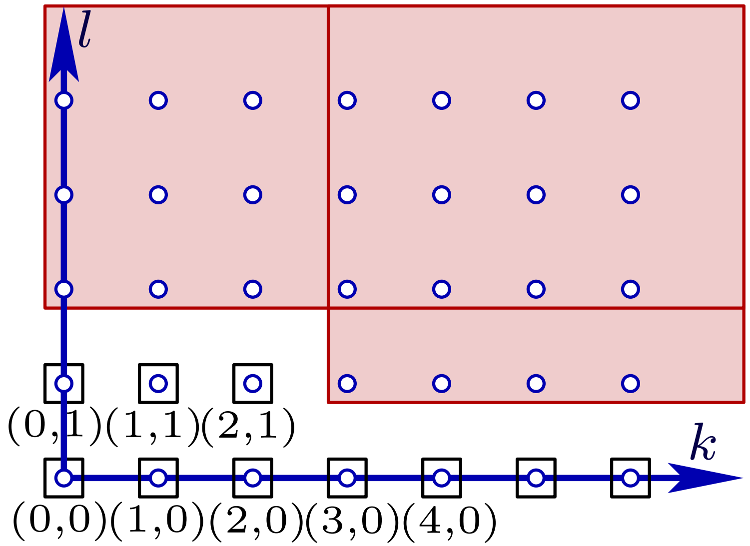
Differential Invariants, Recurrence Relations and Homogeneous Models in Branches and in Subbranches

JOËL MERKER

www.imo.universite-paris-saclay.fr/∼merker/

Institut de Mathématique d'Orsay

Bibliothèque mathématique Jacques Hadamard



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Geometric Structures

- **GRIEG project:**

SCREAMS

Symmetry, Curvature Reduction, and Equivalence Method**S**

- **Finite-dimensional Lie group G:**

□ G acts on a manifold N.

□ G displaces submanifolds:

$$M \subset N \qquad \rightsquigarrow \qquad G \cdot M \subset N.$$

- **Examples:**

□ Second order, third order, $\ldots$, ODEs.

□ (k, l, m)-distributions.

□ Cauchy-Riemann submanifolds.

□ Para-CR structures, or completely integrable PDE systems.

Today's Goal: Divide by 1

Divide by Invariants $I, J, K, \ldots$

- **Functions:** In Cartan's equivalence method:

☐ Objects are *functions*;

☐ Differential forms have coefficients which are *functions*;

☐ Computations are often *nonlinear*;

☐ Invariants (relative or absolute) are *functions*.

- **Divisions:** When some invariant $I \neq 0$ is nonzero, one has to divide by I.

- **Example:** Gaussian metric on a surface:

$$ds^2 = \left|(du, dv)\right|^2 = E\,du^2 + 2F\,dudv + G\,dv^2,$$

having

$$\textsf{Curvature} = \frac{1}{\boldsymbol{(EG - F^2)^2}} \left\{ E \left[\frac{\partial E}{\partial v} \cdot \frac{\partial G}{\partial v} - 2\,\frac{\partial F}{\partial u} \cdot \frac{\partial G}{\partial v} + \left(\frac{\partial G}{\partial u} \right)^2 \right] + \right.$$
$$+ F \left[\frac{\partial E}{\partial u} \cdot \frac{\partial G}{\partial v} - \frac{\partial E}{\partial v} \cdot \frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial v} \cdot \frac{\partial F}{\partial v} + 4\,\frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v} - 2\,\frac{\partial F}{\partial u} \cdot \frac{\partial G}{\partial u} \right] -$$
$$+ G \left[\frac{\partial E}{\partial u} \cdot \frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial u} \cdot \frac{\partial F}{\partial v} + \left(\frac{\partial E}{\partial v} \right)^2 \right] -$$
$$\left. - 2 \left(EG - F^2 \right) \left[\frac{\partial^2 E}{\partial v^2} - 2\,\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 G}{\partial u^2} \right] \right\},$$

and, in the denominator appears $E\,G - F^2$ which is > 0.

• **Example:** With a function φ of 5 variables, consider a hypersurface $M^5 \subset \mathbb{C}^3$ graphed as:

$$v \,=\, \varphi(x_1, x_2, y_1, y_2, u),$$

introduce the *slant function*:

$$\boldsymbol{k} = \frac{\varphi_{z_2\overline{z}_1} + \varphi_{z_2\overline{z}_1}\varphi_u\varphi_u - \sqrt{-1}\varphi_{\overline{z}_1}\varphi_{z_2u} - \varphi_{\overline{z}_1}\varphi_{z_2u}\varphi_u + \sqrt{-1}\varphi_{z_2}\varphi_{\overline{z}_1u} + \varphi_{z_2}\varphi_{\overline{z}_1}\varphi_{uu} - \varphi_{z_2}\varphi_{\overline{z}_1u}\varphi_u}{-\varphi_{z_1\overline{z}_1} - \varphi_{z_1\overline{z}_1}\varphi_u\varphi_u + \sqrt{-1}\varphi_{\overline{z}_1}\varphi_{z_1u} + \varphi_{\overline{z}_1}\varphi_{z_1u}\varphi_u - \sqrt{-1}\varphi_{z_1}\varphi_{\overline{z}_1u} - \varphi_{z_1}\varphi_{\overline{z}_1}\varphi_{uu} + \varphi_{z_1}\varphi_{\overline{z}_1u}\varphi_u},$$

and with the vector field:

$$\overline{\mathscr{L}}_1 = \frac{\partial}{\partial \overline{z}_1} + \frac{\varphi_{z_1}}{\sqrt{-1} - \varphi_u}\frac{\partial}{\partial u},$$

constant Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$ are 2-*nondegenerate* iff:

$$0 \neq \overline{\mathscr{L}}_1(\boldsymbol{k}).$$

☐ In terms of $J^3\varphi$, this $\overline{\mathscr{L}}_1(\boldsymbol{k})$ has 1 page long expression.

☐ Always, one has to divide by $\overline{\mathscr{L}}_1(\boldsymbol{k})$.

Theorem. [Pocchiola 2013] *Cartan's method conducts to construct five invariant 1-forms $\{\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}\}$ together with four 1-forms π^1, π^2, $\overline{\pi}^1$, $\overline{\pi}^2$ satisfying:*

$$\begin{aligned} d\rho &= \left(\pi^1 + \overline{\pi}^1\right) \wedge \rho + i\, \kappa \wedge \overline{\kappa}, \\ d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa}, \\ d\zeta &= \left(\pi^1 - \overline{\pi}^1\right) \wedge \zeta + i\, \pi^2 \wedge \kappa + \\ &\quad + \boldsymbol{R}\, \rho \wedge \zeta + i\, \frac{1}{\overline{\mathsf{c}}^3}\, \overline{\boldsymbol{J}}_0\, \rho \wedge \overline{\kappa} + \frac{1}{\mathsf{c}}\, \mathbf{W}_0\, \kappa \wedge \zeta. \end{aligned}$$

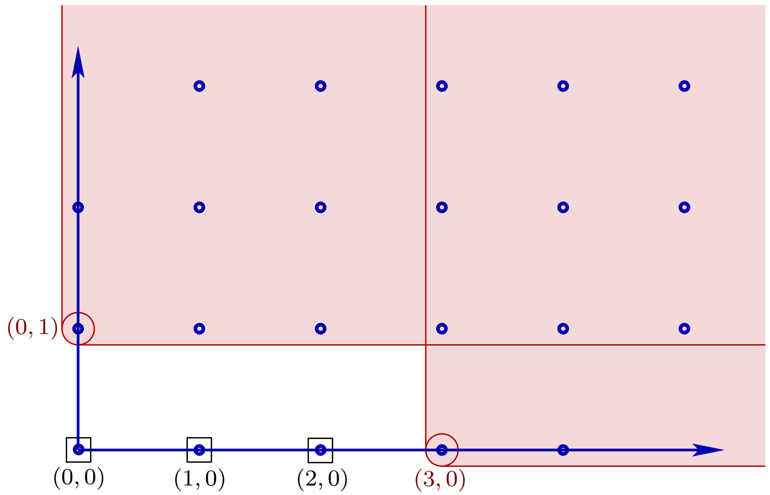
There are four remaining group parameters c, e, $\overline{\mathsf{c}}$, $\overline{\mathsf{e}}$, *and* $\boldsymbol{R}$ *is a secondary invariant:*

$$\boldsymbol{R} \;:=\; \mathsf{Re}\left[i\,\frac{\mathsf{e}}{\mathsf{c}\mathsf{c}}\,\boldsymbol{W}_0 + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\Big(-\,\frac{i}{2}\,\overline{\mathscr{L}}_1\big(\boldsymbol{W}_0\big) + \frac{i}{2}\,\Big(-\,\frac{1}{3}\,\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)} + \frac{1}{3}\,\overline{P}\Big)\,\boldsymbol{W}_0\Big)\right],$$

expressed in terms of two primary relative invariants:

$$\begin{aligned}
\boldsymbol{W}_0 \;:=\; & -\frac{1}{3}\,\frac{\mathscr{K}\Big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\Big)}{\overline{\mathscr{L}}_1(k)^2} + \frac{1}{3}\,\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1(k)\big)\;\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)^3} + \\
& + \frac{2}{3}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\overline{k})\big)}{\mathscr{L}_1(\overline{k})} + \frac{2}{3}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)} + \frac{i}{3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}}_1(k)}, \\
\overline{J}_0 \;:=\; & \frac{1}{6}\,\frac{\overline{\mathscr{L}}_1\Big(\overline{\mathscr{L}}_1\Big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\Big)\Big)}{\overline{\mathscr{L}}_1(k)} - \frac{5}{6}\,\frac{\overline{\mathscr{L}}_1\Big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\Big)\;\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)^2} - \frac{1}{6}\,\frac{\overline{\mathscr{L}}_1\Big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\cdots}{\overline{\mathscr{L}}_1(k)} \\
& + \frac{20}{27}\,\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)^3}{\overline{\mathscr{L}}_1(k)^3} + \frac{5}{18}\,\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)^2}{\overline{\mathscr{L}}_1(k)^2}\,\overline{P} + \frac{1}{6}\,\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\;\overline{\mathscr{L}}_1\big(\overline{P}\big)}{\overline{\mathscr{L}}_1(k)} - \frac{1}{9}\,\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\cdots}{\overline{\mathscr{L}}_1(k)} \\
& - \frac{1}{6}\,\overline{\mathscr{L}}_1\Big(\overline{\mathscr{L}}_1\big(\overline{P}\big)\Big) + \frac{1}{3}\,\overline{\mathscr{L}}_1\big(\overline{P}\big)\,\overline{P} - \frac{2}{27}\,\overline{P}\,\overline{P}\,\overline{P}.
\end{aligned}$$

• **Example:** In [M.-Nurowski 2020], homogeneous models of para-CR structures analog to such $M^5 \subset \mathbb{C}^3$ were classified.



These are system of two PDEs:

$$z_y = F(x, y, z, z_x, z_{xx}) \quad \& \quad z_{xxx} = H(x, y, z, z_x, z_{xx}),$$

satisfying the compatibility condition:

$$D^3 F = \Delta H,$$

in terms of the two total differentiation operators:

$$D := \partial_x + p\,\partial_z + r\,\partial_p + H\,\partial_r \quad \& \quad \Delta := \partial_y + F\,\partial_z + DF\,\partial_p + D^2 F\,\partial_r,$$

By luck:

$$\overline{\mathscr{L}}_1(\boldsymbol{k}) \quad \longleftrightarrow \quad F_{z_x z_x},$$

so denominator complexity decreases serendipitously.

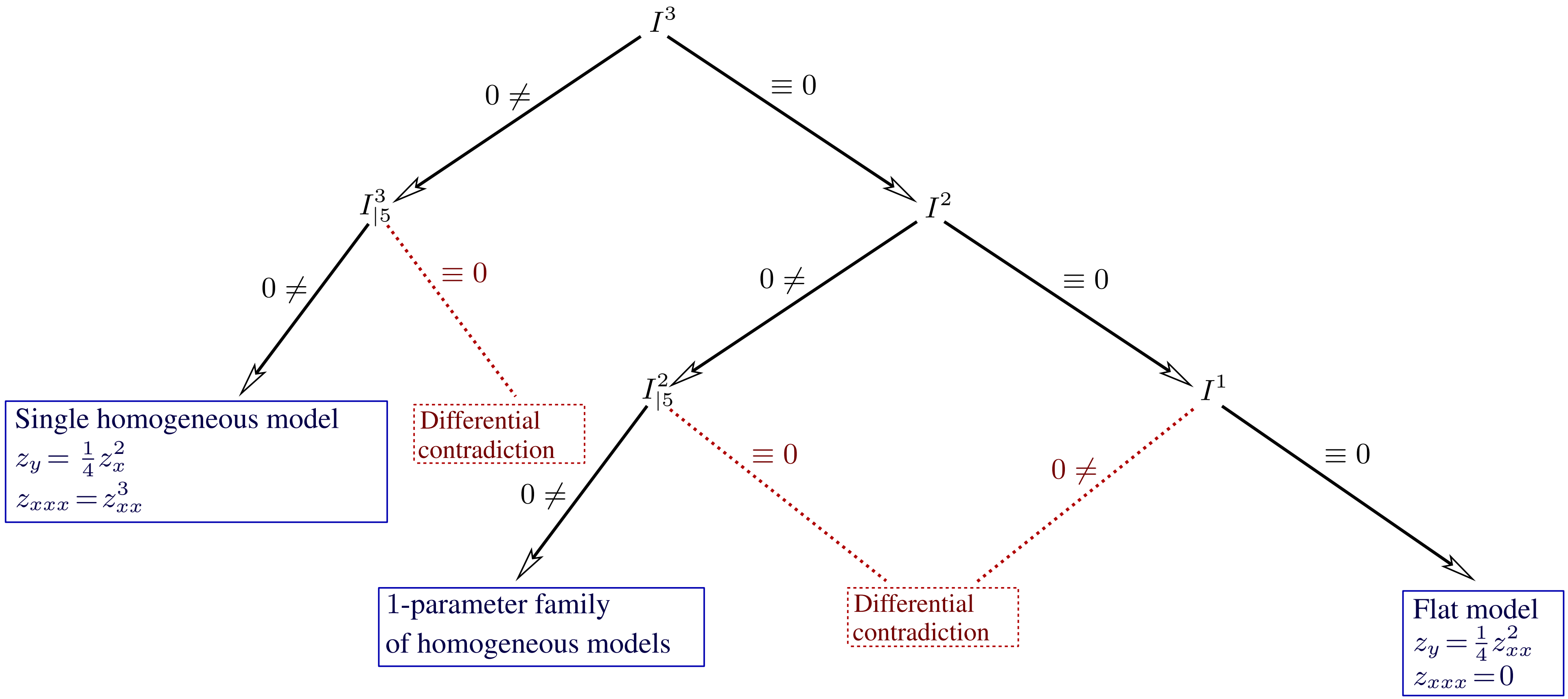
Theorem. [M.-Nurowski 2020] *On the bundle* $\mathscr{G}^9 = M^5 \times G^4$ *with* $M^5 \ni (x, y, z, p, r)$ *times* $\mathbb{R}^4 \ni (\rho, \phi, f_2, \overline{f}_2)$, *there exist four* 1*-forms* Ω_1, Ω_2, Ω_3, Ω_4 *with* θ^1, θ^2, θ^3, θ^4, θ^5, Ω_1, Ω_2, Ω_3, Ω_4 *linearly independent at every point which satisfy the following para-CR invariant exterior differential system:*

