On uniqueness of submaximally symmetric parabolic geometries

Dennis The

Department of Mathematics & Statistics
UiT The Arctic University of Norway

20 August 2021

SCREAM opening workshop,
GRIEG project 2019/34/H/ST1/00636

Apologies

Sorry we couldn't make it to Poland this year. Hope to see you all in Norway next year!

Symmetry gaps

Let $\mathfrak{M}$ & $\mathfrak{S}$ be the max & submax sym dim for structures below:

Structure	G	P	$\mathfrak{M}$	$\mathfrak{S}$
2-dim projective	A_2	P_1	8	3
2nd order ODE	A_2	$P_{1,2}$	8	3
(2,3,5)-distributions	G_2	P_1	14	7
5-dim G_2-contact	G_2	P_2	14	7
3-dim projective	A_3	P_1	15	8
4-dim split-conformal	A_3	P_2	15	9
5-dim Legendrian contact	A_3	$P_{1,3}$	15	8
(3,6)-distributions	B_3	P_3	21	11
57-dim E_8-contact	E_8	P_8	248	147

Kruglikov–T. (2014): Framework for sym gaps; found many $\mathfrak{S}$.

Locally, $\exists!$ maximally symmetric structure (the "flat" model).

Q: Locally classify all submaximally symmetric structures.

($\exists$ techniques for classification, e.g. Cartan reduction, but they are cumbersome to apply beyond low dimensions.)

Outline

1. Examples and main theorem
2. Recap of framework for symmetry gaps
3. Main thm – proof ideas

Examples and main theorem

- 2nd order ODE $y'' = f(x, y, y')$, $(A_2, P_{1,2}), \mathfrak{M} = 8, \mathfrak{S} = 3$:

	Tresse (relative) invariants	
Submax sym model ($\mathfrak{S} = 3$)	I_1	$I_2 = f_{y'y'y'y'}$
$y'' = e^{y'}$	$\neq 0$	$\neq 0$
$y'' = (y')^a$ $(a \in \mathbb{C}\backslash\{0, 1, 2, 3\})$	$\neq 0$	$\neq 0$
$y'' = 6yy' - 4y^3 + c(y' - y^2)^{3/2}$ $(c \in \mathbb{C}\backslash\{0\})$	$\neq 0$	$\neq 0$
$y'' = \frac{3(y')^2}{2y} + y^3$	$\neq 0$	0

Note $y'' = (xy' - y)^3$ has $I_1 \neq 0$ a.e. ($I_1 = 0$ along $xy' = y$) and $I_2 = 0$. It has 3-dim intransitive symmetry.

- $(2, 3, 5)$-distributions, (G_2, P_1), $\mathfrak{M} = 14$, $\mathfrak{S} = 7$. Monge form: $\langle \partial_x + p\partial_y + q\partial_p + f\partial_z, \partial_q \rangle$, with $f = f(x, y, p, q, z)$, $f_{qq} \neq 0$.

Submax sym model ($\mathfrak{S} = 7$)	Cartan quartic
$f = q^m$ $(m \notin \{-1, 0, \frac{1}{3}, \frac{2}{3}, 1, 2\})$	N
$f = \log(q)$	N

- G_2-contact structures, (G_2, P_2), $\mathfrak{M} = 14$, $\mathfrak{S} = 7$.
  T. 2021: Locally, $\exists!$ G_2-contact str. with 7-dim sym.

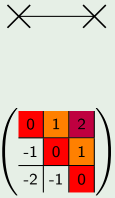
Parabolic subalgebras and gradings

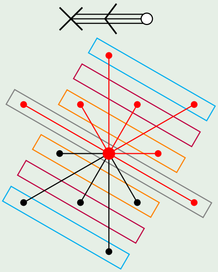
$\mathfrak{g}$: s.s. Lie algebra; $(\mathfrak{g}, \mathfrak{p}) \rightsquigarrow \mathbb{Z}$-grading: $\mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$,

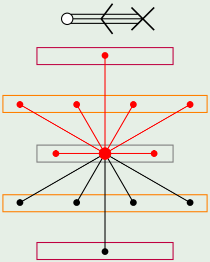
$\exists$ grading element Z with $\mathfrak{g}_j = \{x \in \mathfrak{g} : [Z, x] = jx\}$.

