachr., 165:107–131, 1994)
$\rightsquigarrow$ variational approach focused on a different functional than we discuss here.

Tractor calculus

— The tractor bundle $\mathcal{T}$ is isomorphic, depending on the choice of the metric $g \in [g]$, to the direct sum $[\mathcal{T}]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$.

— The tractor bundle $\mathcal{T}$ admits an invariant connection $\nabla^{\mathcal{T}}$,

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + \mathrm{P}_{ab}\alpha \\ \nabla_a \tau - \mathrm{P}_{ab}\mu^b \end{pmatrix}.$$

where $\boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2]$ is the *conformal metric* and P_{ab} the Schouten tensor. Further, we have $\nabla^{\mathcal{T}}$-parallel Lorenzian metric h on $\mathcal{T}$,

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

— Problem 2 is solved if we build the Frenet frame using tractors.

— The tractor bundle $\mathcal{T}$ is isomorphic, depending on the choice of the metric $g \in [g]$, to the direct sum $[\mathcal{T}]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$.

— The tractor bundle $\mathcal{T}$ admits an invariant connection $\nabla^{\mathcal{T}}$,

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + \mathsf{P}_{ab}\alpha \\ \nabla_a \tau - \mathsf{P}_{ab}\mu^b \end{pmatrix}.$$

where $\boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2]$ is the *conformal metric* and P_{ab} the Schouten tensor. Further, we have $\nabla^{\mathcal{T}}$-parallel Lorenzian metric h on $\mathcal{T}$,

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

— Problem 2 is solved if we build the Frenet frame using tractors.

Tractor calculus

— The tractor bundle $\mathcal{T}$ is isomorphic, depending on the choice of the metric $g \in [g]$, to the direct sum $[\mathcal{T}]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$.

— The tractor bundle $\mathcal{T}$ admits an invariant connection $\nabla^{\mathcal{T}}$,

$$\nabla_a^{\mathcal{T}} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab} \tau + \mathsf{P}_{ab} \alpha \\ \nabla_a \tau - \mathsf{P}_{ab} \mu^b \end{pmatrix}.$$

where $\boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2]$ is the *conformal metric* and P_{ab} the Schouten tensor. Further, we have $\nabla^{\mathcal{T}}$-parallel Lorenzian metric h on $\mathcal{T}$,

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

— Problem 2 is solved if we build the Frenet frame using tractors.

Passing from parametrized curves to tractors

— The naive tractor frame for a fixed parametrization $t : \Gamma \to \mathbb{R}$ with $u = \sqrt{U^a U_a}$ starts with tractors

$$\boldsymbol{T} := \begin{pmatrix} 0 \\ 0 \\ u^{-1} \end{pmatrix}, \quad \boldsymbol{U} := \tfrac{d}{dt}\boldsymbol{T} = \begin{pmatrix} 0 \\ u^{-1}U^a \\ * \end{pmatrix}, \quad \boldsymbol{U}' := \tfrac{d^2}{dt^2}\boldsymbol{T} = \begin{pmatrix} -u \\ * \\ * \end{pmatrix},$$

where $*$ denotes unspecified terms and

$$\boldsymbol{U} := \tfrac{d}{dt}\boldsymbol{T}, \qquad \boldsymbol{U}' := \tfrac{d^2}{dt^2}\boldsymbol{T}, \qquad \ldots, \qquad \boldsymbol{U}^{(i)} := \tfrac{d^{i+1}}{dt^{i+1}}\boldsymbol{T}.$$

— We consider Gram matrices of $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \ldots, \boldsymbol{U}^{(i)})$, e.g.

$$\mathrm{Gram}(\boldsymbol{T},\boldsymbol{U},\boldsymbol{U}',\boldsymbol{U}'') = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\alpha \\ -1 & 0 & \alpha & \frac{1}{2}\alpha' \\ 0 & -\alpha & \frac{1}{2}\alpha' & \beta \end{pmatrix}, \qquad \begin{array}{l} \alpha = \boldsymbol{U}' \cdot \boldsymbol{U}', \\ \beta = \boldsymbol{U}'' \cdot \boldsymbol{U}''. \end{array}$$

