Poisson transforms adapted to BGG-complexes

Christoph Harrach

University of Vienna

Symmetry, **C**urvature **R**eduction, and **E**quiv**A**lence **M**ethods
August 20, 2021

Overview

1. Parabolic geometries v.s. symmetric spaces
   - BGG-complexes on homogeneous parabolic geometries
   - Intertwining operators adapted to BGG-complexes
2. Poisson transforms
   - Construction of PT
   - PT and Differential operators
   - Poisson transforms for complex hyperbolic space

Conformal compactification

Let (S^{n+1}, c) be the conformal sphere of dimension $n+1$ with the natural action of $\hat{G} := \mathrm{SO}_0(n+2, 1)$. Breaking the symmetry via a tractor $I^A \in \Gamma(\mathcal{T})$ with $I^2 = 1$ we obtain a natural action of the group $G = \mathrm{Stab}_{\hat{G}}(I)$ isomorphic to $\mathrm{SO}_0(n+1, 1)$ on S^{n+1}.

Let $K \subset G$ maximal compact and $P \subset G$ parabolic. The two open G-orbits $S_\pm$ are isomorphic to the real hyperbolic space G/K with common boundary S_0 being the closed G-orbit isomorphic to the conformal n-sphere G/P.

More generally: G connected semisimple Lie group, $K \subset G$ maximal compact subgroup, $P \subset G$ parabolic subgroup, G/K of non-compact type. Then we can view G/P as (part of) the boundary of the symmetric space G/K at infinity.

The topology and geometry of these two spaces are quite different:

- G/K: complete Riemannian manifold; invariant inner products and connections on every vector bundle; various local invariants and natural differential operators.
- G/P: compact manifold; local invariants and natural differential operators are rare

The boundary relation was exploited to find joint eigenfunctions of invariant DO on G/K. However: the geometry of G/P was mostly disregarded.

One important example of natural differential operators on parabolic geometries is given by BGG-sequences. We recall their construction in the case of homogeneous parabolic geometries G/P.

BGG-complex I

Let $\mathbb{V}$ be a G-representation and $V := G \times_P \mathbb{V}$ its associated vector bundle. Then V naturally carries a flat G-invariant connection ∇^V, inducing a family of covariant exterior derivatives

$$d^V \colon \Omega^k(G/P, V) \to \Omega^{k+1}(G/P, V)$$

with $d^V \circ d^V = 0$.

The cotangent bundle $T^*(G/P)$ is naturally a bundle of Lie algebras. Thus, the differentials in Lie algebra homology induce G-equivariant bundle maps

$$\partial^* = \partial^*_k \colon \Lambda^k T^*(G/P) \otimes V \to \Lambda^{k-1} T^*(G/P) \otimes V,$$

called the *Kostant codifferential*. Since $\ker(\partial^*_k) \supset \operatorname{im}(\partial^*_{k+1})$ are G-subbundles we can define the G-bundle

$$\mathcal{H}_k(G/P, V) := \ker(\partial^*_k)/\operatorname{im}(\partial^*_{k+1}).$$

BGG-complex II

For all k there is a unique natural differential operator

$$L = L_k \colon \Gamma(\mathcal{H}_k(G/P, V)) \to \Gamma(\ker(\partial^*))$$

with $\pi \circ L = 0$ and $\partial^* \circ d^V \circ L = 0$ ("splitting operator").

$$
\begin{array}{ccccccccc}
\cdots & \underset{\partial^*}{\overset{d^V}{\rightleftarrows}} & \Omega^k(G/P,V) & \underset{\partial^*}{\overset{d^V}{\rightleftarrows}} & \Omega^{k+1}(G/P,V) & \underset{\partial^*}{\overset{d^V}{\rightleftarrows}} & \cdots \\
 & & \cup\!\!| & & \cup\!\!| & & \\
\cdots & \overset{d^V \circ L}{\nearrow} & \Gamma(\ker(\partial^*)) & \overset{d^V \circ L}{\nearrow} & \Gamma(\ker(\partial^*)) & \overset{d^V \circ L}{\nearrow} & \cdots \\
 & & \pi \downarrow \; \uparrow L & & \pi \downarrow \; \uparrow L & & \\
\cdots & \underset{D^V_{k-1}}{\dashrightarrow} & \Gamma(\mathcal{H}_k(G/P,V)) & \underset{D^V_k}{\dashrightarrow} & \Gamma(\mathcal{H}_{k+1}(G/P,V)) & \underset{D^V_{k+1}}{\dashrightarrow} & \cdots
\end{array}
$$