⚠ Grading is auxilliary! Filtration $\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_j$ is important.

Example ($A_2/P_{1,2}$, G_2/P_1, G_2/P_2)







Curved versions:

2nd order ODE | $(2, 3, 5)$-distrib. | G_2-contact str.

Rank 3 examples

- 5-dim Legendrian contact $(M^5, \mathcal{C} = \mathcal{E} \oplus \mathcal{F})$, $(\mathrm{SL}(4), P_{1,3})$, $\mathfrak{M} = 15$, $\mathfrak{S} = 8$. Basic invariants:
  - $\tau_{\mathcal{E}}, \tau_{\mathcal{F}}$: obstruct Frobenius-integrability;
  - $\mathcal{W}$: binary quartic (analogue of Weyl curvature).

  When $\tau_{\mathcal{F}} = 0$, can describe as PDE $u_{ij} = f_{ij}(x^k, u, u_\ell)$.
  FACT: $\exists$ three inequivalent models with 8-dim symmetry, each with exactly one of these invariants being nonzero. When $\tau_{\mathcal{E}} = \tau_{\mathcal{F}} = 0$, the model (Doubrov–Medvedev–T. 2020) is:

$$u_{xx} = (u_y)^2, \quad u_{xy} = 0, \quad u_{yy} = 0.$$

- Real CR hypersurfaces in $\mathbb{C}^3$ with Levi form that is:
  - positive-def: $(\mathrm{SU}(1,3), P_{1,3})$, $\mathfrak{M} = 15$, $\mathfrak{S} = 7$; several parametric families of submax models (Loboda 2001).
  - indefinite: $(\mathrm{SU}(2,2), P_{1,3})$, $\mathfrak{M} = 15$, $\mathfrak{S} = 8$; $\exists!$ submax model:

$$\mathfrak{Im}(w + \bar{z}_1 z_2) = |z_1|^4 \quad \text{(Winkelmann hypersurface)}.$$

Parabolic geometries

Starting point: $\exists$ equivalence of categories between regular, normal parabolic geometries and underlying geometric structures (Tanaka, Morimoto, Čap–Schichl). Upshot: study symmetry "upstairs".

Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P).

- Curvature: $K = d\omega + \frac{1}{2}[\omega, \omega]$, $\kappa(x, y) = K(\omega^{-1}(x), \omega^{-1}(y))$, $\kappa : \mathcal{G} \to \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \bigwedge^2\mathfrak{g}_+ \otimes \mathfrak{g}$. Flat if $\kappa = 0$.
- Regular: $\mathrm{im}(\kappa)$ valued in the positive subspace wrt Z.
- Normal: $\partial^*\kappa = 0$, with ∂^* the Lie alg homology differential.
- Harmonic curvature: $\kappa_H := \kappa \mod \mathrm{im}(\partial^*)$, valued in $H_2(\mathfrak{g}_+, \mathfrak{g})^1$, i.e. positive part of $H_2(\mathfrak{g}_+, \mathfrak{g}) := \frac{\ker \partial^*}{\mathrm{im}\, \partial^*}$ (completely reducible, so only the $\mathfrak{g}_0$-action is relevant).

Thm: $\boxed{(\mathcal{G} \to M, \omega) \text{ is flat iff } \kappa_H = 0}$.

Submax sym dim: $\boxed{\mathfrak{S} := \max\{\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \mid \kappa_H \not\equiv 0\}}$.

Given: P-irrep $\mathbb{V} \subset H_2(\mathfrak{g}_+, \mathfrak{g})^1$. Say $(\mathcal{G} \to M, \omega)$ is type $(G, P, \mathbb{V})$ if it is of type (G, P) and $\mathrm{im}(\kappa_H) \subset \mathbb{V}$. Analogously define $\mathfrak{S}_{\mathbb{V}}$.

