Symmetry gaps for higher order ODEs

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The symmetry gap problem

For systems of m ODEs of fixed order $n+1$, $\mathbf{u}$ is $\mathbb{R}^m$-valued function of t, admitting finite dim (infinitesimal) symmetry algebra:

$$\mathbf{u}^{(n+1)} = \mathbf{f}(t, \mathbf{u}, \mathbf{u}', \ldots, \mathbf{u}^{(n)}), \tag{1}$$

Q. What is the next largest realizable (*submaximal*) sym dim $\mathfrak{S}$?

Example (parabolic geometries)

n	m	Pseudogroup	max $\mathfrak{M}$	$\mathfrak{S}$
1	1	point	8	3
2	1	contact	10	5
1	≥ 2	contact	$(m+2)^2-1$	m^2+5

Table: Submax sym dim for ODE of order $n+1$

The main result

We considered the gap problem for higher order ODEs:

- scalar ODEs of order ≥ 4 ($m = 1,\ n \geq 3$)
- vector ODEs of order ≥ 3 ($m, n \geq 2$)

Doubrov–Komrakov–Morimoto 1999: These are (non-parabolic Cartan geometries) with $\mathfrak{M} = m^2 + (n+1)m + 3$.

Theorem (K–The 2021)

Fix (n, m) with $m = 1, n \geq 3$ or $m, n \geq 2$. Among the ODEs (1), the submaximal contact symmetry dimension is

$$\mathfrak{S} = \begin{cases} \mathfrak{M} - 1, & \text{if } m = 1,\ n \in \{4, 6\}; \\ \mathfrak{M} - 2, & \text{otherwise.} \end{cases}$$

Recover the classical result for scalar cases and resolve the problem for vector cases.

Approach

For $m = 1$, the problem was resolved using methods relying on the complete classification of Lie algebras of contact vector fields on plane. Those methods are not feasible for $m \geq 2$.

Our approach is based on a categorically equivalent reformulation of ODEs $\mathcal{E}$ given by (1) as *regular, normal Cartan geometries* $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P), for some appropriate Lie group G and closed subgroup $P \subset G$.

For parabolic geometries, the gap problem was resolved by Kruglikov–The 2013. In particular, they established a universal algebraic upper bound $\mathfrak{U}$ on $\mathfrak{S}$. We adapt their approach to the (non-parabolic) ODE setup.

The trivial ODE (flat model)

Abstractly, the contact sym algebra $\mathfrak{g}$ for $\mathbf{u}^{(n+1)} = 0$:
$\mathfrak{g} = \mathfrak{q} \ltimes V, \quad$ where $\quad \mathfrak{q} := \mathfrak{sl}_2 \times \mathfrak{gl}_m, \quad V := \mathbb{V}_n \otimes W.$
$\mathbb{V}_n$, $\mathfrak{sl}_2$-irrep of dim $n+1$ and $W = \mathbb{R}^m$, the standard rep of $\mathfrak{gl}_m$.
The *grading element* $\mathrm{Z} = -\frac{1}{2}\left(\mathrm{H} + (n+2)\,\mathrm{id}_m\right)$,

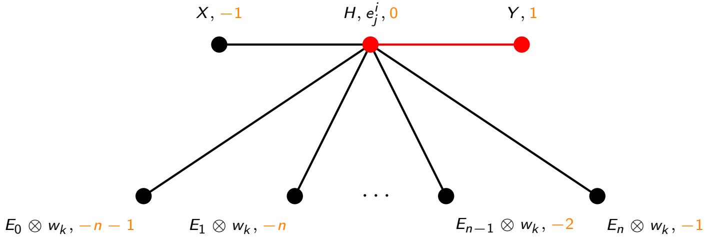

Figure: Grading on $\mathfrak{g}$

$\mathrm{X}, \mathrm{H}, \mathrm{Y}$– standard $\mathfrak{sl}_2$-triple and E_i for $\mathbb{V}_n$ with $[\mathrm{X}, E_i] = E_{i-1}$ and $[\mathrm{Y}, E_n] = 0.$

Filtration: $\mathfrak{g}^i := \sum_{j \geq i} \mathfrak{g}_j$, so $\mathfrak{p} := \mathfrak{g}^0 = \langle \mathsf{H}, e^i_j, \mathsf{Y} \rangle$, $\mathfrak{p}_+ := \mathfrak{g}^1 = \langle \mathsf{Y} \rangle$.
At the group level, let

- $m = 1$: $G = GL_2 \ltimes \mathbb{V}_n$ and $P = ST_2 \subset GL_2$, the subgroup of lower triangular matrices;
- $m \geq 2$: $G = (SL_2 \times GL_m) \ltimes V$ and $P = ST_2 \times GL_m$.