$$
\begin{aligned}
d\theta^1 &= -\theta^1 \wedge \Omega_1 + \theta^2 \wedge \theta^4, \\
d\theta^2 &= \theta^2 \wedge (\Omega_2 - \tfrac{1}{2}\Omega_1) - \theta^1 \wedge \Omega_3 + \theta^3 \wedge \theta^4, \\
d\theta^3 &= 2\theta^3 \wedge \Omega_2 - \theta^2 \wedge \Omega_3 + \frac{e^{3\phi}}{\rho^3} I^1\, \theta^1 \wedge \theta^4 + \frac{e^{-\phi}}{\rho} I^3\, \theta^2 \wedge \theta^3 + \\
&\quad \frac{1}{8\rho^3}\Big(2e^{\phi}\bar{f}^2 I^3{}_{|5} + \rho(I^3{}_{|52} + 2I^3{}_{|4}) - 4e^{-\phi} f^2 I^3\Big)\, \theta^1 \wedge \theta^3, \\
d\theta^4 &= -\theta^2 \wedge \theta^5 - \theta^4 \wedge (\tfrac{1}{2}\Omega_1 + \Omega_2) - \theta^1 \wedge \Omega_4, \\
d\theta^5 &= -2\theta^5 \wedge \Omega_2 + \theta^4 \wedge \Omega_4 + \frac{e^{-3\phi}}{\rho^3} I^2\, \theta^1 \wedge \theta^2 - \frac{e^{\phi}}{2\rho} I^3{}_{|5}\, \theta^4 \wedge \theta^5 + \\
&\quad \frac{1}{8\rho^3}\Big(2e^{\phi}\bar{f}^2 I^3{}_{|5} + \rho(I^3{}_{|52} + 2I^3{}_{|4}) - 4e^{-\phi} f^2 I^3\Big)\, \theta^1 \wedge \theta^5,
\end{aligned}
$$

where I^1, I^2, I^3 are explicit relative differential invariants on the base M:

$$
\begin{aligned}
I^1 &:= -\tfrac{1}{54}\big(9D^2H_r - 27DH_p - 18DH_rH_r + 18H_pH_r + 4H_r^3 + 54H_z\big),\\
I^2 &:= \frac{40F_{ppp}^3 - 45F_{pp}F_{ppp}F_{pppp} + 9F_{pp}^2F_{ppppp}}{54\,F_{pp}^3},\\
I^3 &:= \frac{2F_{ppp} + F_{pp}H_{rr}}{3\,F_{pp}},
\end{aligned}
$$

and where $(\cdot)|_i$ for $i = 1, \dots, 5$ denote directional derivatives along the vector fields X_i dual to θ^i.



Theorem. [M.-Nurowski 2020] *Homogeneous models for* 2*-nondegenerate PDE five variables para-CR structures are classified by the following list of mutually inequivalent models:*

(i) $z_y = \frac{1}{4}(z_x)^2 \quad \& \quad z_{xxx} = 0;$

(ii) $z_y = \frac{1}{4}(z_x)^2 \quad \& \quad z_{xxx} = (z_{xx})^3;$

(iiia) $z_y = \frac{1}{4}(z_x)^b$ & $z_{xxx} = (2-b)\frac{(z_{xx})^2}{z_x}$ *with* $z_x > 0$ *for any real* $b \in [1, 2);$

(iiib) $z_y = f(z_x) \quad \& \quad z_{xxx} = h(z_x)\big(z_{xx}\big)^2$, *where the function f is determined by the implicit equation:*

$$\big(z_x^2 + f(z_x)^2\big)\exp\Big(2b\arctan\frac{bz_x - f(z_x)}{z_x + bf(z_x)}\Big) = 1 + b^2$$

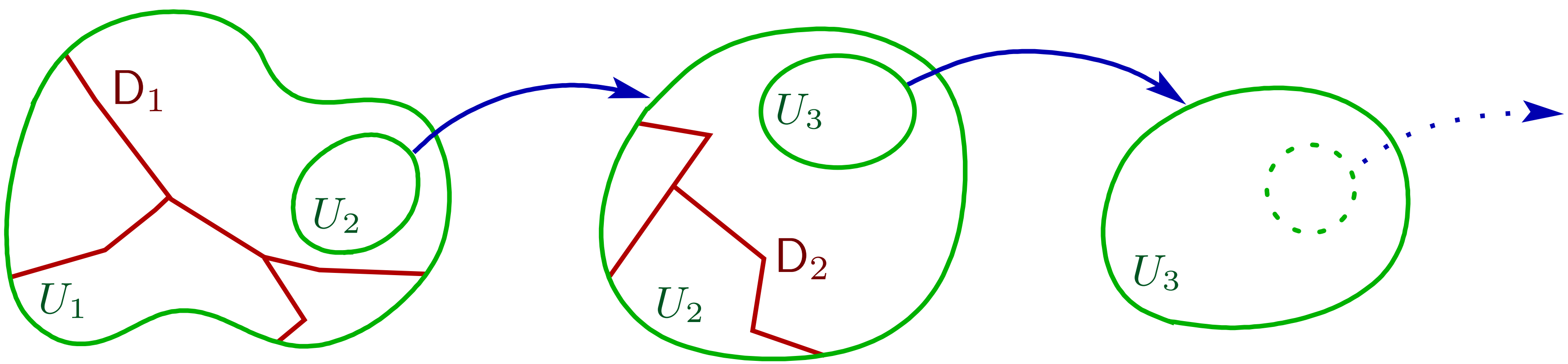
and where:

$$h(z_x) := \frac{(b^2-3)z_x - 4bf(z_x)}{\big(f(z_x) - bz_x\big)^2},$$

for any real $b > 0$.

- **Conclusion:**
- Differential Geometry handles *functions*.
- Differential Invariants are *functions*.
- Any equivalence method forces to divide by (nonzero) *functions*.
- It also forces to *differentiate* several times such quotients.
- This causes exponential growth of complexity.
- **Question:** *But what a function is?*

Genericity Assumptions



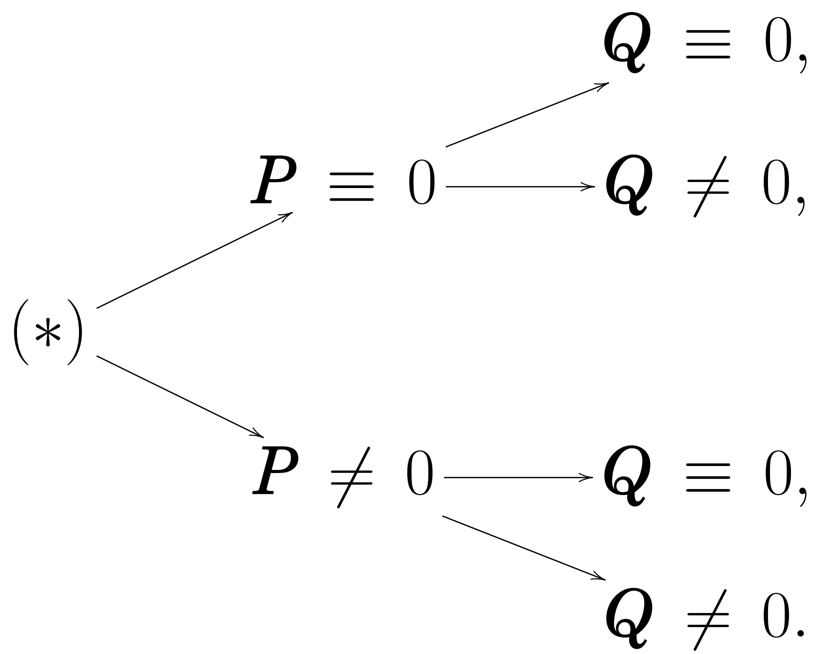
Relocalizing finitely many times in neighborhoods of generic points

Often, certain (relative) differential invariants are encountered:

$$\boldsymbol{P} = \boldsymbol{P}\Big(x, y, u, \{u_{x^j y^k}\}_{1 \leqslant j+k \leqslant n}\Big), \qquad \boldsymbol{Q}, \qquad \boldsymbol{R}, \qquad \dots$$

Their zero-sets $\{\boldsymbol{P} = 0\}$, $\{\boldsymbol{Q} = 0\}$, ..., are invariant under G.

They are responsible for the creation of *branches* and of further *sub-branches*:



- **Lie's principle of thought:**

☐ Either a (relative) differential invariant is identically zero.

☐ Or it is assumed to be nowhere zero, after restriction to a subset.

[Mixed cases are excluded from exploration.]

- **Two cases, two difficulties:**

☐ $P \neq 0$. Divide by $P \rightsquigarrow$ formulas explode.

☐ $P \equiv 0$. How to express consequences?

Power Series Equivalence Method

- **Illustrate only in 2D:** Surface S^2 in $\mathbb{R}^3$ or in $\mathbb{C}^3$:

$$
\begin{aligned}
u &= F^0(x,y) \\
&= \sum_{j+k\geqslant 1} F^0_{j,k}\,\frac{x^j}{j!}\,\frac{y^k}{k!}
\end{aligned}
$$

under $\mathsf{Eucl}(\mathbb{R}^3)$, under $\mathsf{Aff}(\mathbb{R}^3)$, *etc*.

- **Small hypothesis:** Asssume $G \supset \mathsf{translations}$, hence $F(0,0) = 0$.

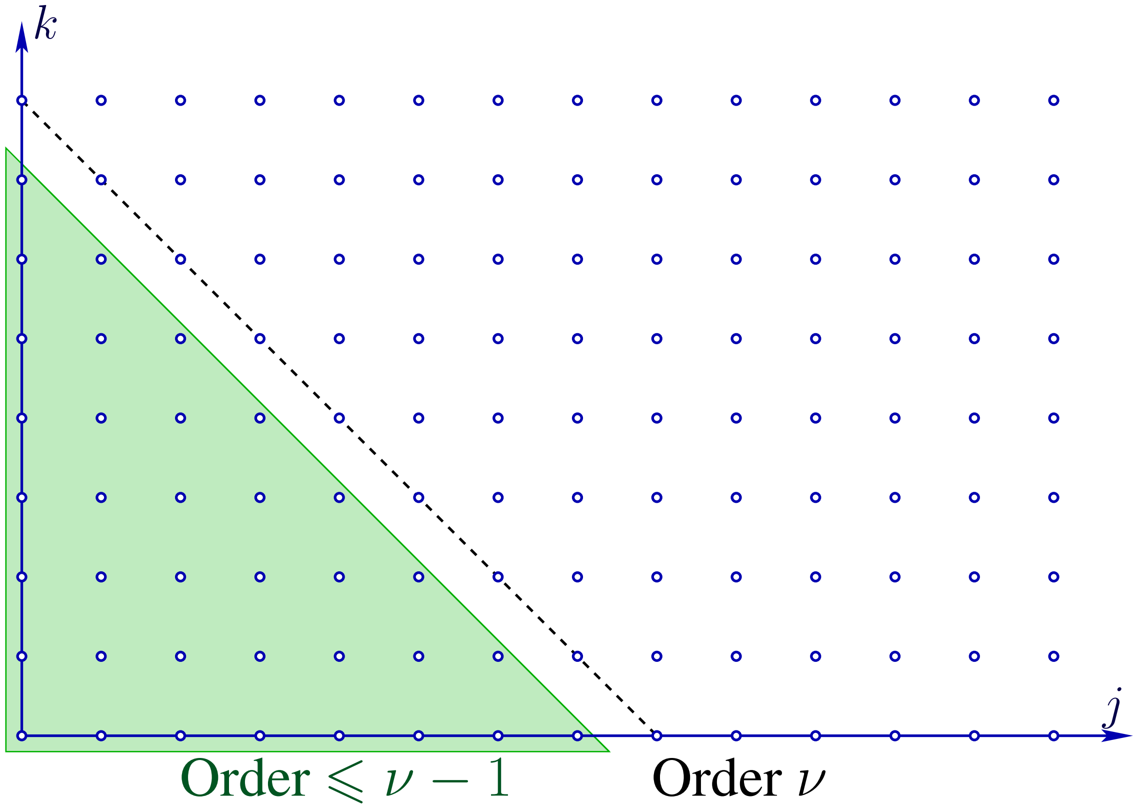
- **Normalize step by step:**

$$
\begin{aligned}
u &= F^0(x,y) = F^{\mathsf{initial}}(x,y), \\
u &= F^1(x,y) = \sum_{j+k\geqslant 1} F^1_{j,k}\,\frac{x^j}{j!}\,\frac{y^k}{k!}, \\
u &= F^2(x,y) = \sum_{j+k\geqslant 1} F^2_{j,k}\,\frac{x^j}{j!}\,\frac{y^k}{k!}, \\
u &= F^3(x,y) = \sum_{j+k\geqslant 1} F^3_{j,k}\,\frac{x^j}{j!}\,\frac{y^k}{k!}.
\end{aligned}
$$

- **Keep memory:**

$$
\begin{aligned}
F^1_{j,k} &= F^1_{j,k}\big(F^0_{\bullet,\bullet}\big), \\
F^2_{j,k} &= F^2_{j,k}\big(F^1_{\bullet,\bullet}\big) = F^2_{j,k}\big(F^0_{\bullet,\bullet}\big), \\
F^3_{j,k} &= F^3_{j,k}\big(F^2_{\bullet,\bullet}\big) = F^3_{j,k}\big(F^1_{\bullet,\bullet}\big) = F^3_{j,k}\big(F^0_{\bullet,\bullet}\big).
\end{aligned}
$$

- **Proceed order by order:**



• **In a current branch:** Assume the surface on the left is:

$$u = \underbrace{\sum_{j+k\leqslant\nu-1} F^{\mathsf{n}}_{j,k} \frac{x^j}{j!}\frac{y^k}{k!}}_{\text{normalized}} + \sum_{j+k=\nu} \mathbf{F_{j,k}} \frac{x^j}{j!}\frac{y^k}{k!} + \mathrm{O}_{x,y}(\nu+1),$$

that the group is reduced to a subgroup stabilizing order $\leqslant \nu - 1$ normalizations:

$$G^{\nu-1}_{\mathsf{stab}} \subset G,$$

that a similar surface is given on the right:

$$\overline{u} = \underbrace{\sum_{j+k\leqslant\nu-1} F^{\mathsf{n}}_{j,k} \frac{\overline{x}^j}{j!}\frac{\overline{y}^k}{k!}}_{\text{same normalization}} + \sum_{j+k=\nu} \overline{\mathbf{F}}_{\mathbf{j},\mathbf{k}} \frac{\overline{x}^j}{j!}\frac{\overline{y}^k}{k!} + \mathrm{O}_{\overline{x},\overline{y}}(\nu+1),$$

and try to normalize the $\overline{\mathbf{F}}_{\mathbf{j},\mathbf{k}}$.

