On the conformal transformation between two anisotropic fluid spacetimes

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Overview

- Conformal Cyclic Cosmology
- Obstructions to conformally Einstein metrics
- Anisotropic fluid spacetimes
- Obstructions to conformally anisotropic fluid metric
- Conformal class with two anisotropic fluid metrics

Notation and conventions:

- abstract index notation, e.g. $v^a \in \Gamma\left(TM\right)$, $v_a \in \Gamma\left(T^*M\right)$
- signature of a 4-dimensional metric $(-,+,+,+)$
- symmetrization and antisymmetrization brackets

$$T_{\ldots(ab)\ldots} := \frac{1}{2}T_{\ldots ab\ldots} + \frac{1}{2}T_{\ldots ba\ldots},$$
$$T_{\ldots[ab]\ldots} := \frac{1}{2}T_{\ldots ab\ldots} - \frac{1}{2}T_{\ldots ba\ldots},$$

Einstein field equations

Spacetime is a 4-dimensional manifold M with the metric g_{ab} which satisfies

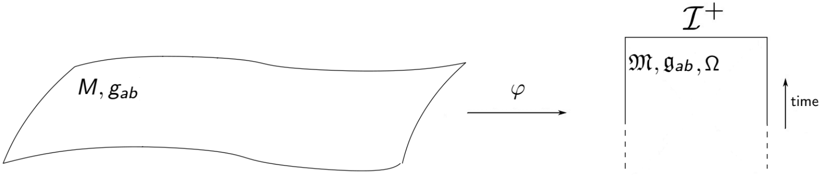
$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = T_{ab}, \tag{1}$$

where R_{ab} is a Ricci tensor, R is a scalar curvature, Λ is a **positive** constant and T_{ab} is an energy-momentum tensor.

Asymptotically de Sitter spacetime (M, g_{ab}) Let

- $\mathfrak{M}$ be a manifold with boundary $\mathcal{I} := \partial\mathfrak{M}$ and metric $\mathfrak{g}_{ab}$
- Ω be a smooth function such that:
  - $\Omega > 0$ on $\mathfrak{M} \backslash \mathcal{I}$
  - $\Omega = 0$, $d\Omega \neq 0$ on $\mathcal{I}$
- there exists $\varphi : M \to \mathfrak{M}$ such that $\varphi(M) = \mathfrak{M} \backslash \mathcal{I}$ and $\varphi^*(\mathfrak{g}_{ab}) = \Omega^2 g_{ab}$
- each null geodesic of (M, g_{ab}) acquires two distinct endpoints on $\mathcal{I}$ (on $\mathcal{I}^-$ and $\mathcal{I}^+$)
- $T_{ab} = 0$ in a neighbourhood of $\mathcal{I}^+$ in $\varphi^{-1}(\mathfrak{M})$

Asymptotically de Sitter spacetime – compactification

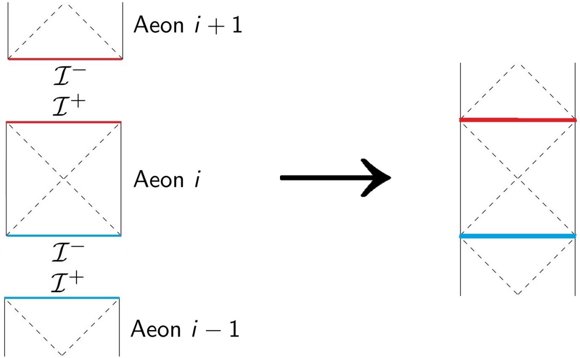


Future endpoints of all null geodesics of of (M, g_{ab}) form a spacelike hypersurface $\mathcal{I}^+$

Conformal Cyclic Cosmology (CCC)

Summary:

- the universe consist of aeons
- each aeon is a conformally conformally compactifiable spacetime with spacelike $\mathcal{I}^-$ and $\mathcal{I}^+$
- two consecutive aeons are matched along null infinities
- the Weyl tensor vanishes at the matching surface



Mathing of two aeons

Let $(\widehat{M}, \widehat{g}_{ab})$ and $(\widecheck{M}, \widecheck{g}_{ab})$ be two aeons with the same conformal extension, i.e.