In either case, let $G_0 := \{g \in P : \mathrm{Ad}_g(\mathfrak{g}_0) \subset \mathfrak{g}_0\}$.

Doubrov–Komrakov–Morimoto 1999: All ODEs $\mathcal{E}$ (1) are filtered G_0-structures, and there is an equivalence of categories between filtered G_0-structures and regular, normal Cartan geometries $(\mathcal{G} \to \mathcal{E}, \omega)$ of fixed type (G, P).

For $(\mathcal{G} \to \mathcal{E}, \omega)$:
(infinitesimal) symmetries: $\mathfrak{inf}(\mathcal{G}, \omega) := \left\{ \xi \in \Gamma(\mathcal{G})^P : \mathcal{L}_\xi \omega = 0 \right\}$.
$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$, Curvature two-form, is determined by the P-equivariant *curvature function* κ valued in $\wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$:
$\kappa(A, B) := K(\omega^{-1}(A), \omega^{-1}(B)), \quad A, B \in \mathfrak{g}$.

ODEs as Cartan geometries

ω is *regular* if $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ for all i, j and *normal* : $\partial^* \kappa = 0$, where ∂^* is the adjoint of the usual standard cohomology differential with respect to some natural inner product on $\mathfrak{g}$.

Since $(\partial^*)^2 = 0$, then for *regular, normal* Cartan geometries one obtains the (P-equivariant) *harmonic curvature* function $\kappa_H : \mathcal{G} \to \frac{\ker \partial^*}{\operatorname{im} \partial^*}$.
$\kappa_H \equiv 0$ iff the geometry is locally equivalent to the flat model (the trivial ODE).

Čap–Doubrov–The (2020): The P-module $\frac{\ker \partial^*}{\operatorname{im} \partial^*}$ is completely reducible, i.e. $\mathfrak{g}^1$ acts trivially.

Doubrov 2001, Medvedev 2010 have identified the the effective part $\mathbb{E} \subsetneq H^2(\mathfrak{g}_-, \mathfrak{g})$ such that $\operatorname{im}(\kappa_H) \subset \mathbb{E}$ for any regular, normal Cartan geometry of type (G, P) associated to ODE (for fixed n, m).

Upper bound $\mathfrak{U}$ on $\mathfrak{S}$

$\mathfrak{S} := \max\{\dim \mathfrak{inf}(\mathcal{G}, \omega) : (\mathcal{G} \to \mathcal{E}, \omega) \text{ regular, normal of type } (G, P),\ \kappa_H \not\equiv 0\}$.

Definition (Tanaka prolongation algebra)

For $\mathfrak{a}_0 \subset \mathfrak{g}_0$, TPA is the graded subalgebra $\mathfrak{a} := \mathrm{pr}(\mathfrak{g}_-, \mathfrak{a}_0)$ of $\mathfrak{g}$ with $\mathfrak{a}_- := \mathfrak{g}_-$ and $\mathfrak{a}_1 := \{X \in \mathfrak{g}_1 : [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_0\}$. Given ϕ in some $\mathfrak{g}_0$-module, let $\mathfrak{ann}(\phi) \subset \mathfrak{g}_0$ be its annihilator and define $\mathfrak{a}^\phi := \mathrm{pr}(\mathfrak{g}_-, \mathfrak{ann}(\phi))$.

$$\mathfrak{U} := \max\left\{\dim \mathfrak{a}^\phi :\ 0 \neq \phi \in \mathbb{E}\right\}.$$

Theorem (K–The 2021)

For a regular, normal Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) associated to an ODE, $\mathfrak{S} \leq \mathfrak{U} < \dim \mathfrak{g}$.

In fact, for a G_0-irrep $\mathbb{U} \subset \mathbb{E}$, $\mathrm{im}(\kappa_H) \subset \mathbb{U}$ with $\kappa_H \not\equiv 0$, define $\mathfrak{S}_{\mathbb{U}}$ analogously to $\mathfrak{S}$ and set $\mathfrak{U}_{\mathbb{U}} := \max\{\dim \mathfrak{a}^\phi : 0 \neq \phi \in \mathbb{U}\}$. Then $\mathfrak{S}_{\mathbb{U}} \leq \mathfrak{U}_{\mathbb{U}}$. For $\mathbb{E} = \bigoplus_i \mathbb{U}_i$, G_0-irreps $\mathbb{U}_i$ (G_0 is reductive), we have $\mathfrak{U} = \max_i \mathfrak{U}_{\mathbb{U}_i}$.