Passing from parametrized curves to tractors

— The naive tractor frame for a fixed parametrization $t : \Gamma \to \mathbb{R}$ with $u = \sqrt{U^a U_a}$ starts with tractors

$$\boldsymbol{T} := \begin{pmatrix} 0 \\ 0 \\ u^{-1} \end{pmatrix}, \quad \boldsymbol{U} := \tfrac{d}{dt}\boldsymbol{T} = \begin{pmatrix} 0 \\ u^{-1}U^a \\ * \end{pmatrix}, \quad \boldsymbol{U}' := \tfrac{d^2}{dt^2}\boldsymbol{T} = \begin{pmatrix} -u \\ * \\ * \end{pmatrix},$$

where $*$ denotes unspecified terms and

$$\boldsymbol{U} := \tfrac{d}{dt}\boldsymbol{T}, \qquad \boldsymbol{U}' := \tfrac{d^2}{dt^2}\boldsymbol{T}, \qquad \dots, \qquad \boldsymbol{U}^{(i)} := \tfrac{d^{i+1}}{dt^{i+1}}\boldsymbol{T}.$$

— We consider Gram matrices of $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \dots, \boldsymbol{U}^{(i)})$, e.g.

$$\mathrm{Gram}(\boldsymbol{T},\boldsymbol{U},\boldsymbol{U}',\boldsymbol{U}'') = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\alpha \\ -1 & 0 & \alpha & \frac{1}{2}\alpha' \\ 0 & -\alpha & \frac{1}{2}\alpha' & \beta \end{pmatrix}, \qquad \begin{array}{l} \alpha = \boldsymbol{U}' \cdot \boldsymbol{U}', \\ \beta = \boldsymbol{U}'' \cdot \boldsymbol{U}''. \end{array}$$

— Another parameter $\tilde{t} = g(t)$ yields the new frame $\widetilde{\boldsymbol{T}}, \widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{U}}', \ldots$ where $\frac{d}{d\tilde{t}} = g'^{-1}\frac{d}{dt}$. Which quantities can we **normalize** by a suitable reparametrization? Firstly,

$$\widetilde{\boldsymbol{U}}' \cdot \widetilde{\boldsymbol{U}}' = g'^{-2}\left(\boldsymbol{U}' \cdot \boldsymbol{U}' - 2\mathcal{S}(g)\right) \quad \rightsquigarrow \quad \text{normalization} \quad \widetilde{\boldsymbol{U}}' \cdot \widetilde{\boldsymbol{U}}' = 0$$

yields a projective class of distingushed parameters. Here $\mathcal{S}(g)$ denotes the Schwarzian derivative. Secondly, put

$$\Delta_i := \det\left(\text{Gram}\left(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \ldots, \boldsymbol{U}^{(i-2)}\right)\right),$$

with $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = 1$ and Δ_4 the first nontrivial. Then

$$\widetilde{\Delta}_i = g'^{-i(i-3)}\Delta_i \quad \rightsquigarrow \quad \text{generic normalization} \quad \Delta_i = \pm 1.$$

Summary: relative and absolute invariants of curves

- Δ_i are **relative invariants** of the curve Γ,
- in fact $\Delta_4 \geq 0$ i.e. generically we can reparametrize to

$$\widetilde{\Delta}_i = 1 \quad \leadsto \quad \underline{\text{conformal arc length}} \text{ parametrization,}$$

- the (parametrization independent) condition $\Delta_4 = 0$ defines a class of **conformal circles**, i.e. distinguished curves in conformal geometry.

— Construction of the tractor Frenet frame:

- find the conformal arc length parametrization,
- build the frame $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \ldots, \boldsymbol{U}^{(i)})$,
- use the Gramm-Schmidt ortonormalization to find its (Lorenzian) orthonormal version,
- derive corresponding Frenet formulae analogously as in Lorenzian geometry $\leadsto$ **absolute invariants** = curvatures/torsions.

Summary: relative and absolute invariants of curves

- Δ_i are **relative invariants** of the curve Γ,
- in fact $\Delta_4 \geq 0$ i.e. generically we can reparametrize to

$$\widetilde{\Delta}_i = 1 \quad \leftsquigarrow \quad \underline{\text{conformal arc length}} \text{ parametrization,}$$

- the (parametrization independent) condition $\Delta_4 = 0$ defines a class of **conformal circles**, i.e. distinguished curves in conformal geometry.

— Construction of the tractor Frenet frame:

- find the conformal arc length parametrization,
- build the frame $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \ldots, \boldsymbol{U}^{(i)})$,
- use the Gramm-Schmidt ortonormalization to find its (Lorenzian) orthonormal version,
- derive corresponding Frenet formulae analogously as in Lorenzian geometry $\rightsquigarrow$ **absolute invariants** = curvatures/torsions.