Main theorem

Theorem (T. 2021)

Let G be a complex or split-real simple Lie group, P a parabolic subgroup. Let $(\mathcal{G} \to M, \omega)$ be a reg./nor. parabolic geometry of type $(G, P, \mathbb{V})$, where $\mathbb{V} \subset H_2(\mathfrak{g}_+, \mathfrak{g})^1$ is a P-irrep. Suppose that $\mathrm{diminf}(\mathcal{G}, \omega) = \mathfrak{S}_{\mathbb{V}}$, and $\mathrm{rank}(G) \geq 3$ or $(G, P) = (G_2, P_2)$. Then about any $u \in \mathcal{G}$ with $\kappa_H(u) \neq 0$, the geometry is locally homogeneous and is:

- *complex case: locally unique;*
- *split-real case: locally one of at most two possibilities.*

NB. The result is constructive in the "Cartan sense". (More later.)

Framework for studying symmetry gaps

Key algebraic ingredient #1: Kostant theory

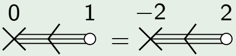
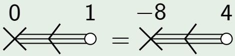
Given $(\mathfrak{g}, \mathfrak{p})$, we have $\mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$. Kostant (1961) $\Rightarrow H_2(\mathfrak{g}_+, \mathfrak{g})^1 \cong_{\mathfrak{g}_0} H^2_+(\mathfrak{g}_-, \mathfrak{g})$, and this is easily computed using:

Theorem (Simplified Kostant thm for $\mathfrak{g}$ $\mathbb{C}$-simple with highest weight λ)

$H^k(\mathfrak{g}_-, \mathfrak{g}) \cong_{\mathfrak{g}_0} \bigoplus_{w \in W^{\mathfrak{p}}(k)} \mathbb{V}_{-w\bullet\lambda}$. *Also have explicit lowest weight vectors ϕ_0.*

- $\mathbb{V}_\mu$ is the $\mathfrak{g}_0$-irrep with lowest weight μ.
- $w \bullet \lambda := w(\lambda + \rho) - \rho$ (affine action of Weyl group W).
- $W^{\mathfrak{p}}(k) :=$ length k words of the Hasse subset $W^{\mathfrak{p}} \subset W$.
- Efficient Dynkin diagram recipes, cf. Baston–Eastwood (1989).

Example (G_2/P_1: $Z = Z_1$, $W^{\mathfrak{p}}(1) = \{(1)\}$, $W^{\mathfrak{p}}(2) = \{(12)\}$)

<table>
<tr><th>Calculation</th><th>Lowest wt</th><th>Interpretation</th></tr>
<tr><td>(1) • 
</td><td>$2\lambda_1 - 2\lambda_2 = -2\alpha_1 - 2\alpha_2$</td><td>$H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0$ ($\therefore \mathrm{pr}(\mathfrak{g}_-) \cong \mathfrak{g}$.)</td></tr>
<tr><td>(12) • 
</td><td>$8\lambda_1 - 4\lambda_2 = +4\alpha_1$</td><td>$H^2_+(\mathfrak{g}_-, \mathfrak{g}) \cong S^4\mathfrak{g}_1 \cong S^4(\mathfrak{g}_{-1})^*$</td></tr>
</table>

Key algebraic ingredient #2: Tanaka prolongation

Definition (Extrinsic Tanaka prolongation)

Given ϕ in a $\mathfrak{g}_0$-rep, let $\mathfrak{a} := \mathfrak{a}^\phi \subset \mathfrak{g}$ be the graded Lie subalg with:

1. $\mathfrak{a}_{\leq 0} := \mathfrak{g}_- \oplus \mathfrak{ann}(\phi)$, *and*
2. $\mathfrak{a}_k := \{x \in \mathfrak{g}_k : [x, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{k-1}\}, \quad \forall k > 0.$

Of interest:

- $0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})$;
- $0 \neq \phi \in \mathbb{V} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$, where $\mathbb{V}$ is a (irreducible) submodule;
- $0 \neq \phi \in \mathcal{O} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$, where $\mathcal{O}$ is a G_0-orbit.