- **Group reductions:**

$$G^{\nu-1}_{\mathsf{stab}} \subset G^{\nu-2}_{\mathsf{stab}} \subset \cdots \subset G^{2}_{\mathsf{stab}} \subset G^{1}_{\mathsf{stab}} \subset G.$$

- **Compute the action for $j+k=\nu$:**

$$0 = - * \mathbf{F}_{\mathbf{j},\mathbf{k}} + * \overline{\mathbf{F}}_{\mathbf{j},\mathbf{k}} + \text{freedom to normalize.}$$

- **Two examples:**

$$\begin{aligned}
0 &= -a^2_{1,1}\,F_{2,1} + a^2_{1,1}\,a_{2,2}\,\overline{F}_{2,1},\\
0 &= -a^2_{1,1}\,F_{3,0} + a^3_{1,1}\,\overline{F}_{3,0} + 3\,a^2_{1,1}\,a_{2,1}\,\overline{F}_{2,1} + 3\,a_{1,1}\,\boxed{b_1}.
\end{aligned}$$

Divide by 1

- **Question:** *What do you prefer, to divide by:*

large differential polynomials?

Or to divide by:

one ?

- **Quickly:** Sketch an equivalence method which divides only by 1.

[Mathematical Advertising]

- **Lagrange:** A *function* is a power series.
- **Lie:** *Infinitezimalize!*
- **Example:**

$$0 = -a_{1,1}^2 F_{2,1} + a_{1,1}^2 a_{2,2} \overline{F}_{2,1}.$$

In the branch where these are nonzero, one can normalize:

$$\overline{F}_{2,1} := 1 =: F_{2,1}.$$

- **Hidden behind is:** A complicated differential invariant:

$$
\begin{aligned}
F_{2,1} &= F_{2,1}\big(F_{\bullet,\bullet}^{\text{initial}}\big) \\
&= \boldsymbol{I}_{2,1}.
\end{aligned}
$$

- **Function-theoretic Equivalence Method:** Forces to divide by:

$$\boldsymbol{I}_{2,1} = \text{complicated expression.}$$

- **Power Series Equivalence Method:** Forces to divide by:

$$1 = \text{value at } (x,y) = (0,0) \text{ of } \boldsymbol{I}_{2,1}.$$

- **Conclusion:** Division by 1 works!

- **Summary:**

☐ Division by I causes computational explosion.

☐ In the branch $I \neq 0$, by working at the origin, it suffices to divide by 1.

☐ But what about the branch $I \equiv 0$?

☐ Everything breaks down because only $\boldsymbol{I}(0) = 0$ is seen at the origin.

Identically Vanishing Differential Invariant $I \equiv 0$

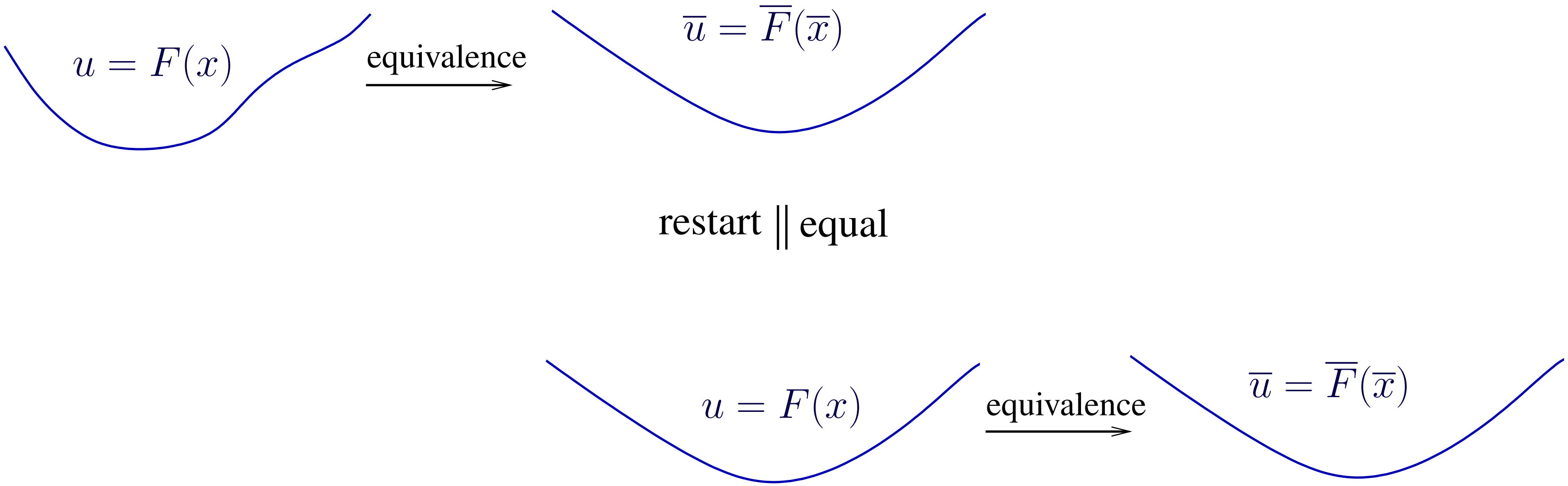
- **Remind the two examples:**

$$
\begin{aligned}
0 &= -\,a_{1,1}^2\,F_{2,1} + a_{1,1}^2\,a_{2,2}\,\overline{F}_{2,1},\\
0 &= -\,a_{1,1}^2\,F_{3,0} + a_{1,1}^3\,\overline{F}_{3,0} + 3\,a_{1,1}^2\,a_{2,1}\,\overline{F}_{2,1} + 3\,a_{1,1}\,\boxed{b_1}.
\end{aligned}
$$

- **Normalize:** Use free group parameter b_1 to make:

$$
\overline{F}_{3,0} := 0.
$$

- **Restart:**



• **Thus:** Can assume both:

$$F_{3,0} \,=\, 0 \,=\, \overline{F}_{3,0}.$$

• **Reduce stability group:**

$$G_{\mathsf{stab}}^{\nu}\Big|_{\mathsf{step\,1}} \subset G_{\mathsf{stab}}^{\nu-1},$$

and continue to normalize all freely normalizable $\overline{F}_{j,k}$:

$$G_{\mathsf{stab}}^{\nu} \subset \cdots \subset G_{\mathsf{stab}}^{\nu}\Big|_{\mathsf{step\,2}} \subset G_{\mathsf{stab}}^{\nu}\Big|_{\mathsf{step\,1}} \subset G_{\mathsf{stab}}^{\nu-1}.$$

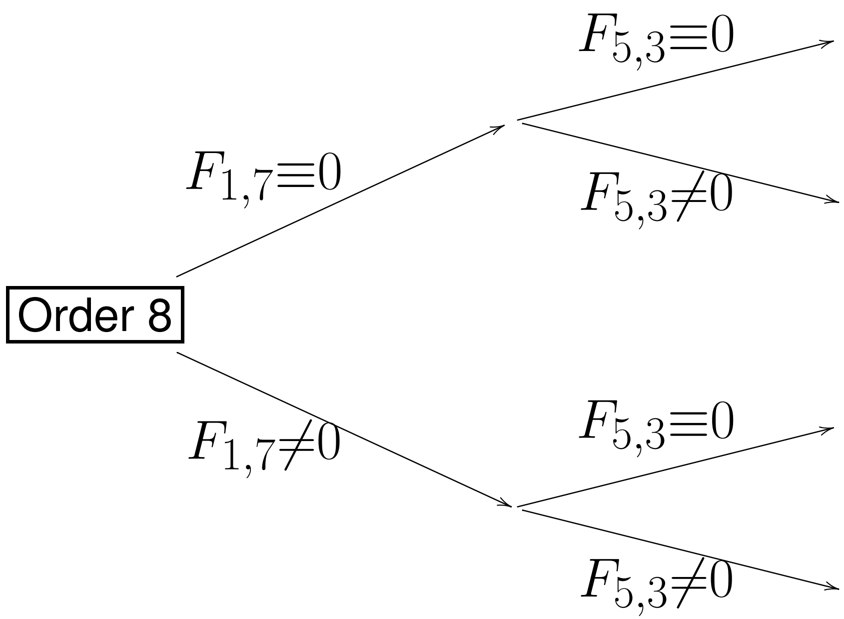
• **Remaining:**

$$F_{j,k} \,=\, \mathsf{nonzero}\,\overline{F}_{j,k},$$

are relative invariants.

• **Method at order ν:**

- ☐ Normalize all possible $F_{j+k=\nu}$ to be 0.
- ☐ Leave aside relative invariants $F_{j,k} \propto \overline{F}_{j,k}$.
- ☐ Create dichotomic branching, for instance:



Proposition. [True in examples] *At order ν, if a relative invariant:*

$$\begin{aligned} \boldsymbol{I}_{j,k} &= F_{j,k}\big(F^{\mathsf{initial}}_{\bullet,\bullet}\big) \\ &\equiv 0 \end{aligned}$$

vanishes identically, then the graphing function $F(x,y)$ satisfies a PDE:

$$F_{x^j y^k} \equiv \frac{\mathsf{something}}{\mathsf{allowed\ denominator}}.$$

Example. A surface $u = F(x,y)$ under $\mathsf{Aff}(\mathbb{R}^3)$ with rank 1 Hessian:

$$0 \equiv \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} \qquad \text{and} \qquad 0 \neq F_{xx},$$

leads to:

$$F_{yy} \equiv \frac{F_{xy}^2}{F_{xx}}.$$

$$\begin{array}{lllll} \boxed{F_{xx} \neq 0 \equiv F_{xx}F_{yy} - F_{xy}^2} \longrightarrow \boldsymbol{S} \neq 0 \longrightarrow \boxed{\boldsymbol{W} \equiv 0} & \longrightarrow & \boldsymbol{X} \equiv 0 & & \\ & \searrow & & & \\ & & \boldsymbol{X} \neq 0 & \longrightarrow & \boldsymbol{Y} \equiv 0 \\ & & & \searrow & \\ & & & & \boldsymbol{Y} \neq 0 \end{array}$$

In such a branch, another relative invariant comes:

$$\boldsymbol{W} := \boldsymbol{I}_{3,1} := \mathsf{invariantization}\big(F_{xxxy}\big),$$

equal to:

$$\boldsymbol{W} = \frac{F_{xx}^2\, \mathbf{F_{xxxy}} - F_{xx}\, F_{xy}\, F_{xxxx} + 2\, F_{xy}\, F_{xxx}^2 - 2\, F_{xx}\, F_{xxx}\, F_{xxxx}}{\mathsf{nonvanishing\ denominator}}.$$

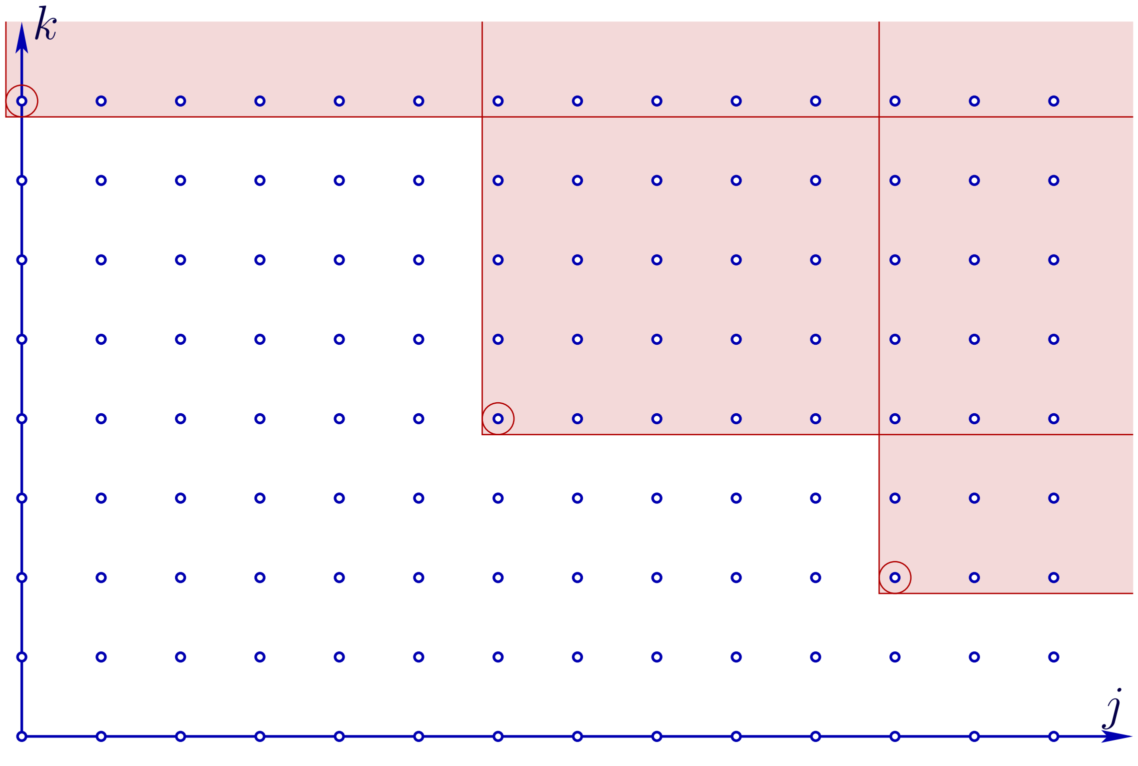
and assuming $\boldsymbol{W} \equiv 0$, we may solve:

$$F_{xxxy} \equiv \frac{F_{xy}\,F_{xxxx}}{F_{xx}} - 2\,\frac{F_{xy}\,F_{xxx}^2}{F_{xx}^2} + 2\,\frac{F_{xxx}\,F_{xxy}}{F_{xx}}.$$