Summary: relative and absolute invariants of curves

- Δ_i are **relative invariants** of the curve Γ,
- in fact $\Delta_4 \geq 0$ i.e. generically we can reparametrize to

$$\widetilde{\Delta}_i = 1 \quad \leadsto \quad \underline{\text{conformal arc length}} \text{ parametrization,}$$

- the (parametrization independent) condition $\Delta_4 = 0$ defines a class of **conformal circles**, i.e. distinguished curves in conformal geometry.

— Construction of the tractor Frenet frame:

- find the conformal arc length parametrization,
- build the frame $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \ldots, \boldsymbol{U}^{(i)})$,
- use the Gramm-Schmidt ortonormalization to find its (Lorenzian) orthonormal version,
- derive corresponding Frenet formulae analogously as in Lorenzian geometry $\leadsto$ **absolute invariants** = curvatures/torsions.

Motivation: curves in the Euclidean/Riemannian space

Conformal curves

Variational approach to conformal curves

The second variation of α

Conformally invariant functionals for curves

— Obvious candidates for such functionals are

$$\alpha = \boldsymbol{U}' \cdot \boldsymbol{U}', \quad \beta = \boldsymbol{U}'' \cdot \boldsymbol{U}'', \quad \gamma = \boldsymbol{U}''' \cdot \boldsymbol{U}'''$$

or their combinations, e.g. $-\Delta_4 = \beta - \alpha^2 \geq 0$.

— The **simplest functional** is

$$\alpha = \boldsymbol{U}' \cdot \boldsymbol{U}' = 3u^{-2} U'_c U'^c + 2u^{-2} U_c U''^c - 6u^{-4} (U_c U'^c)^2 + 2\mathrm{P}_{cd} U^c U^d$$

is of the 3rd order. (That is, the tangent vector U^a is of the first order.)

- The order can be reduced by adding an exact term:

$$\alpha - 2U^r \nabla_r \left(u^{-2} U_c U'^c \right) = u^{-2} U'_c U'^c - 2u^{-4} (U_c U'^c)^2 + 2\mathrm{P}_{ab} U^a U^b.$$

- The right hand side is exactly (conformally noninvariant) BE-functional $\rightsquigarrow$ BE-functional and α have the same family critical curves.

Conformally invariant functionals for curves

— Obvious candidates for such functionals are

$$\alpha = \boldsymbol{U}' \cdot \boldsymbol{U}', \quad \beta = \boldsymbol{U}'' \cdot \boldsymbol{U}'', \quad \gamma = \boldsymbol{U}''' \cdot \boldsymbol{U}'''$$

or their combinations, e.g. $-\Delta_4 = \beta - \alpha^2 \geq 0$.

— The **simplest functional** is

$$\alpha = \boldsymbol{U}' \cdot \boldsymbol{U}' = 3u^{-2} U'_c U'^c + 2u^{-2} U_c U''^c - 6u^{-4} (U_c U'^c)^2 + 2\mathrm{P}_{cd} U^c U^d$$

is of the 3rd order. (That is, the tangent vector U^a is of the first order.)

- The order can be reduced by adding an exact term:

$$\alpha - 2U^r \nabla_r \left(u^{-2} U_c U'^c\right) = u^{-2} U'_c U'^c - 2u^{-4} (U_c U'^c)^2 + 2\mathrm{P}_{ab} U^a U^b.$$

- The right hand side is exactly (conformally noninvariant) BE-functional $\rightsquigarrow$ BE-functional and α have the same family critical curves.

— Obvious candidates for such functionals are

$$\alpha = \boldsymbol{U}' \cdot \boldsymbol{U}', \quad \beta = \boldsymbol{U}'' \cdot \boldsymbol{U}'', \quad \gamma = \boldsymbol{U}''' \cdot \boldsymbol{U}'''$$

or their combinations, e.g. $-\Delta_4 = \beta - \alpha^2 \geq 0$.

— The **simplest functional** is

$$\alpha = \boldsymbol{U}' \cdot \boldsymbol{U}' = 3u^{-2} U'_c U'^c + 2u^{-2} U_c U''^c - 6u^{-4} (U_c U'^c)^2 + 2\mathrm{P}_{cd} U^c U^d$$

is of the 3rd order. (That is, the tangent vector U^a is of the first order.)