Definition

If $\mathfrak{a}^\phi_+ = 0$, $\forall \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})$, then $(\mathfrak{g}, \mathfrak{p})$ is prolongation-rigid (PR).

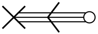
Kruglikov–T. (2014): If $\mathfrak{p} \subset \mathfrak{g}$ is maximal parabolic, i.e. single cross, then $(\mathfrak{g}, \mathfrak{p})$ is PR.

Kruglikov-T. (2014):

- Fix any (G, P). Then $\mathfrak{S} \leq \mathfrak{U}$ for some universal upper bound

$$\mathfrak{U} := \max\{\dim \mathfrak{a}^\phi : 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\}.$$

  (Analogously, $\mathfrak{S}_{\mathbb{V}} \leq \mathfrak{U}_{\mathbb{V}}$ or $\mathfrak{S}_{\mathcal{O}} \leq \mathfrak{U}_{\mathcal{O}}$.)
- Complex or split-real simple G setting:
  - Efficient Dynkin diagram recipes to compute $\mathfrak{U}$, e.g.



$$\rightsquigarrow \quad \mathfrak{U} = 7.$$

  - $\mathfrak{S} = \mathfrak{U}$, but some $\mathfrak{S} < \mathfrak{U}$ exceptions only when $\mathrm{rank}(G) = 2$.
  - Non-exceptional cases: any submax sym structure is locally homogeneous near $u \in \mathcal{G}$ with $\kappa_H(u) \neq 0$.

In fact, we proved a much stronger result (KT 2014 / 2016):

$$\mathfrak{s}(u) \subset \mathfrak{a}^{\kappa_H(u)}, \quad \forall u \in \mathcal{G},$$

where $\mathfrak{s}(u) := \mathrm{gr}(\mathfrak{f}(u))$, with $\mathfrak{f}(u) := \omega_u(\inf(\mathcal{G}, \omega))$.

Exhibiting homogeneous models

Q: How to exhibit a homogeneous model?

Example ((2, 3, 5)-distributions)

- Coordinate model: $\mathcal{D} := \langle \partial_x + p\partial_y + q\partial_p + q^m\partial_z, \partial_q \rangle$, where $m \in \mathbb{C}\backslash\{-1, 0, \frac{1}{3}, \frac{2}{3}, 1, 2\}$; syms $\mathbf{X}_1, ..., \mathbf{X}_7$.
- Lie-theoretic model: $(\mathfrak{f}, \mathfrak{f}^0)$ infinitesimally effective pair, with $\mathfrak{f}^0$-invariant filtration

$$\mathfrak{f} = \mathfrak{f}^{-3} \supset \mathfrak{f}^{-2} \supset \mathfrak{f}^{-1} \supset \mathfrak{f}^0 \supset 0$$

- Cartan-theoretic model: ?

Algebraic models

Any homogeneous parabolic geometry over $M = F/F^0$ that is "infinitesimally effective" admits a description as:

Definition (Cartan-theoretic description of homog. structures)

An algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ *is a Lie algebra* $(\mathfrak{f}, [\cdot,\cdot]_{\mathfrak{f}})$ *such that:*

M1: $\mathfrak{f} \subset \mathfrak{g}$ *is a filtered subspace, with filtrands* $\mathfrak{f}^i := \mathfrak{f} \cap \mathfrak{g}^i$, *and* $\mathfrak{s} := \mathrm{gr}(\mathfrak{f})$ *satisfying* $\mathfrak{s}_- = \mathfrak{g}_-$. *(Thus,* $\mathfrak{f}/\mathfrak{f}^0 \cong \mathfrak{g}/\mathfrak{p}$.*)*

M2: $\mathfrak{f}^0$ *inserts trivially into* $\kappa(x,y) := [x,y] - [x,y]_{\mathfrak{f}}$. *(Thus,* $\kappa \in \bigwedge^2(\mathfrak{f}/\mathfrak{f}^0)^* \otimes \mathfrak{g} \cong \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$.*)*

M3: κ *is regular and normal, i.e.* $\kappa \in \ker(\partial^*)_+$.