Computations of $\mathfrak{U}$ and $\mathfrak{U}_{\mathbb{U}}$

$\mathfrak{g} = (\mathfrak{sl}_2 \times \mathfrak{gl}_m) \ltimes (\mathbb{V}_n \otimes W)$. We refine the grading to a *bi-grading* $(\mathsf{Z}_1, \mathsf{Z}_2)$:

$$\mathsf{Z}_1 = -\frac{1}{2}(\mathsf{H} + n\,\mathrm{id}_m), \quad \mathsf{Z}_2 = -\,\mathrm{id}_m \quad \text{with} \quad \mathsf{Z} = \mathsf{Z}_1 + \mathsf{Z}_2.$$

Have induced bi-gradings on the effective part $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$. Given $(a, b) \in \mathbb{Z}^2$, let $\mathbb{E}_{a,b} := \{\phi \in \mathbb{E} : \mathsf{Z}_1 \cdot \phi = a\phi,\ \mathsf{Z}_2 \cdot \phi = b\phi\}$, similarly, $\mathfrak{g}_{a,b}$.

Lemma

Let $0 \neq \phi \in \mathbb{E}$. Then $\mathfrak{a}_1^\phi \neq 0$ iff ϕ lies in the direct sum of all $\mathbb{E}_{a,b}$ for (a, b) that is a multiple of $(n, 2)$.

Proof.

Note that $\mathfrak{a}_1^\phi \neq 0$ iff $\mathfrak{a}_1^\phi = \mathfrak{g}_1 = \mathbb{R}Y$. Since $[\mathsf{Y}, \mathfrak{g}_{0,-1}] = 0$, then this occurs iff $[\mathsf{Y}, \mathsf{X}] = -\mathsf{H} \in \mathfrak{a}_0^\phi := \mathfrak{ann}(\phi)$. But $\mathsf{H} = -2\mathsf{Z}_1 + n\mathsf{Z}_2$, so $\mathsf{H} \in \mathfrak{ann}(\phi)$ iff ϕ lies in the direct sum of the claimed modules. □

Definition (Prolongation rigidity)

We say that a $\mathfrak{g}_0$-submodule $\mathbb{U} \subset \mathbb{E}$ is PR if $\mathfrak{a}_1^{\phi} = 0$ for any $0 \neq \phi \in \mathbb{U}$.

$\mathbb{E}$ has been computed: Doubrov 2001 ($m = 1, n \geq 3$), Medvedev 2011 ($m \geq 2, n = 1$) and Doubrov–Medvedev ($m \geq 2, n \geq 2$).

n	$\mathfrak{g}_0$-irrep $\mathbb{U}$	Bi-grade	Range
≥ 3	$\mathbb{W}_i$	$(i, 0)$	$3 \leq i \leq n+1$
3	$\mathbb{B}_3$	$(1, 2)$	–
3	$\mathbb{B}_4$	$(2, 2)$	–
4	$\mathbb{B}_6$	$(4, 2)$	–
≥ 4	$\mathbb{A}_2$	$(1, 1)$	–
≥ 5	$\mathbb{A}_3$	$(2, 1)$	–
≥ 6	$\mathbb{A}_4$	$(3, 1)$	–

Table: Effective part $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$ for scalar ODEs of order $n+1 \geq 4$

Lemma

(a) $\mathbb{E}$ *is not PR iff* $n = 4$ *or* 6. *In particular,* $(n, \mathbb{U}) = (4, \mathbb{B}_6)$ *and* $(6, \mathbb{A}_4)$ *are not PR.*

(b) *If* $\mathbb{U} \subset \mathbb{E}$ *is a* $\mathfrak{g}_0$*-irrep, then* $\mathfrak{U}_{\mathbb{U}} = \begin{cases} n+4, & \text{if } (n, \mathbb{U}) = (4, \mathbb{B}_6) \text{ or } (6, \mathbb{A}_4); \\ n+3, & \text{otherwise.} \end{cases}$