- The order can be reduced by adding an exact term:

$$\alpha - 2U^r \nabla_r \left(u^{-2} U_c U'^c \right) = u^{-2} U'_c U'^c - 2u^{-4} (U_c U'^c)^2 + 2\mathrm{P}_{ab} U^a U^b.$$

- The right hand side is exactly (conformally noninvariant) BE-functional $\rightsquigarrow$ BE-functional and α have the same family critical curves.

Euler-Lagrange equations of critical curves for α I.

— **The setup for variation:** fix endpoints $x_1, x_2 \in M$ and tangent vectors at endpoints $A_i \in T_{x_i}M$.

- Given a curve $c(t)$ parametrised on interval $[t_1, t_2]$ with $c(t_i) = x_i \in M$ and tangent vectors at endpoints $U(t_i) = A_i$, we put

$$\mathcal{J}(c) = \int_{t_1}^{t_2} \boldsymbol{U}' \cdot \boldsymbol{U}'.$$

- Given such a curve, we consider a variational vector field Z along $c(t)$ such that

$$Z(x_i) = Z'(x_i) = 0 \quad \text{and} \quad [U, Z] = \nabla_U Z - \nabla_Z U = 0,$$

where we extended Z and U to some neighbourhood of $c(t)$.

— Further consider $\boldsymbol{Z} := \nabla_Z \boldsymbol{T} = \begin{pmatrix} 0 \\ u^{-1} Z^a \\ -u^{-3} U^r Z'_r \end{pmatrix}$ where $Z' = \nabla_U Z$.

- Since $\nabla_Z \boldsymbol{U}' = \nabla_U \nabla_U \boldsymbol{Z} + Z^r U^s \Omega_{rs}(\boldsymbol{U}) \cdot \boldsymbol{U}'$, integration by parts yields

$$\begin{aligned}\nabla_Z \mathcal{J}(c) &= \nabla_Z \int_{t_1}^{t_2} \boldsymbol{U}' \cdot \boldsymbol{U}' = 2 \int_{t_1}^{t_2} \boldsymbol{Z} \cdot \boldsymbol{U}''' + Z^r U^s \Omega_{rs}(\boldsymbol{U}) \cdot \boldsymbol{U}' = \\ &= 2 \int_{t_1}^{t_2} \boldsymbol{Z} \cdot (\boldsymbol{U}''' + \alpha \boldsymbol{U}' + \alpha' \boldsymbol{U}) + Z^r U^s \Omega_{rs}(\boldsymbol{U}) \cdot \boldsymbol{U}' = 0\end{aligned}$$

  for every Z (modulo boundary terms).

- Further we assume **conformally flat case**, i.e. $\Omega_{ab} = 0$. Since the tractor field $\boldsymbol{U}''' + \alpha \boldsymbol{U}' + \alpha' \boldsymbol{U}$ has zero top slot, we obtain the tractor version of the Euler-Lagrange equations

$$\boldsymbol{U}''' + \alpha \boldsymbol{U}' + \alpha' \boldsymbol{U} + \Phi \boldsymbol{T} = 0 \quad \text{for a function } \Phi.$$

  In fact, one can show that $\Phi = -\Delta_4 = \beta - \alpha^2$.

— Recall curves on n-dimensional conformal manifolds have conformal curvatures and $K_1, \ldots, K_{n-1}$. Alternatively, $K_2, \ldots, K_{n-1}$ are refered to as (higher) conformal torsions.

- The condition $\boldsymbol{U}'' \in \langle \boldsymbol{U}', \boldsymbol{U}, \boldsymbol{T} \rangle$ means $K_1 = \ldots = K_{n-1} = 0$; these curves are *conformal circles*.
- The condition $\boldsymbol{U}''' \in \langle \boldsymbol{U}'', \boldsymbol{U}', \boldsymbol{U}, \boldsymbol{T} \rangle$ means $K_2 = \ldots = K_{n-1} = 0$.
  - Our condition $\boldsymbol{U}''' + \alpha \boldsymbol{U}' + \alpha' \boldsymbol{U} + \Phi \boldsymbol{T} = 0$ is even more restrictive: it equivalently means

$$\alpha = const, \ \Phi = const, \quad K_1 = const, \quad K_2 = \ldots = K_{n-1} = 0 \quad (1)$$

— **Conclusion:** on locally flat manifolds, critical curves for the simplest conformal functional $\alpha = \boldsymbol{U}' \cdot \boldsymbol{U}'$ given by (1) are loxodromas (spirals on a sphere); these, in a more general sense, include circles and lines.