Given (G, P), let $\mathcal{M}$ be the set of all algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$.

- $\mathcal{M}$ is partially ordered: $\mathfrak{f} \leq \mathfrak{f}'$ iff $\mathfrak{f} \hookrightarrow \mathfrak{f}'$ as Lie algs.
- $\mathcal{M}$ admits a P-action: i.e. $p \cdot \mathfrak{f} = \mathrm{Ad}_p \mathfrak{f}$.

Upshot: Classify maximal elements in $(\mathcal{M}, \leq)$ with $\kappa_H \neq 0$.

Necessary constraints

Proposition

Let $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ *be an algebraic model. Then*

1. $(\mathfrak{f}, [\cdot,\cdot]_{\mathfrak{f}})$ *is a filtered Lie alg, and* $\mathfrak{s} = \mathrm{gr}(\mathfrak{f}) \subset \mathfrak{g}$ *is a graded Lie subalg.*
2. $\mathfrak{f}^0 \cdot \kappa = 0$, *i.e.* $[z, \kappa(x,y)] = \kappa([z,x],y) + \kappa(x,[z,y])$, $\forall x, y \in \mathfrak{f}$, $\forall z \in \mathfrak{f}^0$.
3. $\mathfrak{s} \subset \mathfrak{a}^{\kappa_H}$, *i.e.* $\mathfrak{f}$ *is a "constrained filtered sub-deformation" of* $\mathfrak{a}^{\kappa_H}$.

Proof.

1. Recall $\mathfrak{f}^i := \mathfrak{f} \cap \mathfrak{g}^i$. Hence, $[\mathfrak{f}^i, \mathfrak{f}^j]_{\mathfrak{f}} \subset \mathfrak{f}^{i+j}$ follows from regularity.
2. Use Jacobi identity for $[\cdot,\cdot]_{\mathfrak{f}} = [\cdot,\cdot] - \kappa(\cdot,\cdot)$.
3. ∂^* is $\mathfrak{p}$-equiv., so $\mathrm{im}(\partial^*)$ is $\mathfrak{p}$-inv. Then (2) $\Rightarrow$ $\mathfrak{f}^0 \cdot \kappa_H = 0$, so $\mathfrak{s}_0 \cdot \kappa_H = 0$, since $\mathfrak{g}_+$ is trivial on $H_2(\mathfrak{g}_+, \mathfrak{g})$. For $k > 0$, $[\mathfrak{s}_k, \mathfrak{g}_{-1}] = [\mathfrak{s}_k, \mathfrak{s}_{-1}] \subset \mathfrak{s}_{k-1}$. Let $\mathfrak{a} := \mathfrak{a}^{\kappa_H}$, so $\mathfrak{s}_0 \subset \mathfrak{a}_0 := \mathfrak{ann}(\kappa_H)$. Inductively, $\mathfrak{s}_k \subset \mathfrak{a}_k$, $\forall k > 0$.

The canonical curved model

Fix $(G, P, \mathbb{V})$ complex or split-real, where $\mathbb{V} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$ is a $\mathfrak{g}_0$-irrep. with lowest weight vector ϕ_0.

Define $\mathfrak{f} := \mathfrak{a}^{\phi_0} \subset \mathfrak{g}$ as a filtered subspace, but with bracket

$$[\cdot,\cdot]_{\mathfrak{f}} := [\cdot,\cdot] - \phi_0(\cdot,\cdot),$$

where we view ϕ_0 as a harmonic 2-cochain. Well-defined?