$$\widehat{g}_{ab} = \widehat{\Omega}^{-2}\mathfrak{g}_{ab}, \quad \widecheck{g}_{ab} = \widecheck{\Omega}^{-2}\mathfrak{g}_{ab}, \quad (2)$$

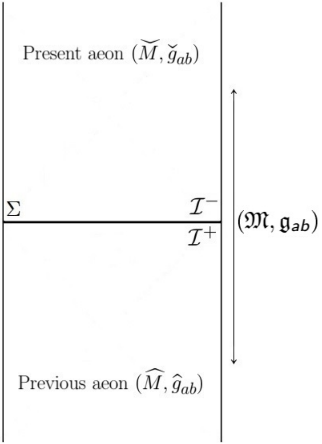
Moreover

$$\mathfrak{M} = \widehat{M} \cup \Sigma \cup \widecheck{M},$$

where a common boundary

$$\Sigma = \{\widehat{\Omega} = 0\} = \{\widecheck{\Omega}^{-1} = 0\}$$

is a future null infinity of the previous aeon $\mathcal{I}^{+}\left(\widehat{M}\right)$ and a past null infinity of the present aeon $\mathcal{I}^{-}\left(\widecheck{M}\right)$.



Conformal Cyclic Cosmology (CCC)

Reciprocal hypothesis Because of the conformal freedom we can impose

$$\check{\Omega}\hat{\Omega} = -1 \tag{3}$$

Hence

$$\check{g}_{ab} = \hat{\Omega}^4 \hat{g}_{ab} \tag{4}$$

i.e. the metric from the present aeon is determined by the metric from the previous aeon if one can provide a unique $\hat{\Omega}$.
Simplified Brinkmann's question Find $\mathfrak{g}_{ab}$ and $\hat{\Omega}$ such that

$$\hat{g}_{ab} = \hat{\Omega}^{-2}\mathfrak{g}_{ab}, \quad \check{g}_{ab} = \hat{\Omega}^{2}\mathfrak{g}_{ab}, \tag{5}$$

solve the Einstein field equations.
Alternative view Assume $(\hat{M}, \hat{g}_{ab})$ and $(\check{M}, \check{g}_{ab})$ are spacetimes with the energy-momentum tensor of the same type. What are the restrictions on $\hat{\Omega}$?

Schouten, Cotton and Bach tensors

The Riemann tensor can be decomposed in the following way,

$$R_{abcd} = C_{abcd} + 2\left(g_{c[a}P_{b]d} + g_{d[b}P_{a]c}\right), \tag{6}$$

where P_{ab} is the Schouten tensor,

$$P_{ab} := \frac{1}{2}R_{ab} - \frac{R}{12}g_{ab}. \tag{7}$$

It can be used to define the Cotton (A_{abc}) and Bach (B_{ab}) tensors,

$$\begin{aligned} A_{abc} &:= 2\nabla_{[b}P_{c]a}, \\ B_{ab} &:= -\nabla^{c}A_{abc} + P^{dc}C_{dacb}. \end{aligned} \tag{8}$$

Conformal properties of C_{abcd}, P_{ab}, A_{abc} and B_{ab}

Let

$$\widehat{g}_{ab} = e^{2\psi}\breve{g}_{ab}. \tag{9}$$

Then

$$\begin{aligned}
\widehat{C}^{a}{}_{bcd} &= \breve{C}^{a}{}_{bcd}, \\
\widehat{P}_{ab} &= \breve{P}_{ab} - \breve{\nabla}_a\psi_b + \psi_a\psi_b - \frac{1}{2}\breve{g}_{ab}\breve{g}^{cd}\psi_c\psi_d, \\
\widehat{A}_{abc} &= \breve{A}_{abc} + \psi^d\,\breve{C}_{dabc}, \\
\widehat{B}_{ab} &= e^{-2\psi}\,\breve{B}_{ab}
\end{aligned} \tag{10}$$

where $\psi_a := \partial_a\psi$.

Conformal Einstein space

Conformal Einstein space Let $\widehat{g}_{ab}$ be the Einstein metric,

$$\widehat{R}_{ab} = c\widehat{g}_{ab} \iff \widehat{P}_{ab} = \frac{c}{6}\widehat{g}_{ab} \tag{11}$$

Therefore

$$\widehat{A}_{abc} = 0, \quad \widehat{B}_{ab} = 0. \tag{12}$$

Necessary conditions for $\breve{g}_{ab}$ to be conformally Einstein metric $\widehat{g}_{ab} = e^{2\psi}\breve{g}_{ab}$:

$$\breve{A}_{abc} + \psi^{d}\breve{C}_{dabc} = 0, \quad \breve{B}_{ab} = 0 \tag{13}$$

for some gradient ψ_a.