What are critical curves?

— Recall curves on n-dimensional conformal manifolds have conformal curvatures and $K_1, \ldots, K_{n-1}$. Alternatively, $K_2, \ldots, K_{n-1}$ are refered to as (higher) conformal torsions.

- The condition $\boldsymbol{U}'' \in \langle \boldsymbol{U}', \boldsymbol{U}, \boldsymbol{T} \rangle$ means $K_1 = \ldots = K_{n-1} = 0$; these curves are *conformal circles*.
- The condition $\boldsymbol{U}''' \in \langle \boldsymbol{U}'', \boldsymbol{U}', \boldsymbol{U}, \boldsymbol{T} \rangle$ means $K_2 = \ldots = K_{n-1} = 0$.
  - Our condition $\boldsymbol{U}''' + \alpha \boldsymbol{U}' + \alpha' \boldsymbol{U} + \Phi \boldsymbol{T} = 0$ is even more restrictive: it equivalently means

$$\alpha = const, \ \Phi = const, \quad K_1 = const, \quad K_2 = \ldots = K_{n-1} = 0 \quad (1)$$

— **Conclusion:** on locally flat manifolds, critical curves for the simplest conformal functional $\alpha = \boldsymbol{U}' \cdot \boldsymbol{U}'$ given by (1) are <u>loxodromas</u> (spirals on a sphere); these, in a more general sense, include circles and lines.

Alternative equation for critical curves

— In order to build tractor Frenet frame, we observe $\boldsymbol{U}'' + \alpha\boldsymbol{U} \perp \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle$ where

$$\boldsymbol{U}'' + \alpha\boldsymbol{U} = \begin{pmatrix} 0 \\ N^a \\ * \end{pmatrix}$$

where $N \perp U$ is the *Fialkow normal*. Explicitly,

$$N^a = u^{-1}U''^a - 3u^{-3}(U_r U'^r)U'^a + u^{-1}(\ldots)U^a - u^{-1}U^r \mathrm{P}_r{}^a.$$

— Using the Filakow normal, we recover the following:

- The equation for conformal circles: $N = 0$
- The condition for the conformal arc length: $||N|| = 1$.
- The equation for loxodromas: $N'^a \propto U^a$ and $\alpha \in \mathbb{R}$.

Alternative equation for critical curves

— In order to build tractor Frenet frame, we observe $\boldsymbol{U}'' + \alpha \boldsymbol{U} \perp \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle$ where

$$\boldsymbol{U}'' + \alpha \boldsymbol{U} = \begin{pmatrix} 0 \\ N^a \\ * \end{pmatrix}$$

where $N \perp U$ is the *Fialkow normal*. Explicitly,

$$N^a = u^{-1} U''^a - 3u^{-3}(U_r U'^r) U'^a + u^{-1}(\ldots) U^a - u^{-1} U^r \mathrm{P}_r{}^a.$$

— Using the Filakow normal, we recover the following:

- The equation for conformal circles: $N = 0$
- The condition for the conformal arc length: $||N|| = 1$.
- The equation for loxodromas: $N'^a \propto U^a$ and $\alpha \in \mathbb{R}$.

The index form of the second variation of α

— Assuming the **locally flat case**, critical curves of $\mathcal{J}(c)$ are charecterized by

$$\boldsymbol{U}''' + \alpha \boldsymbol{U}' - \Phi \boldsymbol{T} = 0, \qquad \Phi = \beta - \alpha^2 \geq 0, \ \alpha, \beta \in \mathbb{R}.$$

— Let us compute the second variation at such curve (modulo boundary terms):

$$\nabla_Z \nabla_Z \mathcal{J}(c) = \int_{t_1}^{t_2} \boldsymbol{Z}'' \cdot \boldsymbol{Z}'' - \alpha \boldsymbol{Z}' \cdot \boldsymbol{Z}' - \Phi \boldsymbol{Z} \cdot \boldsymbol{Z}.$$

— Is there really a chance for local extremals? No – a suitable reparametrization can increase or decrease $\mathcal{J}(c)$. Further observe:

- Variation vector field Z tangent to the curve $\rightsquigarrow$ reparametrization of $c(t)$.
- Thus we have the following question: are there local extremals of $\mathcal{J}(c)$ with respect to **normal variations** only?
- Clearly second (normal) variation cannot be definite for circles – any variation in the plane of the circle is zero.