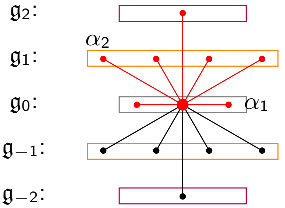
- $\mathrm{im}(\phi_0) \subset \mathfrak{g}_- \subset \mathfrak{a}$ almost always.
  Exceptions when $\mathrm{rank}(G) = 2$.
- Is Jacobi identity satisfied? Yes
- Get $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ with $\kappa_H = \phi_0$, $\dim \mathfrak{f} = \mathfrak{U}_{\mathbb{V}}$, maximal in $(\mathcal{M}, \leq)$. Called "canonical curved model of type $(G, P, \mathbb{V})$"; have $\mathfrak{S}_{\mathbb{V}} = \mathfrak{U}_{\mathbb{V}}$. This relies on the following (KT 2014):

Prop: $\dim \mathfrak{a}^{\phi} = \dim \mathfrak{a}^{\phi_0}$ iff $[\phi] \in G_0 \cdot [\phi_0]$ (so $\mathfrak{U}_{\mathbb{V}} = \dim \mathfrak{a}^{\phi_0}$).

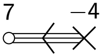
Q: Classify (up to P-action) all $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ with $\mathfrak{s} = \mathrm{gr}(\mathfrak{f}) = \mathfrak{a}^{\phi_0}$.

Main theorem – proof ideas

(G_2, P_2) case



Let $\{Z_1, Z_2\}$ be dual to $\{\alpha_1, \alpha_2\}$. Set $Z := Z_2$.
$\mathfrak{g}_0 = \langle Z, h_{\alpha_1}, e_{\alpha_1}, e_{-\alpha_1} \rangle \cong \mathfrak{gl}_2$

$H^2_+(\mathfrak{g}_-, \mathfrak{g}) =$  $=: \mathbb{V}_\mu$

$\mu = -7\lambda_1 + 4\lambda_2 = -2\alpha_1 + \alpha_2$ (degree $+1$)
$\phi_0 = e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \otimes e_{-3\alpha_1-\alpha_2}$
$\mathfrak{a} = \mathfrak{g}_- \oplus \mathfrak{a}_0, \quad \mathfrak{a}_0 = \langle Z_1 + 2Z_2 \rangle \oplus \mathfrak{g}_{-\alpha_1}$

GOAL: Classify $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ with $\mathrm{gr}(\mathfrak{f}) = \mathfrak{a}$. Step 1: Determine subspace $\mathfrak{f} \subset \mathfrak{g}$.

- Let $T \in \mathfrak{f}^0$ with $\mathrm{gr}_0(T) = Z_1 + 2Z_2$. Since $(Z_1 + 2Z_2)(\alpha) \neq 0$, $\forall \alpha \in \Delta(\mathfrak{g}_+)$, use P_+-action $\overset{\text{normalize}}{\rightsquigarrow}$ $T = Z_1 + 2Z_2 \in \mathfrak{f}^0$.
- Define $\mathfrak{a}^\perp = \langle Z_2 \rangle \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_+$, so $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ (ad_T-invariant). Write $\mathfrak{f}$ as a graph over $\mathfrak{a}$, i.e. $\mathfrak{f} \ni x = a + \mathfrak{d}(a)$ for $a \in \mathfrak{a}$ and $\mathfrak{d} \in \mathfrak{a}^* \otimes \mathfrak{a}^\perp$, positive degree and $\boxed{T \cdot \mathfrak{d} = 0}$, i.e. $\mathfrak{d}$ is a sum of weight vectors for weights that are multiples of $\mu = -2\alpha_1 + \alpha_2$. Weights of $\mathfrak{a}^*$ (and $\mathfrak{a}^\perp$):

$$0, \quad \alpha_1, \quad \alpha_2, \quad \alpha_1 + \alpha_2, \quad 2\alpha_1 + \alpha_2, \quad 3\alpha_1 + \alpha_2, \quad 3\alpha_1 + 2\alpha_2.$$

Must have $\mathfrak{d} = 0$, so $\mathfrak{f} = \mathfrak{a}$ as filtered subspaces of $\mathfrak{g}$.

(G_2, P_2) continued

Step 2: Determine curvature $\kappa \in \ker(\partial^*)_+ \subset \bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{g}$.