Conformally anisotropic fluid metrics

Assume that the Ricci tensor has the form dictated by the energy-momentum tensor of anisotropic fluid type and obtain the necessary conditions analogous to (13).

Energy-momentum tensor of an anisotropic fluid Let (M, g_{ab}) be a spacetime with

$$T_{ab} = (\rho + p)\, u_a u_b + p g_{ab} + \pi_{ab} \tag{14}$$

where

- u^a is a timelike unit vector field (four-velocity of the fluid)
- ρ and p are scalar fields (density and isotropic pressure)
- π_{ab} is the anisotropic pressure tensor

Decomposition of the derivative of u^a

Let

- $h_a{}^b := \delta_a{}^b + u_a u^b$ be a projector onto $\Sigma \perp u^a$
- $\omega_{ab} := h_{[a}{}^c h_{b]}{}^d \nabla_c u_d$ be the **vorticity tensor**
- $\theta := h^{ab} \nabla_a u_b$ be the **expansion scalar**
- $\sigma_{ab} := h_{(a}{}^c h_{b)}{}^d \nabla_c u_d - \frac{1}{3} h_{ab} \theta$ be the **shear tensor**
- $\dot{u}_a := u^b \nabla_b u_a$ be the **acceleration vector**

Decomposition of the derivative of u^a

Ultimately

$$\nabla_a u_b = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - u_a \dot{u}_b \tag{15}$$

Continuity equation for T_{ab}

In the present setting $\nabla_b T_a{}^b = 0$ reduces to

$$\begin{aligned} u^a \nabla_a \rho + (\rho + p)\,\theta + \pi_{ab}\sigma^{ab} &= 0, \\ (\rho + p)\,\dot{u}_a + h_a{}^b\left(\nabla_b p + \nabla_c \pi^c{}_b\right) &= 0. \end{aligned} \tag{16}$$

First equation can be interpreted as a rate of change of entropy of the system, hence

$$\pi_{ab} = -\lambda\sigma_{ab}, \quad \lambda = \lambda\,(\rho, p) \tag{17}$$

will be imposed to keep it positive.

Let $(\widehat{M}, \widehat{g}_{ab})$ be an anisotropic fluid spacetime with

- vanishing vorticity $\widehat{\omega}_{ab} = 0$
- vanishing acceleration $\dot{\widehat{u}}_a = 0$
- ρ and p constant on $\Sigma \perp \widehat{u}^a$

In that case shear $\widehat{\sigma}_{ab}$ is an obstruction to conformal flatness.

Cotton and Bach tensors We have

$$\begin{aligned} \widehat{u}^a \widehat{A}_{abc} &= 0, \\ \widehat{u}^a \widehat{h}_b{}^c \widehat{B}_{ac} &= 0. \end{aligned} \tag{18}$$

Conformal anisotropic fluid metric

Let $\widehat{g}_{ab} = e^{2\psi} \breve{g}_{ab}$ and

$$\breve{u}^a = e^{\psi} \widehat{u}^a \implies \breve{g}^{cd} \breve{u}_c \breve{u}_d = -1 \tag{19}$$

Then

$$\begin{aligned} \breve{u}^a \left(\breve{A}_{abc} + \psi^d \breve{C}_{dabc} \right) &= 0, \\ \breve{u}^a \breve{h}_b{}^c \breve{B}_{ac} &= 0. \end{aligned} \tag{20}$$

Electric and magnetic parts of the Weyl tensor

Timelike vector $\breve{u}^a$ induces a decomposition of the Weyl tensor into $\breve{E}_{ab}$ and $\breve{H}_{ab}$,

$$\breve{E}_{ab} := \breve{u}^c \breve{u}^d \breve{C}_{acbd}, \quad \breve{H}_{ab} := \frac{1}{2} \breve{u}^c \breve{u}^d \breve{\eta}_{ackl} \breve{C}^{kl}{}_{bd} \tag{21}$$

where $\breve{\eta}_{ackl}$ the covariant Levi-Civita tensor and $\breve{E}_{ab}, \breve{H}_{ab} \perp \breve{u}^a$.