- **General example:** Suppose 3 invariants already vanish:

$$F_{y^8} \equiv \frac{\text{something}}{\text{denominator}},$$
$$F_{x^6y^4} \equiv \frac{\text{something}}{\text{denominator}},$$
$$F_{x^{11}y^2} \equiv \frac{\text{something}}{\text{denominator}},$$

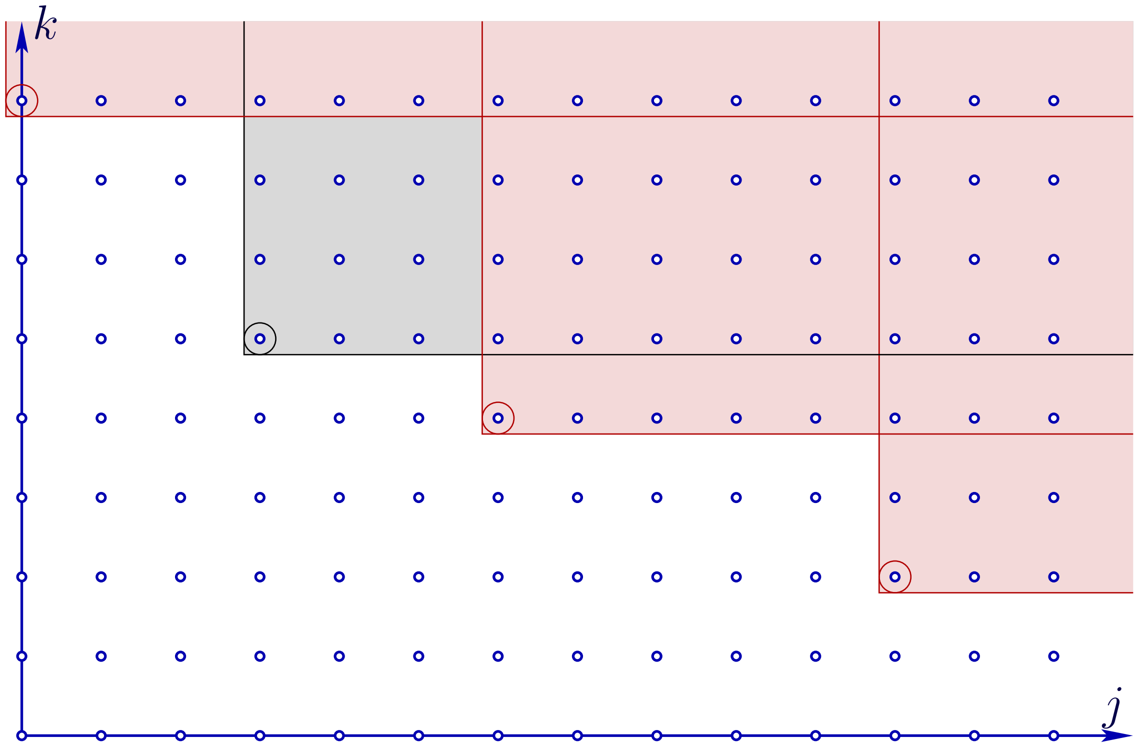
with 3 red-dependent quadrants:



Then by computing, encounter another (relative) invariant $\boldsymbol{I}_{3,5}$ and open the new sub-branch:

$$\boldsymbol{I}_{3,5}(F) \equiv 0$$

which creates a new quadrant of dependent monomials:



- **Question:** *How to compute the dependent $F_{j,k}$ in the red region?*
- **Answer:** Compute invariants explicitly, and differentiate.

Example. Consider:

$$\boldsymbol{I}_{3,1}\big(F_{\bullet,\bullet}\big) \equiv 0,$$

equivalent to:

$$F_{xxxy} \equiv \frac{F_{xy}\,F_{xxxx}}{F_{xx}} - 2\,\frac{F_{xy}\,F_{xxx}^2}{F_{xx}^2} + 2\,\frac{F_{xxx}\,F_{xxy}}{F_{xx}}.$$

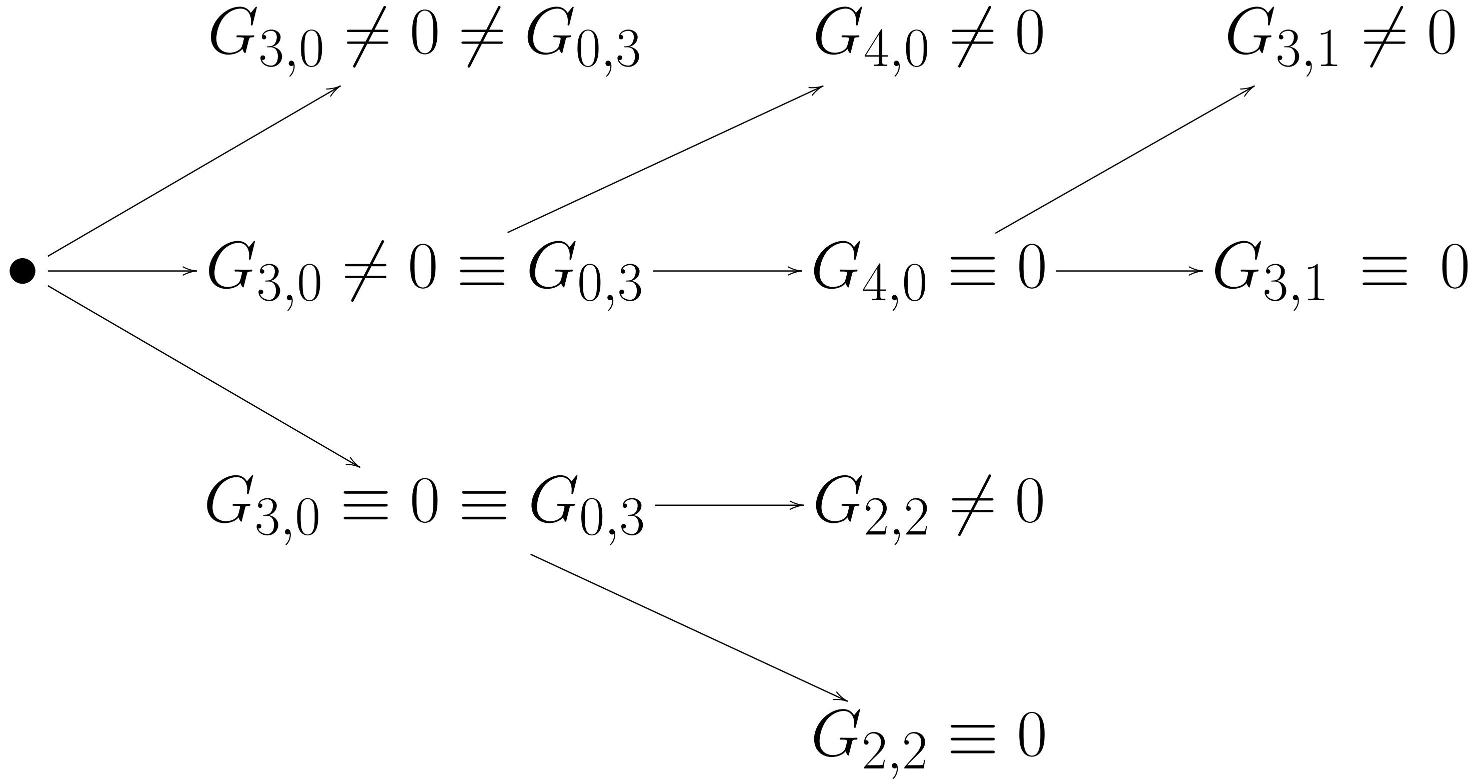
□ Determine all $F_{x^{j\geqslant 3}y^{k\geqslant 1}}$ by differentiations and replacements.

□ Evaluate at $(x, y) = (0, 0)$.

Example. For surfaces $S^2 \subset \mathbb{C}^3$ with rank 2 Hessian:

$$0 \neq \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$$

the branching tree is:


$G_{3,0} \neq 0 \neq G_{0,3}$
$G_{3,0} \neq 0 \equiv G_{0,3}$
$G_{3,0} \equiv 0 \equiv G_{0,3}$
$G_{4,0} \neq 0$
$G_{4,0} \equiv 0$
$G_{2,2} \neq 0$
$G_{2,2} \equiv 0$
$G_{3,1} \neq 0$
$G_{3,1} \equiv 0$

and one relative invariant $G_{3,0} \equiv I_{3,0}$ is:

$$
\begin{aligned}
I_{3,0} = &-\frac{1}{2\sqrt{2}(F_{2,0}+2F_{1,1}+F_{0,2})^{\frac{3}{2}}(F_{1,1}^2-F_{2,0}F_{0,2})^{\frac{3}{2}}} \\
&\times \Big\{ -4F_{0,3}F_{1,1}^3 + 6F_{0,2}F_{1,1}^2F_{1,2} + 3F_{0,2}F_{0,3}F_{1,1}F_{2,0} - 6F_{0,3}F_{1,1}^2F_{2,0} - 3F_{0,2}^2F_{1,2}F_{2,0} + 9F_{0,2}F_{1,1}F_{1,2}F_{2,0} \\
&+ 3F_{0,2}F_{0,3}F_{2,0}^2 - 3F_{0,3}F_{1,1}F_{2,0}^2 + 9F_{0,2}F_{1,2}F_{2,0}^2 + 3F_{1,1}F_{1,2}F_{2,0}^2 - F_{0,3}F_{2,0}^3 + 4F_{0,3}F_{1,1}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} \\
&- 6F_{0,2}F_{1,1}F_{1,2}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} - F_{0,2}F_{0,3}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} + 6F_{0,3}F_{1,1}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} \\
&- 9F_{0,2}F_{1,2}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} + 3F_{0,3}F_{2,0}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} + 3F_{1,2}F_{2,0}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}} \\
&- 3F_{0,2}^2F_{1,1}F_{2,1} - 9F_{0,2}^2F_{2,0}F_{2,1} - 9F_{0,2}F_{1,1}F_{2,0}F_{2,1} - 6F_{1,1}^2F_{2,0}F_{2,1} + 3F_{0,2}F_{2,0}^2F_{2,1} \\
&+ 3F_{0,2}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{2,1} - 9F_{0,2}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{2,1} - 6F_{1,1}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{2,1} + F_{0,2}^3F_{3,0} \\
&+ 3F_{0,2}^2F_{1,1}F_{3,0} + 6F_{0,2}F_{1,1}^2F_{3,0} + 4F_{1,1}^3F_{3,0} - 3F_{0,2}^2F_{2,0}F_{3,0} - 3F_{0,2}F_{1,1}F_{2,0}F_{3,0} \\
&+ 3F_{0,2}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{3,0} + 6F_{0,2}F_{1,1}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{3,0} + 4F_{1,1}^2\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{3,0} \\
&- F_{0,2}F_{2,0}\sqrt{F_{1,1}^2 - F_{0,2}F_{2,0}}F_{3,0}\Big\}
\end{aligned}
$$

For $I_{0,3}$, just switch indices.

Theorem. [Chen-M. 2020] *In the second branch* $B_{2.1}$ *where* $G_{3,0} \neq 0 \equiv G_{0,3}$ *and* $G_{4,0} \neq 0$*, the following holds.*

(1) *The graphed equation normalizes as:*

$$u = x\,y + \frac{x^3}{6} + \frac{y^3}{6} + \frac{x^{24}}{24} + I_{3,1}\frac{x^3 y}{6} + I_{2,2}\frac{x^2y^2}{4} + I_{1,3}\frac{xy^3}{6} + I_{0,4}\frac{y^4}{24} + \sum_{j+k\geqslant 5} I_{j,k}\left(F_{\bullet,\bullet}\right)\frac{x^j}{j!}\,\frac{y^k}{k!}$$

where all $I_{j,k}$ are differential invariants.

(2) *The algebra of differential invariants is generated by $I_{3,1}$, $I_{2,2}$, $I_{5,0}$ and all their invariant derivatives: $\mathscr{D}_1^{\alpha_1}\mathscr{D}_2^{\alpha_2}(\bullet)$, with $\alpha_1, \alpha_2 \in \mathbb{N}$. In particular, $I_{4,1}$ can be solved:*

$$I_{4,1} = -8\,I_{3,1}^2 + 2\,I_{5,0}\,I_{3,1} + \mathscr{D}_x\,I_{3,1} + \tfrac{7}{2}\,I_{2,2} - 2\,I_{3,1}.$$

(3) *The moduli space of all possible homogeneous models is exactly described, in the space $\mathbb{C}^3 \ni \big(I_{3,1}, I_{2,2}, I_{5,0}, I_{4,1}, I_{3,2}\big)$, by the complex-algebraic variety of dimension 1 defined by the $4+3$ equations:*

$$(E41)\;\; 0 = I_{4,1} + 8I_{3,1}^2 - \tfrac{7}{2}I_{2,2} + 2I_{3,1} - 2I_{5,0}I_{3,1},$$

$$(E42)\;\; 0 = 4I_{3,1}I_{2,2} + 2I_{3,1}^2 - 2I_{4,1}I_{3,1} + I_{3,2},$$

$$(E43)\;\; 0 = 12I_{3,1}I_{2,2} - 3I_{5,0}I_{2,2} + 4I_{2,2} + I_{3,2},$$

$$(E44)\;\; 0 = 6I_{2,2}^2 + 4I_{3,1}I_{2,2} - 3I_{4,1}I_{2,2},$$

$$(F51)\quad 0 = 24\, I_{3,1}^2\, I_{5,0} - 2\, I_{5,0}^2\, I_{3,1} - \tfrac{15}{2}\, I_{2,2}\, I_{5,0} + 7\, I_{5,0}\, I_{3,1} + \tfrac{21}{2}\, I_{2,2} - 64\, I_{3,1}^3 + 36\, I_{2,2}\, I_{3,1} - 40\, I_{3,1}^2 - 6\, I_{3,1},$$

$$(F52)\quad 0 = 30\, I_{2,2}\, I_{3,1} + 72\, I_{2,2}\, I_{3,1}^2 - 18\, I_{2,2}\, I_{5,0}\, I_{3,1} - \tfrac{63}{4}\, I_{2,2}^2 + 56\, I_{3,1}^3 - 14\, I_{3,1}^2\, I_{5,0} + 12\, I_{3,1}^2 + 64\, I_{3,1}^4 - 32\, I_{3,1}^3\, I_{5,0} + 4\, I_{5,0}^2\, I_{3,1}^2,$$

$$(F53)\quad 0 = -I_{3,1}\, (-16\, I_{3,1}^2 + 4\, I_{3,1}\, I_{5,0} + 3\, I_{2,2} - 6\, I_{3,1})\, (-32\, I_{3,1}^2 + 8\, I_{3,1}\, I_{5,0} + 6\, I_{2,2} - 13\, I_{3,1}).$$

- **Strategy for determining homogeneous models:**

☐ Write out the Lie-Fels-Olver recurrence formulas:

$$\mathscr{D}_j \boldsymbol{I}_K^\alpha \;=\; \boldsymbol{I}_{K,j}^\alpha + \sum_{1\leqslant\sigma\leqslant r} \varphi_\sigma{}_K^\alpha\big(\boldsymbol{I}^{(K)}\big)\cdot \boldsymbol{K}_j^\sigma\big(\boldsymbol{I}^{(n_{\rm G}+1)}\big).$$

☐ Assume that all $\boldsymbol{I}_{j,k} = \text{constant}$.