- Thm: Lowest degree part of κ is harmonic. (Čap–Slovak, Thm.3.1.12.)
- $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \cong \mathbb{V}_\mu$, with $\mu = -2\alpha_1 + \alpha_2$ has degree $+1$. (Apply $Z = Z_2$.)
- Since $T \cdot \kappa = 0$, want 2-cochain wts:

$$\sigma = r\mu = \alpha + \beta + \gamma, \quad \alpha, \beta \in \Delta(\mathfrak{g}_+) \text{ distinct}, \ \gamma \in \Delta \cup \{0\}, \ \boxed{r \geq 1}.$$

- Highest weight of $\mathfrak{g}$ is $\lambda = 3\alpha_1 + 2\alpha_2$. Note $-\lambda \leq \gamma < \sigma = r\mu$. Apply Z_1:

$$-3 \leq Z_1(\sigma) = rZ_1(\mu) = -2r \quad \Rightarrow \quad \boxed{r \leq \tfrac{3}{2}}.$$

- μ has degree $+1$ and $\sigma = k\alpha_1 + \ell\alpha_2$ with $k, \ell \in \mathbb{Z}$, so $\boxed{r = 1}$.
- $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \cong \mathbb{V}_\mu$ is a $\mathfrak{g}_0$-irrep, with unique lwv ϕ_0, then $\kappa = c\phi_0$ for $c \neq 0$.
- Use $\mathrm{Ad}_{\exp(tZ)}$ to rescale: over $\mathbb{C}$, we have wlog $c = 1$.

Conclusion: We get only the canonical curved model. □

NB.

1. Over $\mathbb{R}$, we might have at most $c = \pm 1$ after rescaling.
2. Did not use full structure equations for (G_2, P_2) geometries!
3. Made efficient use of G_2 weights.

General case

The general case is similar, but more technical, using specific knowledge coming from Kostant's thm for the lowest weight μ and lwv ϕ_0. Starting point:

$$\mathfrak{a}_0 = \mathfrak{ann}(\phi_0) = \ker(\mu) \oplus \bigoplus_{\gamma \in \Delta(\mathfrak{g}_{0,\leq 0})} \mathfrak{g}_\gamma$$

wrt a certain secondary grading. Let $\mathfrak{a} := \mathfrak{a}^{\phi_0} = \mathfrak{g}_- \oplus \mathfrak{a}_0 \oplus \dots$.

Given a geometry of type $(G, P, \mathbb{V})$, we can appeal to Čap (2005) and equivalently regard it as a geometry on a "minimal twistor space":

$$\begin{array}{c} G/P \\ \big\downarrow \\ G/\bar{P} \end{array}$$

("Normality" and "regularity" are well-behaved in passing down.)

Example ($A_m/P_{1,2} \to A_m/P_1$)

For 2nd order ODE systems, κ_H is comprised of Fels torsion $\mathcal{T}$ (hom. $+2$) and Fels curvature $\mathcal{S}$ (hom. $+3$). If $\mathcal{S} = 0$, the system is geodesic (for some $[\nabla]$).

Benefits of this "twistor simplification" (KT 2014): $\boxed{\mathfrak{a}_+ = 0}$, so $\boxed{\mathfrak{f}^1 = 0}$.

Generic case – strategy

Lemma

Let $\mathfrak{g}$ be complex simple, $\ell := \mathrm{rank}(\mathfrak{g}) \geq 3$, λ its highest root, $\mathfrak{p} \subset \mathfrak{g}$ parabolic. Let $w = (jk) \in W^{\mathfrak{p}}(2)$ with $\mu = -w \bullet \lambda$ satisfies $\mathsf{Z}(\mu) > 0$. Then:

(L1) $\mu = \sum_{i=1}^{\ell} m_i \alpha_i$ *has coefficients m_i of opposite sign. More precisely, $m_i < 0$, $\forall i \neq j, k$, and either $m_j > 0$ or $m_k > 0$.*

(L2) $\exists H_0 \in \ker(\mu)$ *with $f(H_0) \neq 0$ for all $f = \alpha + \beta$ with $(\alpha, \beta) \in \mathcal{R} := \Delta^+ \times (\Delta^+ \cup \{0\})$.*

Given $(G, P, \phi_0 \in \mathbb{V}_\mu)$, classify (wrt P-action) all $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ with $\mathfrak{s} = \mathrm{gr}(\mathfrak{f}) = \mathfrak{a}^{\phi_0}$.