Decomposition of the necessary condition

Equation

$$\breve{u}^a \left(\breve{A}_{abc} + \psi^d \breve{C}_{dabc} \right) = 0.$$

splits into

$$\breve{u}^a \breve{u}^b \breve{h}_i{}^c \breve{A}_{abc} - \breve{E}_{ia} \breve{D}^a \psi = 0, \tag{22}$$

$$\breve{u}^a \breve{h}_i{}^b \breve{h}_j{}^c \breve{A}_{abc} + \breve{\eta}_{jik} \breve{H}_a{}^k \breve{D}^a \psi = 0. \tag{23}$$

where $\breve{D}_a \psi = \breve{h}_a{}^b \breve{\nabla}_b \psi$.

Simplification for invertible $\breve{E}_{ab}$

Suppose that there exists $\tilde{\breve{E}}_{ab}$ such that

$$\tilde{\breve{E}}^{bc}\breve{E}_{ac} = \left|\breve{E}\right| \delta_a{}^b \qquad (24)$$

Then (22) yields

$$\breve{D}_j\psi = \frac{\breve{u}^a\breve{u}^b\tilde{\breve{E}}_j{}^c\breve{A}_{abc}}{\left|\breve{E}\right|}, \qquad (25)$$

so from (23),

$$\breve{u}^a\breve{A}_{abc}\left(\breve{h}_i{}^b\breve{h}_j{}^c\left|\breve{E}\right| + \breve{\eta}_{jik}\breve{H}_d{}^k\breve{u}^b\tilde{\breve{E}}^{dc}\right) = 0. \qquad (26)$$

This gives an **obstruction tensor** to conformally anisotropic fluid metric.

Conformal class with two anisotropic fluid metrics

Let $(\breve{M}, \breve{g}_{ab})$ also be an anisotropic fluid spacetime with the four-velocity of a fluid $\breve{u}^a$ conformally related to $\hat{u}^a$,

$$\breve{u}^a = e^{\psi}\hat{u}^a \tag{27}$$

i.e.

$$\breve{T}_{ab} = (\breve{\rho} + \breve{p})\,\breve{u}_a\breve{u}_b + \breve{p}\breve{g}_{ab} - \breve{\lambda}\breve{\sigma}_{ab}, \tag{28}$$

Then

$$\breve{\omega}_{ab} = 0, \quad \breve{\sigma}_{ab} = e^{-\psi}\hat{\sigma}_{ab}, \quad \dot{\breve{u}}_a = -\breve{D}_a\psi, \tag{29}$$

Let $\breve{\rho}, \breve{p}$ be constant functions on $\Sigma \perp \breve{u}^a$.

Condition on ψ

The continuity equation $\breve{\nabla}_b\breve{T}_a{}^b = 0$ and the transformation rules (29) lead to

$$\left((\breve{\rho} + \breve{p})^2 + 2e^{2\psi}\breve{\lambda}^2\hat{\sigma}_{ab}\hat{\sigma}^{ab}\right)\breve{D}_c\psi = 0, \tag{30}$$

Conformal class with two anisotropic fluid metrics

Equation

$$\left((\breve{\rho}+\breve{p})^2+2e^{2\psi}\breve{\lambda}^2\hat{\sigma}_{ab}\hat{\sigma}^{ab}\right)\breve{D}_c\psi=0,$$

can only be satisfied if

$$\breve{D}_c\psi=0$$

i.e. ψ is constant on $\Sigma\perp\breve{u}^a$ (also $\Sigma\perp\hat{u}^a$).

Evolution equation for ψ

Conformal transformation of $\hat{R}_{ab}\hat{u}^a\hat{u}^b$ yields

$$3\hat{u}^a\hat{u}^b\hat{\nabla}_a\psi_b+\hat{\theta}\hat{u}^a\psi_a+e^{-2\psi}\left(\breve{\Lambda}-\frac{1}{2}\left(\breve{\rho}+3\breve{p}\right)\right)=\hat{\Lambda}-\frac{1}{2}\left(\hat{\rho}+3\hat{p}\right).$$

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Thank you!