(c) $\mathfrak{U} = \begin{cases} \mathfrak{M} - 1 = n+4, & \text{if } n = 4, 6; \\ \mathfrak{M} - 2 = n+3, & \text{otherwise.} \end{cases}$

n	$\mathfrak{g}_0$-irrep $\mathbb{U}$	Bi-grade	$\mathfrak{sl}(W)$-module $\mathbb{U}$	$\mathfrak{sl}(W)$ h.w. λ
≥ 2	$\mathbb{W}_i^{\text{tf}}$, $2 \leq i \leq n+1$	$(i, 0)$	$\mathfrak{sl}(W)$	$\lambda_1 + \lambda_{m-1}$
≥ 2	$\mathbb{W}_i^{\text{tr}}$, $3 \leq i \leq n+1$	$(i, 0)$	$\mathbb{R}\,\text{id}_m$	0
2	$\mathbb{B}_4$	$(2, 2)$	$S^2 W^*$	$2\lambda_{m-1}$
≥ 2	$\mathbb{A}_2^{\text{tf}}$	$(1, 1)$	$(S^2 W^* \otimes W)_0$	$\lambda_1 + 2\lambda_{m-1}$
≥ 3	$\mathbb{A}_2^{\text{tr}}$	$(1, 1)$	W^*	λ_{m-1}

Table: Effective part $\mathbb{E} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$ for $m \geq 2$ ODEs of order $n+1 \geq 3$

Lemma

(a) $\mathbb{E}$ *is not PR iff* $n = 2$. *In particular,* $(n, \mathbb{U}) = (2, \mathbb{A}_2^{\text{tf}})$ *and* $(2, \mathbb{B}_4)$ *are not PR.*

(b) *If* $\mathbb{U} \subset \mathbb{E}$ *is a* $\mathfrak{g}_0$*-irrep, then* $\mathfrak{U}_{\mathbb{U}}$ *is given in Table below.*

(c) $\mathfrak{U} = \mathfrak{M} - 2 = m^2 + (n+1)m + 1$.

n	$\mathfrak{g}_0$-irrep $\mathbb{U}$	$\max_{0\neq\phi\in\mathbb{U}} \dim \mathfrak{ann}(\phi)$	$\mathbb{U}$ PR?	$\mathfrak{U}_{\mathbb{U}}$
≥ 2	$\mathbb{W}_i^{\text{tf}}$	$m^2 - 2m + 3$	✓	$m^2 + (n-1)m + 4$
≥ 2	$\mathbb{W}_i^{\text{tr}}$	m^2	✓	$m^2 + (n+1)m + 1$
2	$\mathbb{B}_4$	$m^2 - m + 1$	×	$m^2 + 2m + 3$
2	$\mathbb{A}_2^{\text{tf}}$	$m^2 - 2m + 3$	×	$m^2 + m + 5$
≥ 3	$\mathbb{A}_2^{\text{tf}}$	$m^2 - 2m + 3$	✓	$m^2 + (n-1)m + 4$
≥ 3	$\mathbb{A}_2^{\text{tr}}$	$m^2 - m + 1$	✓	$m^2 + nm + 2$

Table: Upper bounds $\mathfrak{U}_{\mathbb{U}}$ for $m \geq 2$ ODE of order $n + 1 \geq 3$

Conclusion and remarks

Since $\mathfrak{U}$ is realized by ODEs in Table below, then $\mathfrak{U} = \mathfrak{S}$. Hence the result.

m	n	ODE	Sym dim $= \mathfrak{U}$
1	4	$9u_2^2u_5 - 45u_2u_3u_4 + 40u_3^3 = 0$	8
	6	$10u_3^3u_7 - 70u_3^2u_4u_6 - 49u_3^2u_5^2$ $+280u_3u_4^2u_5 - 175u_4^4 = 0$	10
	$3, 5, \geq 7$	$nu_{n-1}u_{n+1} - (n+1)u_n^2 = 0$	$n+3$
≥ 2	≥ 2	$u_{n+1}^i = u_0^i,\ 1 \leq i \leq m$	$m^2 + (n+1)m + 1$

Unlike the classical approach, ours provides a uniform way to study the symmetry gap problem for both scalar and vector ODEs.

Moreover, it can be used to finer gap problem namely: Among all ODEs (1) with $0 \not\equiv \kappa_H$ valued in a G_0-irrep $\mathbb{U} \subset \mathbb{E}$, determine $\mathfrak{S}_{\mathbb{U}}$? In this direction, it turns out that for vector cases $\mathfrak{S}_{\mathbb{U}} = \mathfrak{U}_{\mathbb{U}}$. Scalar cases are still under investigations.