☐ Left-hand sides become zero:

$$0 \;=\; \boldsymbol{I}_{K,j}^\alpha + \sum_{1\leqslant\sigma\leqslant r} \varphi_\sigma{}_K^\alpha\big(\boldsymbol{I}^{(K)}\big)\cdot \boldsymbol{K}_j^\sigma\big(\boldsymbol{I}^{(n_{\rm G}+1)}\big).$$

☐ Analyze the incoming algebraic equations.

Infinite-Dimensional Lie Groups and Normal Forms

- **Pass to jet spaces:** [Lie, Olver]

$$\begin{array}{rccc} \cdots & & \cdots\cdots\cdots\cdots & \\ \text{on} & J^2M & \subset & J^2N \\ \text{on} & J^1M & \subset & J^1N \\ G \text{ acts on} & M & \subset & N, \end{array}$$

until:

$$r = \dim G \ll \dim J^\kappa M.$$

- **Question:** *What if* $\dim G = \infty$*?*

Assertion. *Thanks to Poincaré-Moser normal forms, the action boils down to a* finite-dimensional *action.*

- **Example:** 2^{nd} order ODEs: [Kamran, Shadwick, Hsu, . . .]

$$y_{xx} = Q\big(x, y, y_x\big),$$

under *fiber-preserving* point transformations:

$$x' = f(x), \qquad\qquad y' = g(x, y).$$

Theorem. [Foo-Hey-M. 2021] *Coordinates* (x, y) *exist in which:*

$$
\begin{aligned}
0 &\equiv Q(x, y, 0), \\
0 &\equiv Q_{y_x}(0, y, 0), \\
0 &\equiv Q_{y_x}(x, 0, 0), \\
0 &\equiv Q_{y_x y_x}(0, y, 0).
\end{aligned}
$$

Furthermore, the stability group of such a normal form enjoys:

$$\dim G_{\mathsf{stab}} \leqslant 3.$$

- **Classification of Homogeneous models:**

☐ [Hsu-Kamran 1989].

☐ [Foo-M.-Heyd 2021].

- **Example:** Hypersurface $M^3 \subset \mathbb{C}^2$ in coordinates:

$$z = x + i\,y, \qquad\qquad w = u + i\,v,$$

under $\mathsf{Bihol}(\mathbb{C}^2)$:

$$z' = \sum_{j+k\geqslant 1} f_{j,k}\, z^j w^k \qquad\qquad w' = \sum_{j+k\geqslant 1} g_{j,k}\, z^j w^k$$

$$= \sum_j z^j\, f_j(w), \qquad\qquad = \sum_j z^j\, g_j(w).$$

Graphed hypersurface:

$$\begin{aligned} v &= F\big(z, \overline{z}, u\big) \\ &= \sum_{j+k+l\geqslant 1} F_{j,k,l}\, z^j \overline{z}^k u^l \\ &= \sum_{j,k} z^j \overline{z}^k\, F_{j,k}(u), \end{aligned}$$

with:

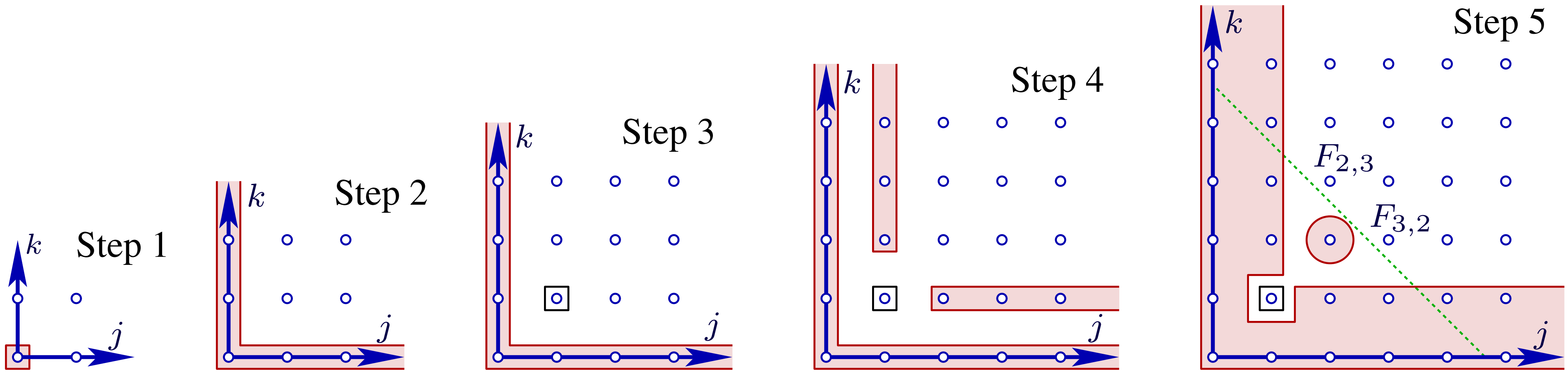
$$\overline{F_{k,j}(u)} = F_{j,k}(u).$$

- **Levi determinant:**

$$0 \neq \mathsf{Levi}(F) := \begin{vmatrix} 0 & F_z & -\frac{1}{2i} + \frac{1}{2} F_u \\ F_{\overline{z}} & F_{z\overline{z}} & \frac{1}{2} F_{\overline{z}u} \\ \frac{1}{2i} + \frac{1}{2} F_u & \frac{1}{2} F_{zu} & \frac{1}{4} F_{uu} \end{vmatrix}.$$

Theorem. [Moser 1974] *Holomorphic coordinates* (z, w) *exist in which:*

$$v = z\overline{z} + z^4\overline{z}^2 F_{4,2}(u) + z^2\overline{z}^4 F_{2,4}(u) + \sum_{\substack{j+k \geqslant 7 \\ j \geqslant 2,\, k \geqslant 2}} z^j \overline{z}^k F_{j,k}(u).$$



Successive annihilations (red dashed regions) of coefficient-functions $F_{j,k}(u)$ in the graphing function $v = \sum_{j,k} z^j \overline{z}^k F_{j,k}(u)$ thanks to Moser's normalization process, with $F_{1,1}(u) \equiv 1$, until first occurence of chains.

• **Equivalently:**

$$0 \equiv F_{j,0}(u) \equiv F_{0,k}(u) \equiv \underset{(j\neq 1)}{F_{j,1}(u)} \equiv \underset{(1\neq k)}{F_{1,k}(u)}, \qquad 1 \equiv F_{1,1}(u),$$

$$0 \equiv F_{3,2}(u) \equiv F_{2,3}(u) \equiv F_{3,3}(u).$$

Theorem. [Moser 1974] The stability subgroup $G \subset \mathsf{Bihol}(\mathbb{C}^2)$ sending:

$$v = F = z\overline{z} + z^4\overline{z}^2 F_{4,2}(u) + z^2\overline{z}^4 F_{2,4}(u) + \sum_{\substack{j+k\geqslant 7 \\ j\geqslant 2,\, k\geqslant 2}} z^j \overline{z}^k F_{j,k}(u),$$

to:

$$v' = F' = z'\overline{z}' + {z'}^4\overline{z}'^2 F'_{4,2}(u') + {z'}^2\overline{z}'^4 F'_{2,4}(u') + \sum_{\substack{j+k\geqslant 7 \\ j\geqslant 2,\, k\geqslant 2}} {z'}^j \overline{z}'^k F'_{j,k}(u'),$$

is finite-dimensional:

$$\mathsf{dim}_{\mathbb{R}}\, G_{\mathsf{stab}} \leqslant 5.$$

Cartan's Classification of $M^3 \subset \mathbb{C}^2$ Redone

• **Cartan 1932:** *Si une hypersurface admettant un groupe pseudo-conforme transitif n'est pas localement équivalente à l'hypersphère, elle est globalement équivalente à l'une des hypersurfaces suivantes ou à l'une de leurs variétés de recouvrement:*

$$1^{\circ}\ (E)\ \ \frac{y-\overline{y}}{2i} = \Big(\frac{x-\overline{x}}{2i}\Big)^m, \quad \text{avec } \frac{x-\overline{x}}{2i} > 0 \qquad (|m|\geqslant 1,\ m\neq 1,2)\ ;$$

$$2^{\circ}\ (F)\ \ \frac{y-\overline{y}}{2i} = e^{\frac{x-\overline{x}}{y-\overline{y}}}\ ;$$

$$3^{\circ}\ (H)\ \ \big(x-\overline{x}\big)^2 + \big(y-\overline{y}\big)^2 + 4\,e^{2m\,\mathrm{arctan}\frac{x-\overline{x}}{y-\overline{y}}} = 0\ ;$$

$$4^{\circ}\ (K)\ \ 1+x\overline{x}-y\overline{y} = \mu\,\big|1+x^2-y^2\big|, \quad \text{avec } \frac{x(1+\overline{y})-\overline{x}(1+y)}{i} > 0 \qquad (\mu>1)\ ;$$

$$5^{\circ}\ (K')\ \ x\overline{x}+y\overline{y}-1 = \mu\,\big|x^2+y^2-1\big|, \ \text{sauf les points réels} \quad (|\mu|<1,\ \mu\neq 0)\ ;$$

$$6^{\circ}\ (L)\ \ x_1\overline{x}_1+x_2\overline{x}_2+x_3\overline{x}_3 = \mu\,\big|x_1\overline{x}_1+x_2\overline{x}_2+x_3\overline{x}_3\big| \qquad (\mu>1).$$

• **Proof:**

☐ Tresse 1896, Segre 1930, Bianchi classification of 3D Lie algebras.

☐ Realizations of Lie algebras of holomorphic vector fields.

☐ Confirmation by Cartan's Equivalence Method.

- **Alternative proof:** Start from Moser's normal form, reduce G^5.

• **Result of computations:** For any non-spherical homogeneous $M^3 \subset \mathbb{C}^2$, the only non-linearly solvable Taylor coefficients are:

$$\begin{aligned} a + i\,b &:= F_{5,2,0}, \\ c &:= F_{4,4,0}, \\ d &:= \mathsf{Im}\, F_{4,2,1}. \end{aligned}$$

Theorem. *In the branch* $F_{4,2,0} = 1$, *homogeneous* $M^3 \subset \mathbb{C}^2$ *are simply transitive and* $1\!:\!1$ *parametrized* [Up to discrete equivalence] *by the variety:*

$$\begin{aligned} 0 &= b\left(576\,c + 64\,d - 25\,a^2 - 25\,b^2\right), \\ 0 &= a\left(576\,c - 64\,d + 25\,a^2 + 25\,b^2\right), \\ 0 &= a\,b\,d, \\ 0 &= 384 + 125\,b^2 d - 125\,a^2 d - 1152\,cd + 3000\,a^2 c + 3000\,b^2 c. \end{aligned}$$

Moreover, 3 holomorphic vector field generators e_1, e_2, e_3 *with:*

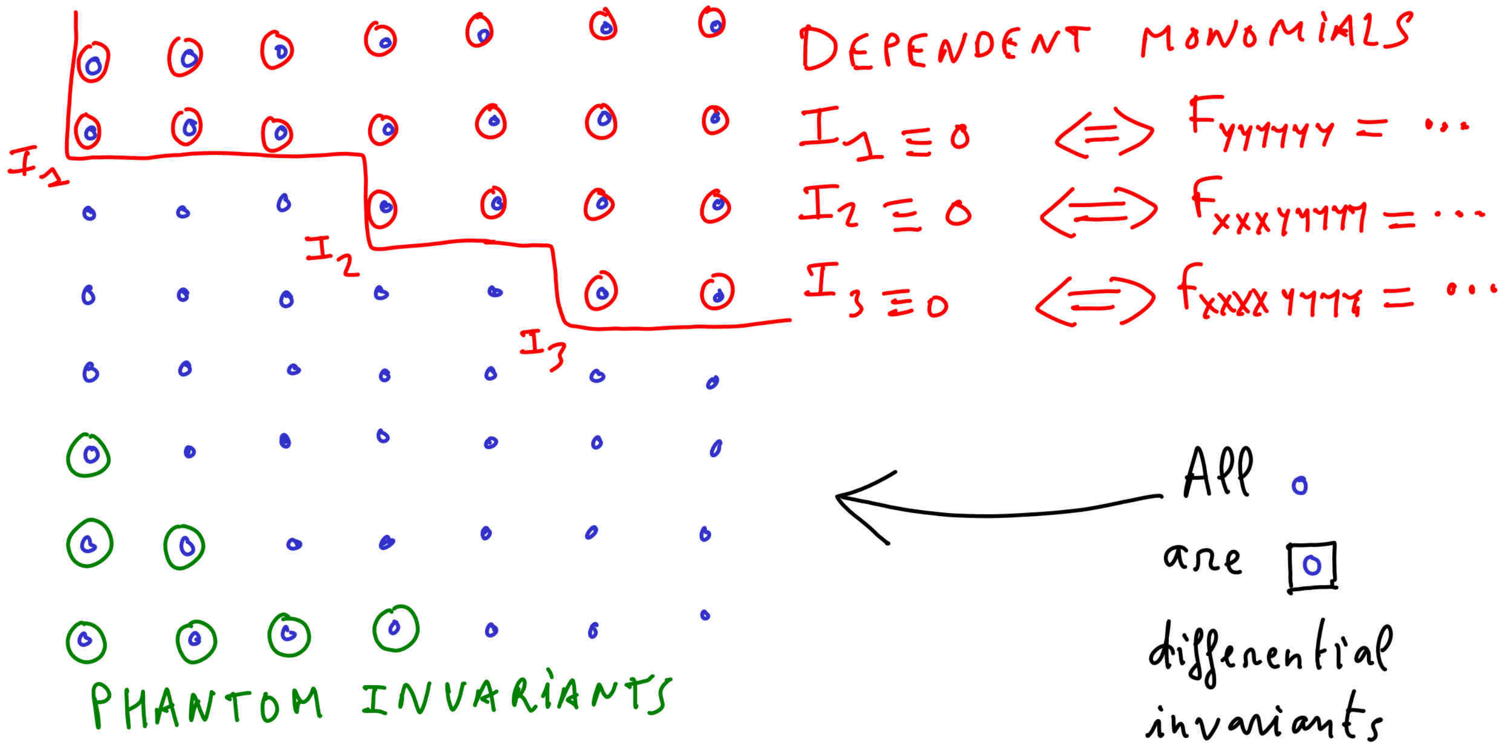
$$e_1\big|_0 = \partial_z, \qquad e_2\big|_0 = i\,\partial_z, \qquad e_3\big|_0 = \partial_w,$$