Strategy:

1. WLOG, pass to the minimal twistor space. Then $\mathfrak{f}^1 = 0$.
2. Pick H_0 as in Lemma. Use P_+-action to normalize to that $H_0 \in \mathfrak{f}^0$.
3. $\ker(\mu) \subset \mathfrak{f}^0$: Set $H' := H_0' + H_+' \in \mathfrak{f}^0$ for $H_0' \in \ker(\mu)$, $H_+' \in \mathfrak{g}^1$. Then

$$[H_0, H']_{\mathfrak{f}} = [H_0, H'] = [H_0, H_+'] \in \mathfrak{g}_+ \cap \mathfrak{f} = \mathfrak{f}^1 = 0 \quad \overset{(L2)}{\Rightarrow} \quad H_+' = 0.$$

General case - 3

4. Show $\boxed{\mathfrak{f} = \mathfrak{a}}$ as filtered subspaces of $\mathfrak{g}$.

$$\mathfrak{a} = \mathfrak{g}_- \oplus \mathfrak{a}_0, \quad \mathfrak{a}_0 := \ker(\mu) \oplus \bigoplus_{\gamma \in \Delta(\mathfrak{g}_{0,\leq 0})} \mathfrak{g}_\gamma$$

$$\mathfrak{a}^\perp := \ker(\mu)^\perp \oplus \mathfrak{g}_{0,+} \oplus \mathfrak{g}_+$$

Have $\mathfrak{f} \ni x = a + \mathfrak{d}(a)$ for $\mathfrak{d} : \mathfrak{a} \to \mathfrak{a}^\perp$ of positive degree; admissible weights are multiples of μ. But by (L1), μ has coeffs of opposite sign!

- $\mathfrak{d}|_{\mathfrak{g}_-} = 0$: immediate - all weights of $\mathfrak{g}_-^* \otimes \mathfrak{a}^\perp$ are non-negative.
- $\mathfrak{d}|_{\mathfrak{a}_0} = 0$: Bit more technical part – see my article.

5. Determine $\kappa \in \ker(\partial^*)_+ \subset \bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{g}$. Get canonical curved model.

- $\mathfrak{f}^0 \cdot \kappa = 0$. Want 2-cochain wts $\sigma = r\mu = \alpha + \beta + \gamma$ with

$$\alpha, \beta \in \Delta(\mathfrak{g}_+) \text{ distinct}, \quad \gamma \in \Delta \cup \{0\}, \quad \boxed{r \geq 1}.$$

- Hw of $\mathfrak{g}$ is $\lambda = \sum_i n_i \alpha_i$, $\boxed{n_i > 0, \forall i}$. Have $-\lambda \leq \gamma < \sigma = r\mu$. Have $\mu = -(jk) \bullet \lambda \equiv -\lambda \operatorname{mod} \{\alpha_j, \alpha_k\}$. Apply Z_i for $i \neq j, k$: $-n_i \leq rZ_i(\mu) = -rn_i$. Thus, $\boxed{r \leq 1}$, so $\boxed{r = 1}$.
- $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \cong \mathbb{V}_\mu$ irrep, $\kappa = c\phi_0$, $c \neq 0$. Wlog $\kappa = \phi_0$ over $\mathbb{C}$.

Summary

- We used a Cartan-theoretic approach to establish a local uniqueness result for submax sym models.
- Coordinate models associated to some of these canonical curved models are known, e.g. in the settings of
  - projective structure $(M^n, [\nabla])$, $n \geq 3$ (Egorov 1951);
  - split-conformal structures $(M^n, [g])$, $n \geq 4$ (Casey–Dunajski–Tod 2013, Kruglikov–T. 2014);
  - G-contact structures (T. 2018);
  - C_3-Monge (Anderson–Nurowski 2017);
  - Legendrian contact structures (Doubrov–Medvedev–T. 2020)
- Advantages of the Cartan-theoretic approach:
  - Efficient / uniform classification strategy.
  - Take advantage of basic rep theory of $\mathfrak{g}$.