— Assuming the **locally flat case**, critical curves of $\mathcal{J}(c)$ are charecterized by

$$\boldsymbol{U}''' + \alpha \boldsymbol{U}' - \Phi \boldsymbol{T} = 0, \qquad \Phi = \beta - \alpha^2 \geq 0,\ \alpha, \beta \in \mathbb{R}.$$

— Let us compute the second variation at such curve (modulo boundary terms):

$$\nabla_Z \nabla_Z \mathcal{J}(c) = \int_{t_1}^{t_2} \boldsymbol{Z}'' \cdot \boldsymbol{Z}'' - \alpha \boldsymbol{Z}' \cdot \boldsymbol{Z}' - \Phi \boldsymbol{Z} \cdot \boldsymbol{Z}.$$

— Is there really a chance for local extremals? No – a suitable reparametrization can increase or decrease $\mathcal{J}(c)$. Further observe:

- Variation vector field Z tangent to the curve $\rightsquigarrow$ reparametrization of $c(t)$.
- Thus we have the following question: are there local extremals of $\mathcal{J}(c)$ with respect to **normal variations** only?
- Clearly second (normal) variation cannot be definite for circles – any variation in the plane of the circle is zero.

The index form of the second variation of α

— Assuming the **locally flat case**, critical curves of $\mathcal{J}(c)$ are charecterized by

$$\boldsymbol{U}''' + \alpha \boldsymbol{U}' - \Phi \boldsymbol{T} = 0, \qquad \Phi = \beta - \alpha^2 \geq 0,\ \alpha, \beta \in \mathbb{R}.$$

— Let us compute the second variation at such curve (modulo boundary terms):

$$\nabla_Z \nabla_Z \mathcal{J}(c) = \int_{t_1}^{t_2} \boldsymbol{Z}'' \cdot \boldsymbol{Z}'' - \alpha \boldsymbol{Z}' \cdot \boldsymbol{Z}' - \Phi \boldsymbol{Z} \cdot \boldsymbol{Z}.$$

— Is there really a chance for local extremals? No – a suitable reparametrization can increase or decrease $\mathcal{J}(c)$. Further observe:

- Variation vector field Z tangent to the curve $\rightsquigarrow$ reparametrization of $c(t)$.
- Thus we have the following question: are there local extremals of $\mathcal{J}(c)$ with respect to **normal variations** only?
- Clearly second (normal) variation cannot be definite for circles – any variation in the plane of the circle is zero.

— For simplicity we restrict to dimension **three**. Then we have orthogonal decomposition

$$\mathcal{T}|_{c(t)} = \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle \oplus \langle \boldsymbol{U}'' + \alpha \boldsymbol{U} \rangle \oplus \langle \boldsymbol{V} \rangle, \quad \boldsymbol{V}' = 0.$$

— A normal variation $Z \perp U$ means $\boldsymbol{Z} \perp \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle$.

$$\boldsymbol{Z} = f(\boldsymbol{U}'' + \alpha \boldsymbol{U}) + h\boldsymbol{V}, \qquad f, h : [t_1, t_2] \to \mathbb{R}$$

hence the second normal variation is

$$\nabla_Z \nabla_Z \mathcal{J}(c) = 2 \int_{t_2}^{t_2} \Phi\big(f''^2 - \alpha f'^2\big) + \big(h''^2 - \alpha h'^2 - \Phi h^2\big)$$

— Another ingredient: Wirtinger's inequality

$$\int_{t_1}^{t_2} h'^2 \geq \frac{\pi^2}{(t_2 - t_1)^2} \int_{t_1}^{t_2} h^2, \quad h(t_1) = h(t_2) = 0.$$

— **Conclusion:** Loxodromas are local minima of $\mathcal{J}(c)$ wrt. to normal variations for $t_2 - t_1$ sufficiently small.