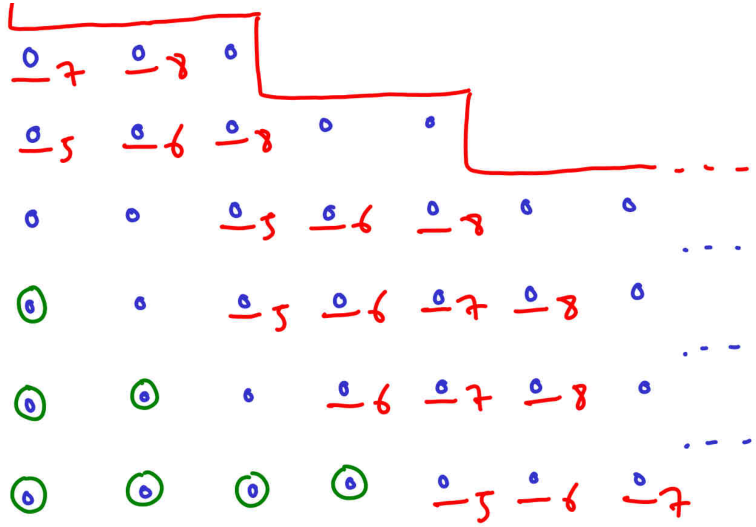
have structure:

$$\begin{aligned} [e_1, e_2] &= \left(-\tfrac{5}{4}\,b\right) e_1 + \left(-\tfrac{5}{4}\,a\right) e_2 + 4\,e_3, \\ [e_1, e_3] &= \left(-\tfrac{2}{3}\,d - 6\,c + \tfrac{25}{96}\,a^2 + \tfrac{25}{96}\,b^2\right) e_2 + \left(\tfrac{5}{2}\,a\right) e_3, \\ [e_2, e_3] &= \left(\tfrac{2}{3}\,d - 6\,c - \tfrac{25}{96}\,a^2 - \tfrac{25}{96}\,b^2\right) e_1 + \left(-\tfrac{5}{2}\,b\right) e_3. \end{aligned}$$

Sketch of General Power Series Method of Equivalence

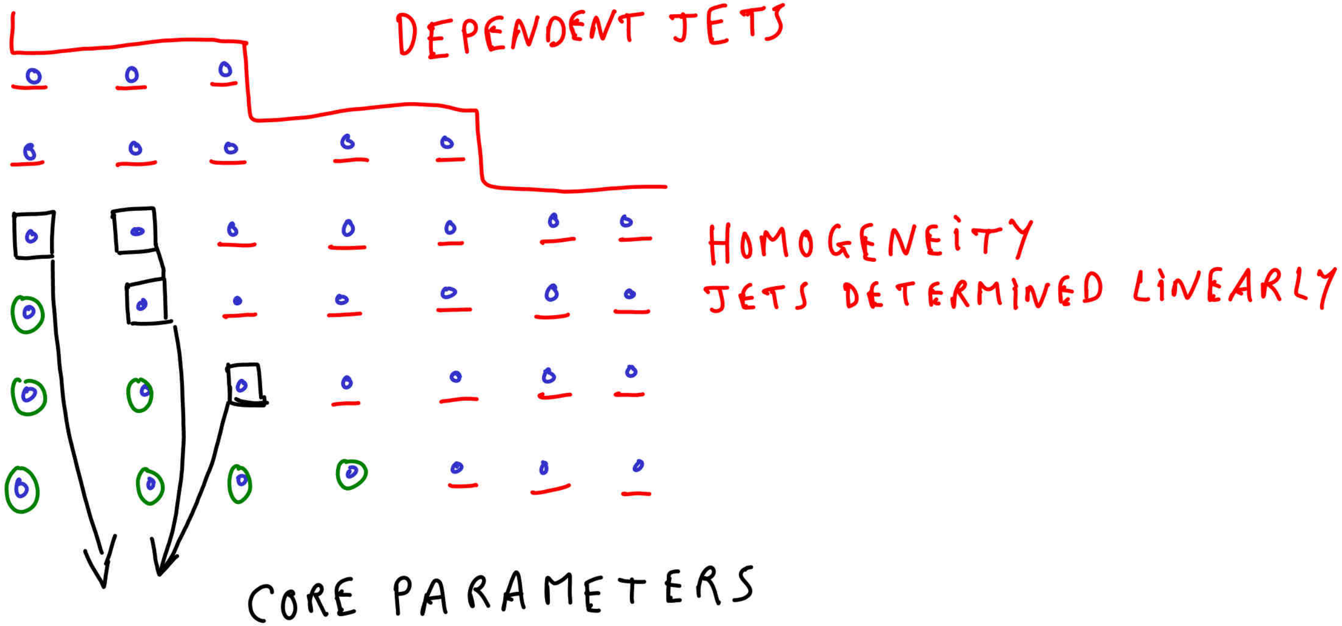
Homogeneous structures with finite-dimensional Lie group



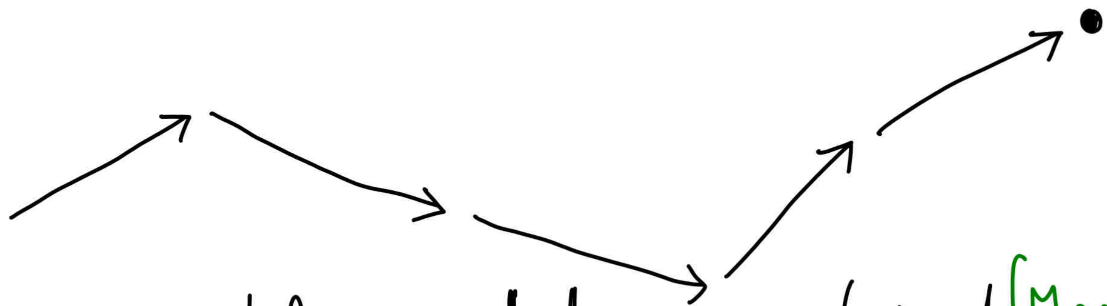


Transitivity: All but finitely many F_{jik} are uniquely determined

It remains:



Thm: [Informal, but very general] In any branch end



Homogeneous models are 1:1 parametrized [MODULI SPACE] by an invariant algebraic variety with explicit

$$\text{Equations}(\text{Cone Parameters}) = 0$$

Homogeneous $\mathfrak{C}_{2,1}$ Hypersurfaces $M^5 \subset \mathbb{C}^3$

In coordinates $\mathbb{C}^3 \ni \big(z, \zeta, w = u + \sqrt{-1}\, v\big)$, the graphed representation of the flat model is [Gaussier-M. 2003]:

$$u \;=\; \frac{z\overline{z} + \frac{1}{2}\,\overline{z}^2\zeta + \frac{1}{2}\,z^2\overline{\zeta}}{1 - \zeta\overline{\zeta}} \;=:\; \boldsymbol{m}\big(z, \zeta, \overline{z}, \overline{\zeta}\big).$$

The 5-dimensional Lie group of its automorphisms fixing the origin writes:

$$z' \;:=\; \lambda\,\frac{z + i\,\alpha\,z^2 + \big(i\,\alpha\,\zeta - i\,\overline{\alpha}\big)\,w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

$$\zeta' \;:=\; \frac{\lambda}{\overline{\lambda}}\,\frac{\zeta + 2i\,\overline{\alpha}\,z - \big(\alpha\overline{\alpha} + i\,\rho\big)\,z^2 + \big(\overline{\alpha}^2 - i\,\rho\,\zeta - \alpha\overline{\alpha}\,\zeta\big)\,w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

$$w' \;:=\; \lambda\overline{\lambda}\,\frac{w}{1 + 2i\,\alpha\,z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $\rho \in \mathbb{R}$ are free.

A general $\mathfrak{C}_{2,1}$ hypersurface $M^5 \subset \mathbb{C}^3$ with $0 \in M$ writes as a perturbation of this model:

$$u \;=\; F\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big) \;=\; \boldsymbol{m}(z, \zeta, \overline{z}, \overline{\zeta}) + G\big(z, \zeta, \overline{z}, \overline{\zeta}, v\big),$$

where:

$$F = \sum_{h,i,j,k,l} z^h \zeta^i \overline{z}^j \overline{\zeta}^k v^l \, F_{h,i,j,k,l} = \sum_{h,i,j,k} z^h \zeta^i \overline{z}^j \overline{\zeta}^k F_{h,i,j,k}(v),$$

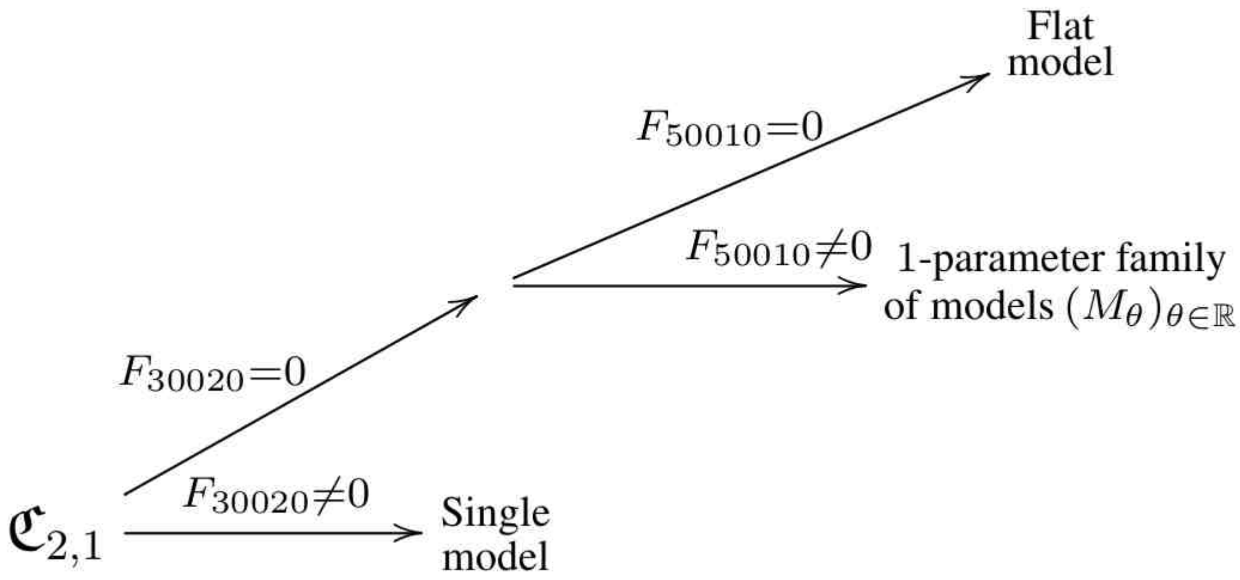
with $\overline{F_{h,i,j,k,l}} = F_{j,k,h,i,l}$, with $0 = F_{0,0,0,0,0}$, and the same for G.

Theorem. [Kolar-Kossovskiy 2019, Foo-M.-Ta 2020] *A convergent Poincaré-Moser normal form is:*

$$\begin{aligned} 0 &\equiv F_{h,i,0,0}(v), & 0 &\equiv F_{3,0,0,1}(v), \\ 0 &\equiv F_{h,i,1,0}(v), & 0 &\equiv F_{4,0,0,1}(v) \equiv F_{3,0,1,1}(v), \\ 0 &\equiv F_{h,i,2,0}(v), & 0 &\equiv F_{4,0,1,1}(v) \equiv F_{3,0,3,0}(v), \end{aligned}$$

with the exceptions $1 \equiv F_{1,0,1,0}(v)$ *and* $\frac{1}{2} \equiv F_{2,0,0,1}(v)$.

• **Question:** What about homogeneous models? [Fels-Kaup 2008: Lie-theoretic methods] Applying the power series method of equivalence, we find the following tree:



In the branch $F_{3,0,0,2,0} = 0$ and $F_{5,0,0,1,0} = 1$, three supplementary (real) normalizations hold:

$$
\begin{aligned}
F_{6,0,0,1,0} &:= 0, & \text{so } \alpha &:= 0, \\
\operatorname{Im} F_{4,0,3,0,0} &:= 0, & \text{so } \rho &:= 0,
\end{aligned}
$$

so that the isotropy is reduced to be zero-dimensional. Notably, a constant value for $F_{3,0,2,1,0} = -15$ is also implied.