The operator $D^V_k := \pi \circ d^V \circ L_k$ is the k-th BGG-operator and the lower line is the BGG-complex.

Intertwining operators

We want to relate the BGG-complex on G/P to differential forms on G/K as follows:

Let $W \to G/K$ be a natural vector bundle. Assume there exists a smooth intertwining operator

$$\Phi\colon \Omega^k(G/P, V) \to \Omega^\ell(G/K, W).$$

which is trivial on $\operatorname{im}(\partial^*)$ and $\operatorname{im}(d^V\partial^*)$. Then Φ factors to a G-equivariant map

$$\underline{\Phi}\colon \Gamma(\mathcal{H}_k(G/P, V)) \to \Omega^k(G/K, W)$$

which satisfies:

1. For $\sigma \in \Gamma(\mathcal{H}_k(G/P, V))$ we have $\underline{\Phi}(\sigma) := \Phi(\alpha)$ for any $\alpha \in \pi^{-1}(\sigma)$,
2. For $\tau \in \Gamma(\mathcal{H}_{k-1}(G/P, V))$ we have $\underline{\Phi}(D^V_{k-1}\tau) = \Phi(d^V\beta)$ for any $\beta \in \pi^{-1}(\tau)$.

Tractor calculus on G/K I

For relating the BGG-complex to geometry on G/K we need to consider tractor bundles and define the differential operators induced by the tractor connection.

Let $\mathbb{W}$ be an irreducible G-representation and $W := G \times_K \mathbb{W}$ the associated vector bundle. This comes with the tractor connection ∇^W and the induced covariant exterior derivative d^W on $\Omega^\bullet(G/K, W)$.

Moreover, there is a unique inner product on $\mathbb{W}$ which is compatible with the Cartan involution θ, i.e.

$$\langle X \cdot w_1, w_2 \rangle = -\langle w_1, \theta(X) \cdot w_2 \rangle \qquad X \in \mathrm{Lie}(G), w_1, w_2 \in \mathbb{W}.$$

This induces a G-invariant bundle metric on $\Lambda^* T^*(G/K) \otimes W$ and thus a Hodge star operator $*^W$ as well as an L^2-inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\Omega^\bullet(G/K, W)$.

Define the *covariant codifferential* δ^W to be the formal adjoint of d^W with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ and the *covariant Laplace* $\Delta^W := d^W \delta^W + \delta^W d^W$.

Intertwining operators adapted to BGG-complexes

Theorem

Let $\mathbb{V}$ be an irreducible G-representation and define $V_K := G \times_K \mathbb{V}$ and $V_P := G \times_P \mathbb{V}$. Let

$$\Phi\colon \Omega^k(G/P, V_P) \to \Omega^\ell(G/K, V_K)$$

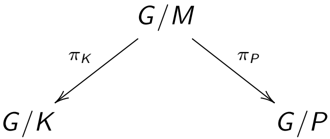
be a smooth intertwining operator. Then the following are equivalent:

1. $\Delta^{V_K} \circ \Phi = 0$
2. $\Phi \circ \partial^* = 0$ *and* $\Phi \circ d^{V_P} \circ \partial^* = 0$.

In this case we say that Φ is BGG-compatible.

Construction of Poisson transforms I

Via the Iwasawa decomposition we obtain $G/K \times G/P \cong G/M$ with $M := K \cap P$ and thus a double fibration



with G-equivariant projections.