— For simplicity we restrict to dimension **three**. Then we have orthogonal decomposition

$$\mathcal{T}|_{c(t)} = \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle \oplus \langle \boldsymbol{U}'' + \alpha \boldsymbol{U} \rangle \oplus \langle \boldsymbol{V} \rangle, \quad \boldsymbol{V}' = 0.$$

— A normal variation $Z \perp U$ means $\boldsymbol{Z} \perp \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle$.

$$\boldsymbol{Z} = f(\boldsymbol{U}'' + \alpha \boldsymbol{U}) + h\boldsymbol{V}, \qquad f, h : [t_1, t_2] \to \mathbb{R}$$

hence the second normal variation is

$$\nabla_Z \nabla_Z \mathcal{J}(c) = 2 \int_{t_2}^{t_2} \Phi\big(f''^2 - \alpha f'^2\big) + \big(h''^2 - \alpha h'^2 - \Phi h^2\big)$$

— Another ingredient: Wirtinger's inequality

$$\int_{t_1}^{t_2} h'^2 \geq \frac{\pi^2}{(t_2 - t_1)^2} \int_{t_1}^{t_2} h^2, \quad h(t_1) = h(t_2) = 0.$$

— **Conclusion:** Loxodromas are local minima of $\mathcal{J}(c)$ wrt. to normal variations for $t_2 - t_1$ sufficiently small.

— For simplicity we restrict to dimension **three**. Then we have orthogonal decomposition

$$\mathcal{T}|_{c(t)} = \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle \oplus \langle \boldsymbol{U}'' + \alpha \boldsymbol{U} \rangle \oplus \langle \boldsymbol{V} \rangle, \quad \boldsymbol{V}' = 0.$$

— A normal variation $Z \perp U$ means $\boldsymbol{Z} \perp \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle$.

$$\boldsymbol{Z} = f(\boldsymbol{U}'' + \alpha \boldsymbol{U}) + h\boldsymbol{V}, \qquad f, h : [t_1, t_2] \to \mathbb{R}$$

hence the second normal variation is

$$\nabla_Z \nabla_Z \mathcal{J}(c) = 2 \int_{t_2}^{t_2} \Phi\big(f''^2 - \alpha f'^2\big) + \big(h''^2 - \alpha h'^2 - \Phi h^2\big)$$

— Another ingredient: Wirtinger's inequality

$$\int_{t_1}^{t_2} h'^2 \geq \frac{\pi^2}{(t_2 - t_1)^2} \int_{t_1}^{t_2} h^2, \quad h(t_1) = h(t_2) = 0.$$

— **Conclusion:** Loxodromas are local minima of $\mathcal{J}(c)$ wrt. to normal variations for $t_2 - t_1$ sufficiently small.

— Let us briefly comment upon the **CR (or Lagrange-contact)** geometry.

- There are CR analogues of α for both transversal and tangent curves to the distribution. In the transversal case, the family of critical curves contains (but is bigger larger than) chains.
- But chains are variational: *Chains in CR geometry as geodesics of a Kropina metric*, CMMM, Adv. Math. $\rightsquigarrow$ a possible relation to tractor-build objects is unclear.

— Back to possible variational characterization of circles: there is a suprise in dimension three. We have Euclidean torsion τ. Critical curves of

$$\int \tau ds$$

are exactly **circles**: *Energy density functions for protein structures*, TCH, J. Mech. Appl. Math.

- An analogue in other dimensions is unclear.

Final comments

— Let us briefly comment upon the **CR (or Lagrange-contact)** geometry.

- There are CR analogues of α for both transversal and tangent curves to the distribution. In the transversal case, the family of critical curves contains (but is bigger larger than) chains.
- But chains are variational: *Chains in CR geometry as geodesics of a Kropina metric*, CMMM, Adv. Math. $\rightsquigarrow$ a possible relation to tractor-build objects is unclear.

— Back to possible variational characterization of circles: there is a suprise in dimension three. We have Euclidean torsion τ. Critical curves of

$$\int \tau \, ds$$

are exactly **circles**: *Energy density functions for protein structures*, TCH, J. Mech. Appl. Math.

- An analogue in other dimensions is unclear.

Final comments

— Let us briefly comment upon the **CR (or Lagrange-contact)** geometry.

- There are CR analogues of α for both transversal and tangent curves to the distribution. In the transversal case, the family of critical curves contains (but is bigger larger than) chains.
- But chains are variational: *Chains in CR geometry as geodesics of a Kropina metric*, CMMM, Adv. Math. $\rightsquigarrow$ a possible relation to tractor-build objects is unclear.

— Back to possible variational characterization of circles: there is a suprise in dimension three. We have Euclidean torsion τ. Critical curves of

$$\int \tau \, ds$$

are exactly **circles**: *Energy density functions for protein structures*, TCH, J. Mech. Appl. Math.

- An analogue in other dimensions is unclear.

Thank you for your attention!