Furthermore, abbreviating:

$$
\theta := \operatorname{Re} F_{4,0,3,0,0},
$$

which is a *free* absolute invariant, all coefficients $F_{h,i,j,k,l} \in \mathbb{C}$ are uniquely determined in terms of $\theta \in \mathbb{R}$, with:

$$F = F^2 + F^3 + F^4 + F^5 + F^6 + F^7 + F^8 + F^9 + F^{10} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(11),$$

where:

$$\begin{aligned} F^2 &= z\overline{z}, \\ F^3 &= \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta}, \\ F^4 &= z\overline{z}\zeta\overline{\zeta}, \\ F^5 &= \tfrac{1}{2}\,z^2\zeta\overline{\zeta}^2 + \tfrac{1}{2}\,\zeta^2\overline{z}^2\overline{\zeta}, \\ F^6 &= -\,15\,z^3\overline{z}^2\overline{\zeta} + z\zeta^2\overline{z}\overline{\zeta}^2 + \zeta\overline{z}^5 + z^5\overline{\zeta} - 15\,z^2\zeta\overline{z}^3, \end{aligned}$$

$$\begin{aligned} F^7 = {} & \tfrac{3}{2}\,z^5\overline{\zeta}^2 + \theta\,z^4\overline{z}^3 - 45\,z^2\zeta\overline{z}^3\overline{\zeta} + \tfrac{1}{2}\,z^2\zeta^2\overline{\zeta}^3 + \theta\,z^3\overline{z}^4 - \tfrac{15}{2}\,z^4\overline{z}\overline{\zeta}^2 + 5\,z\zeta\overline{z}^4\overline{\zeta} \\ & -\,10\,z^3\overline{z}^2\overline{\zeta}^2 + \tfrac{3}{2}\,\zeta^2\overline{z}^5 - 45\,z^3\zeta\overline{z}^2\overline{\zeta} - \tfrac{15}{2}\,z\zeta^2\overline{z}^4 - 10\,z^2\zeta^2\overline{z}^3 + \tfrac{1}{2}\,\zeta^3\overline{z}^2\overline{\zeta}^2 \\ & +\,5\,z^4\zeta\overline{z}\overline{\zeta}, \end{aligned}$$

$$
\begin{aligned}
F^8 \;=\; & z\zeta^3\overline{z}\,\overline{\zeta}^3 - 20\, z^3\zeta^2\overline{z}^2\overline{\zeta} - 75\, z^3\zeta\overline{z}^2\overline{\zeta}^2 - 75\, z^2\zeta^2\overline{z}^3\overline{\zeta} - \tfrac{75}{2}\, z\zeta^2\overline{z}^4\overline{\zeta} \\
& - \tfrac{75}{2}\, z^4\zeta\overline{z}\,\overline{\zeta}^2 - 20\, z^2\zeta\overline{z}^3\overline{\zeta}^2 - \tfrac{1}{5}\,\theta\, z^6\overline{z}\,\overline{\zeta} + 2\,\theta\, z^4\zeta\overline{z}^3 + \tfrac{12}{5}\,\theta\, z^5\overline{z}^2\overline{\zeta} \\
& + 3\,\theta\, z^4\overline{z}^3\overline{\zeta} + 2\,\theta\, z^3\overline{z}^4\overline{\zeta} + 3\,\theta\, z^3\zeta\overline{z}^4 + \tfrac{12}{5}\,\theta\, z^2\zeta\overline{z}^5 - \tfrac{1}{5}\,\theta\, z\zeta\overline{z}^6 - 5\, z^4\overline{z}\,\overline{\zeta}^3 \\
& - 5\, z\zeta^3\overline{z}^4 + 5\,\zeta^2\overline{z}^5\overline{\zeta} + 5\, z^5\zeta\overline{\zeta}^2 - \tfrac{1}{35}\,\theta\,\zeta\overline{z}^7 - \tfrac{1}{35}\,\theta\, z^7\overline{\zeta} - \tfrac{3}{2}\, z^5\overline{\zeta}^3 - 130\, z^5\overline{z}^3 \\
& - \tfrac{325}{6}\, z^4\overline{z}^4 - 130\, z^3\overline{z}^5 - \tfrac{3}{2}\,\zeta^3\overline{z}^5,
\end{aligned}
$$

$$
\begin{aligned}
F^9 \;=\; & \theta\, z^3\overline{z}^4\overline{\zeta}^2 + \theta\, z^4\zeta^2\overline{z}^3 - 165\, z^2\zeta^2\overline{z}^3\overline{\zeta}^2 - 40\, z^3\zeta\overline{z}^2\overline{\zeta}^3 - 165\, z^3\zeta^2\overline{z}^2\overline{\zeta}^2 \\
& - 5\, z^4\zeta^2\overline{z}\,\overline{\zeta}^2 - 5\, z\zeta^2\overline{z}^4\overline{\zeta}^2 - \tfrac{75}{2}\, z^4\zeta\overline{z}\,\overline{\zeta}^3 - \tfrac{75}{2}\, z\zeta^3\overline{z}^4\overline{\zeta} - 40\, z^2\zeta^3\overline{z}^3\overline{\zeta} \\
& + \tfrac{18}{5}\,\theta\, z^5\overline{z}^2\overline{\zeta}^2 + 3\,\theta\, z^4\overline{z}^3\overline{\zeta}^2 + 2\,\theta\, z^6\overline{z}\,\overline{\zeta}^2 + \tfrac{18}{5}\,\theta\, z^2\zeta^2\overline{\zeta}^5 + 2\,\theta\, z\zeta^2\overline{z}^6 \\
& + 3\,\theta\, z^3\zeta^2\overline{\zeta}^4 - 5\sqrt{-1}\, z^6\overline{\zeta} v + 5\sqrt{-1}\,\zeta\overline{z}^6 v + \tfrac{24}{5}\,\theta\, z^2\zeta\overline{z}^5\overline{\zeta} + 12\,\theta\, z^4\zeta\overline{z}^3\overline{\zeta} \\
& + \tfrac{24}{5}\,\theta\, z^5\zeta\overline{z}^2\overline{\zeta} + 12\,\theta\, z^3\zeta\overline{z}^4\overline{\zeta} - \tfrac{1}{5}\,\theta\, z\zeta\overline{z}^6\overline{\zeta} - \tfrac{1}{5}\,\theta\, z^6\zeta\overline{z}\,\overline{\zeta} - 100\sqrt{-1}\, z^3\overline{z}^3\overline{\zeta} v
\end{aligned}
$$

$$
\begin{aligned}
& - 30\sqrt{-1}\, z^5\overline{z}\,\overline{\zeta} v - 75\sqrt{-1}\, z^4\overline{z}^2\overline{\zeta} v - \tfrac{335}{3}\, z^5\overline{z}^3\overline{\zeta} + 5\, z\zeta\overline{z}^7 - \tfrac{6}{35}\,\theta\,\zeta^2\overline{\zeta}^7 - \tfrac{335}{3}\, z^3\zeta\overline{z}^5 \\
& + \tfrac{1}{2}\, z^2\zeta^3\overline{\zeta}^4 + \tfrac{475}{2}\, z^4\overline{z}^4\overline{\zeta} - 455\, z^2\zeta\overline{z}^6 - \tfrac{6}{25}\,\theta^2\, z^5\overline{z}^4 - 190\, z^5\zeta\overline{z}^3 - \tfrac{6}{35}\,\theta\, z^7\overline{\zeta}^2 \\
& + \tfrac{1}{2}\,\zeta^4\overline{z}^2\overline{\zeta}^3 - 455\, z^6\overline{z}^2\overline{\zeta} + 5\, z^7\overline{z}\,\overline{\zeta} - \tfrac{6}{25}\,\theta^2\, z^4\overline{z}^5 - 190\, z^3\overline{z}^5\overline{\zeta} - \tfrac{4}{25}\,\theta^2\, z^6\overline{z}^3 \\
& - \tfrac{4}{25}\,\theta^2\, z^3\overline{z}^6 - \tfrac{15}{2}\, z^5\zeta\overline{\zeta}^3 - \tfrac{15}{2}\,\zeta^3\overline{z}^5\overline{\zeta} - z^5\overline{\zeta}^4 - \zeta^4\overline{z}^5 + \tfrac{25}{4}\,\zeta\overline{z}^8 + \tfrac{25}{4}\, z^8\overline{\zeta} \\
& + 30\sqrt{-1}\, z\zeta\overline{z}^5 v + 100\sqrt{-1}\, z^3\zeta\overline{z}^3 v + 75\sqrt{-1}\, z^2\zeta\overline{z}^4 v + \tfrac{475}{2}\, z^4\zeta\overline{z}^4,
\end{aligned}
$$

$$
\begin{aligned}
F^{10} \;=\;& z\zeta^4\overline{z}\,\overline{\zeta}^4 + \theta\, z^3\zeta^3\overline{z}^4 + \theta\, z^4\overline{z}^3\overline{\zeta}^3 - 210\, z^3\zeta^2\overline{z}^2\overline{\zeta}^3 - 20\, z^4\zeta\overline{z}\,\overline{\zeta}^4 + 105\, z^5\zeta\overline{z}^3\overline{\zeta} \\
& - 210\, z^2\zeta^3\overline{z}^3\overline{\zeta}^2 - \tfrac{1525}{2}\, z^2\zeta\overline{z}^6\overline{\zeta} - \tfrac{1525}{2}\, z^6\zeta\overline{z}^2\overline{\zeta} - \tfrac{255}{2}\, z^4\zeta^2\overline{z}\,\overline{\zeta}^3 \\
& + 1725\, z^4\zeta\overline{z}^4\overline{\zeta} - 20\, z\zeta^4\overline{z}^4\overline{\zeta} - 70\, z^3\overline{\zeta}^3\overline{z}^2\overline{\zeta}^2 - \tfrac{255}{2}\, z\zeta^3\overline{z}^4\overline{\zeta}^2 - 70\, z^2\zeta^2\overline{z}^3\overline{\zeta}^3 \\
& + 105\, z^3\zeta\overline{z}^5\overline{\zeta} + 50\, z^7\zeta\overline{z}\,\overline{\zeta} + 50\, z\zeta\overline{z}^7\overline{\zeta} + \tfrac{3}{175}\,\theta^2\, z\zeta\overline{z}^8 + \tfrac{3}{175}\,\theta^2\, z^8\overline{z}\,\overline{\zeta}
\end{aligned}
$$

$$
\begin{aligned}
& - \tfrac{4}{5}\,\theta^2\, z^3\zeta\overline{z}^6 + 2\,\theta\, z\zeta^3\overline{z}^6 + 2\,\theta\, z^6\overline{z}\,\overline{\zeta}^3 + \tfrac{12}{5}\,\theta\, z^5\overline{z}^2\overline{\zeta}^3 - \tfrac{18}{25}\,\theta^2\, z^5\zeta\overline{z}^4 \\
& - \tfrac{8}{25}\,\theta^2\, z^6\zeta\overline{z}^3 - \tfrac{18}{25}\,\theta^2\, z^4\overline{z}^5\overline{\zeta} - \tfrac{8}{25}\,\theta^2\, z^3\overline{z}^6\overline{\zeta} - \tfrac{72}{175}\,\theta^2\, z^2\zeta\overline{\zeta}^7 - \tfrac{4}{5}\,\theta^2\, z^6\overline{z}^3\overline{\zeta} \\
& - \tfrac{24}{25}\,\theta^2\, z^5\overline{z}^4\overline{\zeta} - \tfrac{24}{25}\,\theta^2\, z^4\zeta\overline{z}^5 + \tfrac{12}{5}\,\theta\, z^2\zeta^3\overline{z}^5 - \tfrac{72}{175}\,\theta^2\, z^7\overline{z}^2\overline{\zeta} - \tfrac{1}{5}\,\theta\, \zeta^2\overline{z}^7\overline{\zeta} \\
& - \tfrac{1}{5}\,\theta\, z^7\zeta\overline{\zeta}^2 - 20\sqrt{-1}\, z^6\overline{\zeta}v + 20\sqrt{-1}\,\zeta^2\overline{z}^6 v + 24\,\theta\, z^3\zeta^2\overline{z}^4\overline{\zeta} + \tfrac{18}{5}\,\theta\, z^6\zeta\overline{z}\,\overline{\zeta}^2
\end{aligned}
$$

$$
\begin{aligned}
& + \tfrac{12}{5}\,\theta\, z^2\zeta\overline{z}^5\overline{\zeta}^2 + \tfrac{18}{5}\,\theta\, z\zeta^2\overline{z}^6\overline{\zeta} + 15\,\theta\, z^4\zeta^2\overline{z}^3\overline{\zeta} + 15\,\theta\, z^3\zeta\overline{z}^4\overline{\zeta}^2 \\
& + \tfrac{12}{5}\,\theta\, z^5\zeta^2\overline{z}^2\overline{\zeta} + 18\,\theta\, z^2\zeta^2\overline{z}^5\overline{\zeta} + 18\,\theta\, z^5\zeta\overline{z}^2\overline{\zeta}^2 + 24\,\theta\, z^4\,\zeta\overline{z}^3\overline{\zeta}^2 \\
& - 150\sqrt{-1}\, z^4\overline{z}^2\overline{\zeta}^2 v - 16\sqrt{-1}\,\theta\, z^3\overline{z}^5 v - 100\sqrt{-1}\, z^3\overline{z}^3\overline{\zeta}^2 v - 90\sqrt{-1}\, z^5\overline{z}\,\overline{\zeta}^2 v + F_{5,0,5,0,0}\, z^5\overline{z}^5 \\
& - \tfrac{15}{2}\, z^5\zeta\overline{\zeta}^4 + 5\,\zeta^3\overline{z}^5\overline{\zeta}^2 + 5\, z^5\zeta^2\overline{\zeta}^3 - 150\, z^5\zeta^2\overline{\zeta}^3 + \tfrac{975}{2}\, z^4\zeta^2\overline{z}^4 + 570\, z^3\zeta^2\overline{z}^5
\end{aligned}
$$

$$
\begin{aligned}
&- 325\, z^2\zeta^2\overline{z}^6 - 435\, z\zeta^2\overline{z}^7 - 435\, z^7\overline{z}\overline{\zeta}^2 - 325\, z^6\overline{z}^2\overline{\zeta}^2 - \tfrac{15}{2}\, \zeta^4\overline{z}^5\overline{\zeta} \\
&+ 570\, z^5\overline{z}^3\overline{\zeta}^2 + \tfrac{975}{2}\, z^4\overline{z}^4\overline{\zeta}^2 - 150\, z^3\overline{z}^5\overline{\zeta}^2 + 9\,\theta\, z^4\overline{z}^6 + \tfrac{164}{7}\,\theta\, z^3\overline{z}^7 + \tfrac{4}{7}\,\theta\,\zeta^3\overline{z}^7 \\
&+ \tfrac{1}{525}\,\theta^2\,\zeta\overline{z}^9 + \tfrac{4}{7}\,\theta\, z^7\overline{\zeta}^3 + \tfrac{1}{525}\,\theta^2\, z^9\overline{\zeta} + 9\,\theta\, z^6\overline{z}^4 + \tfrac{164}{7}\,\theta\, z^7\overline{z}^3 + \tfrac{95}{4}\,\zeta^2\overline{z}^8 \\
&+ \tfrac{95}{4}\, z^8\overline{\zeta}^2 + 90\,\sqrt{-1}\, z\zeta^2\overline{z}^5 v + 100\,\sqrt{-1}\, z^3\zeta^2\overline{z}^3 v + 150\,\sqrt{-1}\, z^2\zeta^2\overline{z}^4 v + 16\,\sqrt{-1}\,\theta\, z^5\overline{z}^3 v \\
&- 30\,\sqrt{-1}\, z^5\zeta\overline{z}\overline{\zeta} v - 150\,\sqrt{-1}\, z^2\zeta\overline{z}^4\overline{\zeta} v + 150\,\sqrt{-1}\, z^4\zeta\overline{z}^2\overline{\zeta} v + 30\,\sqrt{-1}\, z\zeta\overline{z}^5\overline{\zeta} v.
\end{aligned}
$$