The product structure induces a pointwise decomposition

$$\Lambda^k T^*(G/M) \cong \bigoplus_{p+q=k} (\Lambda^p T^*(G/K)) \otimes (\Lambda^q T^*(G/P)).$$

Thus, we have a natural notion of a (p,q)-form on G/M. In particular, for $n := \dim(G/P)$ we can integrate (ℓ, n)-forms over the compact fibre of π_K, which is isomorphic to G/P.

Construction of Poisson transforms II

Fix a differential form $\phi \in \Omega^{\ell,n-k}(G/M)$. For $\alpha \in \Omega^k(G/P)$ we construct $\Phi(\alpha) \in \Omega^\ell(G/K)$ as follows:

1. Consider the pullback $\pi_P^*\alpha \in \Omega^{0,k}(G/M)$.
2. Form the wedge product $\phi \wedge \pi_P^*\alpha \in \Omega^{\ell,n}(G/M)$,
3. Integrate over G/P.

Lemma

If ϕ is G-invariant, then Φ is G-equivariant.

Definition (Poisson transform)

For all $\phi \in \Omega^{\ell,n-k}(G/M)^G$ we call the intertwining operator

$$\Phi\colon \Omega^k(G/P) \to \Omega^\ell(G/K), \qquad \alpha \mapsto \int_{G/P} \phi \wedge \pi_P^*\alpha.$$

the *Poisson transform* associated to the *Poisson kernel* ϕ.

Reduction to representation theory

A Poisson kernel ϕ is a G-invariant differential form on G/M and thus determined by its value in any point.

Let $\mathfrak{g}$ and $\mathfrak{m}$ be the Lie algebras of G and M, respectively. Then

$$\phi(eM) \in \Lambda^\bullet(\mathfrak{g}/\mathfrak{m})^*$$

is M-invariant.

Theorem

There is a bijective correspondence between Poisson transforms

$$\Phi\colon \Omega^k(G/P) \to \Omega^\ell(G/K)$$

and the set of M-invariant elements in $\Lambda^{\ell,n-k}(\mathfrak{g}/\mathfrak{m})^*$.

In particular, we can construct smooth intertwining operators by determining invariant elements in finite dimensional representations of a reductive Lie group.

Poisson transforms and differential operators I

We define the following operators on $\Omega^\bullet(G/M)$:

- The derivative d on $\Omega^\bullet(G/M)$ splits into partial derivatives $d = d_K + d_P$, where d_K raises the first degree and d_P the second.
- The Hodge star $*$ on $\Omega^\bullet(G/K)$ induces

$$*_K\colon \Omega^{p,q}(G/M) \to \Omega^{\dim(G/K)-p,q}(G/M).$$

- Define $\delta_K := (-1)^p *_K^{-1} d_K *_K$ and $\Delta_K := d_K\delta_K + \delta_K d_K$.
- The Kostant codifferential ∂^* on $\Omega^\bullet(G/P)$ induces

$$\partial_P^*\colon \Omega^{p,q}(G/M) \to \Omega^{p,q-1}(G/M).$$

All these operators are G-equivariant, so they induce M-equivariant maps on $(\Lambda^\bullet(\mathfrak{g}/\mathfrak{m})^*)^M$.

Poisson transforms and differential operators II

Theorem

Let $\Phi\colon \Omega^k(G/P) \to \Omega^\ell(G/K)$ be a Poisson transform with associated kernel ϕ.

1. *The compositions $\Phi \circ d$ and $\Phi \circ \partial^*$ are again Poisson transforms with associated kernels $(-1)^{n-k+\ell} d_P \phi$ and $(-1)^{n-k+\ell+1} \partial_P^* \phi$, respectively.*
2. *The compositions $d \circ \Phi$, $* \circ \Phi$, $\delta \circ \Phi$ and $\Delta \circ \Phi$ are again Poisson transforms with associated kernels $d_K \phi$, $*_K \phi$, $\delta_K \phi$ and $\Delta_K \phi$, respectively.*

In particular, we can design intertwining operators adapted to BGG-complexes via computations in finite dimensional representations.

Let $G = \mathrm{SU}(n+1,1)$ so that $G/K \cong H^{n+1}_{\mathbb{C}}$ is the complex hyperbolic space and $G/P \cong S^{2n+1}$ is the CR-sphere.

In this case, the BGG-complex on G/P coincides with the Rumin complex. Explicilty, let $H \subset T(G/P)$ be the contact subbundle and put $Q := T(G/P)/H$. Then for all $1 \leq k \leq 2n$ we have the short exact sequence

$$0 \longrightarrow \Lambda^{k-1}H^* \otimes Q^* \longrightarrow \Lambda^k T^*(G/P) \longrightarrow \Lambda^k H^* \longrightarrow 0.$$

The Kostant codifferential induces a bundle map

$$\underline{\partial}^* : \Lambda^k H^* \to \Lambda^{k-2}H^* \otimes Q$$

which is surjective for $k \leq n$ and injective for $k \geq n+1$. Thus, the homology bundles $\mathcal{H}_k$ are subbundles of $\Lambda^k H^*$ for $k \leq n$ and quotients of $\Lambda^{k-2}H^* \otimes Q$ for $k \geq n+1$.

Theorem (Čap, H., Julg, 2020)

Let $G = \mathrm{SU}(n+1,1)$, $K = U(n+1)$ and $P \subset G$ parabolic.

1. *If $p+q \leq n$, there is a unique BGG-compatible Poisson transform $\Omega^{p+q}(G/P,\mathbb{C}) \to \Omega^{p,q}(G/K)$ (up to multiples). The image of the induced map*

$$\underline{\Phi}_{p,q} \colon \Gamma(\mathcal{H}_{p+q} \otimes \mathbb{C}) \to \Omega^{p,q}(G/K)$$

consist of harmonic, coclosed and primitive (p,q)-forms.

2. *If $p+q \geq n+1$, there is a 2-parameter family of Poisson transforms $\Omega^{p+q}(G/P,\mathbb{C}) \to \Omega^{p,q}(G/K)$. The image of the induced maps*

$$\Phi^{\alpha,\beta}_{p,q} \colon \Gamma(\mathcal{H}_{p+q} \otimes \mathbb{C}) \to \Omega^{p,q}(G/K)$$

consist of harmonic and coprimitive (p,q)-forms and satisfy $\partial^* \Phi^{\alpha,*}_{p+1,q} = \overline{\partial}^* \Phi^{*,\alpha}_{p,q+1}$.

Theorem (Čap, H., Julg, 2020)

For all $0 \leq k \leq 2n+1$ *there is a family of BGG-compatible Poisson transforms* $\Omega^k(G/P) \to \Omega^k(G/K)$ *so that the induced maps*

$$\underline{\Phi}_k : \Gamma(\mathcal{H}_k) \to \Omega^k(G/K)$$

satisfy

1. *their image consist of harmonic and coclosed differential forms which are primitive for* $0 \leq k \leq n$ *and coprimitive for* $n+1 \leq k \leq 2n+1$
2. *for the* k*-th BGG-operator* D_k *we have*

$$d \circ \underline{\Phi}_k = c_k \underline{\Phi}_{k+1} \circ D_k$$

with

$$c_k = \begin{cases} n-k+1 & k \leq n \\ n-k-1 & k \geq n+1. \end{cases}$$

References

- A. Čap, C. Harrach, P. Julg: *A Poisson transform adapted to the Rumin complex*, to appear in J. Topol. Anal.
- C. Harrach: *Poisson transforms adapted to BGG-complexes*, Diff. Geom. Appl., Vol. 64, 2019, pp. 92-113
- C. Harrach: *Poisson transforms for differential forms*, Arch. Math. (Brno), Tomus 52 (2016), pp. 303-311
- P.-Y. Gaillard: *Transformation de Poisson de formes differentielles. Le cas de l'espace hyperbolique.* Comment. Math. Helvetici, Vol. 61 (1986), Birkhäuser.