The general infinitesimal CR automorphism, depending on 5 real constants $a, b, c, d, e \in \mathbb{R}$, is $L = A\,\partial_z + B\,\partial_\zeta + C\,\partial_w$, where:

$$
\begin{aligned}
A^0 &= a + \sqrt{-1}\, b, \\
A^1 &= \big(- c + \sqrt{-1}\, d\big)\, z + \big(- a + \sqrt{-1}\, b\big)\, \zeta, \\
A^2 &= \big(\tfrac{2}{5}\,\theta\, a + 5\,\sqrt{-1}\, e\big)\, z^2 + \big(- \tfrac{2}{5}\,\theta\, a + 5\,\sqrt{-1}\, e\big)\, w + \big(- c + \sqrt{-1}\, d\big)\, z\zeta, \\
A^3 &= \big(- 10\, a - 10\,\sqrt{-1}\, b\big)\, z^3 + \big(10\,\sqrt{-1}\, b + 30\, a\big)\, zw + \big(- \tfrac{2}{5}\,\theta\, a - 5\,\sqrt{-1}\, e\big)\, \zeta w, \\
A^4 &= \big(- 10\, c - 5\,\sqrt{-1}\, d\big)\, w^2 + \big(10\, a + 10\,\sqrt{-1}\, b\big)\, z\zeta w + \big(- 20\, c + 10\,\sqrt{-1}\, d\big)\, z^2 w, \\
A^5 &= \big(- \tfrac{4}{5}\,\theta\, a - 10\,\sqrt{-1}\, e\big)\, z^5 + \big(- 5\, c + 5\,\sqrt{-1}\, d\big)\, z^4\zeta + \big(4\,\theta\, a - 50\,\sqrt{-1}\, e\big)\, z^3 w \\
&\quad + \big(75\,\sqrt{-1}\, e - 6\,\theta\, a\big)\, zw^2 + \big(10\, c - 5\,\sqrt{-1}\, d\big)\, \zeta w^2, \\
A^6 &= \big(- 20\, a - 20\,\sqrt{-1}\, b - \tfrac{1}{5}\,\sqrt{-1}\,\theta\, d + \tfrac{1}{5}\,\theta\, c\big)\, z^6 + \big(- \tfrac{200}{3}\, a + \tfrac{100}{3}\,\sqrt{-1}\, b\big)\, w^3 \\
&\quad + \big(- 2\,\theta\, a + 25\,\sqrt{-1}\, e\big)\, z\zeta w^2 + \big(200\, a + 100\,\sqrt{-1}\, b\big)\, z^2 w^2,
\end{aligned}
$$

$$
\begin{aligned}
A^7 \;=\; & \big(10\sqrt{-1}\,d - 10\,c + \tfrac{2}{7}\sqrt{-1}\,\theta\,e + \tfrac{4}{175}\,\theta^2\,a\big)\,z^7 + \big(100\,c + 50\sqrt{-1}\,d\big)\,z^3w^2 \\
& + \big(-\tfrac{200}{3}\,a - \tfrac{100}{3}\sqrt{-1}\,b\big)\,\zeta w^3 + \big(-\tfrac{1}{5}\sqrt{-1}\,\theta\,d + \tfrac{1}{5}\,\theta\,c\big)\,z^6\zeta + \big(50\,a + 50\sqrt{-1}\,b\big)\,z^4\zeta w \\
& + \big(-50\sqrt{-1}\,d + 70\,c\big)\,z^5w + \big(-100\,c - 50\sqrt{-1}\,d + 20\sqrt{-1}\,\theta\,e\big)\,zw^3,
\end{aligned}
$$

$$
\begin{aligned}
A^8 \;=\; & A_{0,0,4}\,w^4 + \big(-\tfrac{3}{175}\,\theta^2\,c + \tfrac{4}{7}\sqrt{-1}\,\theta\,b - \tfrac{31}{7}\,\theta\,a + \tfrac{3}{175}\sqrt{-1}\,\theta^2\,d - \tfrac{125}{2}\sqrt{-1}\,e\big)\,z^8 \\
& + \big(8\,\overline{A_{0,0,4}} + \tfrac{1}{2}\,\overline{B_{1,0,3}}\big)\,z^2w^3 + \big(-30\sqrt{-1}\,d + 30\,c\big)\,z^5\zeta w + \big(-\tfrac{100}{3}\,c - \tfrac{50}{3}\sqrt{-1}\,d\big)\,z\zeta w^3 \\
& + \big(-50\,c + 50\sqrt{-1}\,d\big)\,z^7\zeta + \big(-2\sqrt{-1}\,\theta\,b + 10\,\theta\,a - 50\sqrt{-1}\,e\big)\,z^6w,
\end{aligned}
$$

where:

$$
\begin{aligned}
B^0 \;=\; & c + \sqrt{-1}\,d, \\
B^1 \;=\; & \big(\tfrac{4}{5}\,\theta\,a - 10\sqrt{-1}\,e\big)\,z + \big(2\sqrt{-1}\,d\big)\,\zeta, \\
B^2 \;=\; & \big(-40\sqrt{-1}\,b - 60\,a\big)\,z^2 + \big(-c + \sqrt{-1}\,d\big)\,\zeta^2 + \big(10\,a - 10\sqrt{-1}\,b\big)\,w + \big(\tfrac{4}{5}\,\theta\,a + 10\sqrt{-1}\,e\big)\,z\zeta, \\
B^3 \;=\; & \big(-30\sqrt{-1}\,d + 30\,c\big)\,z^3 + \big(40\,c + 20\sqrt{-1}\,d\big)\,zw + \big(60\,a\big)\,\zeta w + \big(40\,a - 140\sqrt{-1}\,b\big)\,z^2\zeta,
\end{aligned}
$$

$$
\begin{aligned}
B^4 \;=\; & \big(-14\,\theta\,a + 100\sqrt{-1}\,e + 6\sqrt{-1}\,\theta\,b\big)\,z^4 + \big(2\,\theta\,a + 25\sqrt{-1}\,e\big)\,w^2 + \big(-40\,c + 20\sqrt{-1}\,d\big)\,z\zeta w \\
& + \big(24\,\theta\,a - 300\sqrt{-1}\,e\big)\,z^2w + \big(10\,a + 10\sqrt{-1}\,b\big)\,\zeta^2w + \big(-90\sqrt{-1}\,d + 90\,c\big)\,z^3\zeta \\
& + \big(-60\sqrt{-1}\,b + 60\,a\big)\,z^2\zeta^2,
\end{aligned}
$$

$$
\begin{aligned}
B^5 \;=\; & \big(-860\sqrt{-1}\,b + 900\,a + \tfrac{24}{5}\sqrt{-1}\,\theta\,d - \tfrac{24}{5}\,\theta\,c\big)\,z^5 + \big(40\,c - 40\sqrt{-1}\,d\big)\,z^3\zeta^2 \\
& + \big(12\sqrt{-1}\,\theta\,b - 20\,\theta\,a - 100\sqrt{-1}\,e\big)\,z^4\zeta + \big(-300\,a - 300\sqrt{-1}\,b\big)\,z^3w \\
& + \big(400\,a - 200\sqrt{-1}\,b\big)\,zw^2 + \big(150\sqrt{-1}\,e\big)\,\zeta w^2 + \big(56\,\theta\,a + 200\sqrt{-1}\,e\big)\,z^2\zeta w,
\end{aligned}
$$

$$
\begin{aligned}
B^6 \;=\; & \Big(-770\sqrt{-1}\,d + 690\,c + 4\sqrt{-1}\,\theta\,e + \tfrac{32}{25}\,\theta^2\,a - \tfrac{24}{25}\sqrt{-1}\,\theta^2\,b\Big)\,z^6 \\
& + \Big(-\tfrac{100}{3}\,c + \tfrac{50}{3}\sqrt{-1}\,d\Big)\,w^3 + \Big(-250\,c - 350\sqrt{-1}\,d - 30\sqrt{-1}\,\theta\,e - \tfrac{12}{5}\,\theta^2\,a\Big)\,z^4w \\
& + \Big(400\,a + 200\sqrt{-1}\,b\Big)\,z\zeta w^2 + \Big(-2\,\theta\,a + 25\sqrt{-1}\,e\Big)\,\zeta^2w^2 + \Big(-6\,\theta\,a + 6\sqrt{-1}\,\theta\,b\Big)\,z^4\zeta^2 \\
& + \Big(600\,c + 300\sqrt{-1}\,d - 60\sqrt{-1}\,\theta\,e\Big)\,z^2w^2 + \Big(-1500\,a - 300\sqrt{-1}\,b\Big)\,z^3\zeta w \\
& + \Big(300\sqrt{-1}\,e + 24\,\theta\,a\Big)\,z^2\zeta^2w + \Big(\tfrac{48}{5}\sqrt{-1}\,\theta\,d - 1360\sqrt{-1}\,b + 920\,a - \tfrac{48}{5}\,\theta\,c\Big)\,z^5\zeta,
\end{aligned}
$$

$$
\begin{aligned}
B^7 \;=\; & \Big(\tfrac{1056}{7}\sqrt{-1}\,\theta\,b - 168\,\theta\,a + \tfrac{144}{175}\,\theta^2\,c - \tfrac{144}{175}\sqrt{-1}\,\theta^2\,d\Big)\,z^7 \\
& + \Big(-48\,\overline{A_{0,0,4}} - 6\,\overline{B_{1,0,3}} + 60\,\theta\,a - 750\sqrt{-1}\,e\Big)\,z^3w^2 + \Big(-200\,c\Big)\,\zeta w^3 \\
& + \Big(\tfrac{56}{25}\,\theta^2\,a - \tfrac{48}{25}\sqrt{-1}\,\theta^2\,b + 1460\,c - 1460\sqrt{-1}\,d + 4\sqrt{-1}\,\theta\,e\Big)\,z^6\zeta \\
& + \Big(-\tfrac{24}{5}\,\theta^2\,a - 60\sqrt{-1}\,\theta\,e + 100\,c + 100\sqrt{-1}\,d\Big)\,z^4\zeta w \\
& + \Big(\tfrac{24}{25}\sqrt{-1}\,\theta\,d - \tfrac{24}{25}\,\theta\,c + 900\,a - 900\sqrt{-1}\,b\Big)\,z^5\zeta^2 \\
& + \Big(360\,\theta\,a + 144\sqrt{-1}\,\theta\,b + 5100\sqrt{-1}\,e\Big)\,z^5w + \Big(B_{1,0,3}\Big)\,zw^3 \\
& + \Big(-1000\,a + 200\sqrt{-1}\,b\Big)\,z^3\zeta^2w + \Big(-400\,c + 700\sqrt{-1}\,d\Big)\,z^2\zeta w^2,
\end{aligned}
$$

and where:

$$
\begin{aligned}
C^0 &= \sqrt{-1}\,e, \\
C^1 &= \Big(2\,a - 2\sqrt{-1}\,b\Big)\,z, \\
C^2 &= \Big(c - \sqrt{-1}\,d\Big)\,z^2 + \Big(-2\,c\Big)\,w, \\
C^3 &= \Big(\tfrac{4}{5}\,\theta\,a + 10\sqrt{-1}\,e\Big)\,zw,
\end{aligned}
$$

$$C^4 = \big(10\sqrt{-1}\,b\big)\,w^2 + \big(-10\,a - 10\sqrt{-1}\,b\big)\,z^2w,$$
$$C^5 = \big(2\,c - 2\sqrt{-1}\,d\big)\,z^5 + \big(-20\,c + 10\sqrt{-1}\,d\big)\,zw^2,$$
$$C^6 = \big(-4\,\theta\,a\big)\,w^3 + \big(2\,\theta\,a - 25\sqrt{-1}\,e\big)\,z^2w^2,$$

$$C^7 = \big(\tfrac{2}{35}\sqrt{-1}\,\theta\,d - \tfrac{2}{35}\,\theta\,c\big)\,z^7 + \big(-20\,a - 20\sqrt{-1}\,b\big)\,z^5w + \big(\tfrac{400}{3}\,a + \tfrac{200}{3}\sqrt{-1}\,b\big)\,zw^3,$$

$$C^8 = \big(-25\sqrt{-1}\,d + 10\sqrt{-1}\,\theta\,e\big)\,w^4 + \big(-\tfrac{25}{2}\sqrt{-1}\,d + \tfrac{25}{2}\,c\big)\,z^8 + \big(\tfrac{100}{3}\,c + \tfrac{50}{3}\sqrt{-1}\,d\big)\,z^2w^3 + \big(10\sqrt{-1}\,d - 10\,c\big)\,z^6w,$$

$$C^9 = \big(-\tfrac{2}{525}\sqrt{-1}\,\theta^2\,d + \tfrac{2}{525}\,\theta^2\,c\big)\,z^9 + \big(\tfrac{4}{7}\sqrt{-1}\,\theta\,b + \tfrac{4}{7}\,\theta\,a\big)\,z^7w + \big(4\,\theta\,a - 50\sqrt{-1}\,e\big)\,z^5w^2 + \big(2\,\overline{A_{0,0,4}}\big)\,zw^4,$$

and the related 5 holomorphic vector fields e_1, e_2, e_3, e_4, e_5 have structure:

$$[e_1, e_2] = -\tfrac{4}{5}\,\theta\,e_4 - 4\,e_5, \qquad [e_1, e_3] = 0, \qquad [e_1, e_4] = 2\,e_2, \qquad [e_1, e_5] = \tfrac{2}{5}\,\theta\,e_2 - 20\,e_4,$$
$$[e_2, e_3] = -2\,e_2, \qquad [e_2, e_4] = 0, \qquad [e_2, e_5] = 0,$$
$$[e_3, e_4] = 2\,e_4, \qquad [e_3, e_5] = 2\,e_5,$$
$$[e_4, e_5] = 0.$$

This Lie algebra $\mathfrak{g}$ has the derived series of dimensions 5, 3, 0, with:

$$[\mathfrak{g}, \mathfrak{g}] = \mathsf{Span}\big(-\tfrac{4}{5}\,\theta\,e_4 - 4\,e_5,\ \ 2\,e_2,\ \ \tfrac{2}{5}\,\theta\,e_2 - 20\,e_4\big).$$

These three vector fields form a 3-dimensional Abelian ideal $\mathfrak{a} \subset \mathfrak{g}$, whose value at the origin $0 \in \mathbb{C}^3$ spans a maximally real 3-plane. This is coherent with [Fels-Kaup 2008].

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