Second order PDEs, conformal structure on solutions and dispersionless integrability

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Research program GRIEG
SCREAM workshop 2021

Jet formalism and conformal structures

Let $J^\ell M \to M$ be the bundle, whose points are ℓ-jets of functions $u : M \to \mathbb{R}$. A choice of coordinates x^i on M leads to coordinates (x^i, u_α) on $J^\ell M$, with α being a multi-index of length $|\alpha| \leq \ell$. Note that $\pi_{\ell,\ell-1} : J^\ell M \to J^{\ell-1}M$ is an affine bundle modelled on $S^\ell T^*M$. In particular, the fiber of $\pi_{2,1}$ consists of quadrics on TM.

A (nonlinear) differential operator of order $\leq \ell$ on M is a function $F \in C^\infty(J^\ell M) \subset C^\infty(J^\infty M)$. It defines a PDE (system) $\mathcal{E} = \{F = 0\} \subset J^\ell M$, as well as its prolongation

$$\mathcal{E}^{(\infty)} = \{D_\alpha F = 0\} \subset J^\infty M.$$

Restriction of the $dF \in \Omega^1(J^\infty M)$ to $\pi^{-1}_{\ell,\ell-1}(*)$ is a homogeneous polynomial of degree ℓ on $\pi^*_\infty T^*M$ called the symbol of F:

$$\sigma_F = \sum_{|\alpha|=\ell} (\partial_{u_\alpha} F)\partial_\alpha \in \Gamma(\pi^*_\infty S^\ell TM).$$

Its conformal class depends only on $\mathcal{E}$.

Symbol and Characteristic variety

For a solution $u \in \mathrm{Sol}(\mathcal{E})$ we identify $M_u \simeq j_\infty u(M) \subset J^\infty M$. Restriction of the symbol σ_F to M_u is a symmetric ℓ-vector

$$\sigma_{F|u} \in \Gamma(S^\ell TM_u).$$

The characteristic variety is the zero locus of the symbol: its fiber at $\boldsymbol{x} \in M_u$ is the projective variety

$$\mathrm{Char}(\mathcal{E}, u)_{\boldsymbol{x}} = \{[\theta] \in \mathbb{P}(T^*_{\boldsymbol{x}} M_u) : \sigma_F(\theta) = 0\}.$$

In coordinates to compute the characteristic variety one converts the symbol of linearization of F ("Fourier transform": $\partial_i \mapsto p_i$)

$$\sigma_F = \sum_{|\alpha|=\ell} \sigma_\alpha(u)\partial_\alpha \quad \text{to the polynomial} \quad \sigma_F(p) = \sum_{|\alpha|=\ell} \sigma_\alpha(u) p^\alpha$$

where $p = (p_1, \dots, p_d)$ is a coordinate on the fiber of $T^* M_u$ and $p^\alpha = p_1^{i_1} \cdots p_d^{i_d}$ for a multi-index $\alpha = (i_1, \dots, i_d)$.

PDEs of the second order

For an operator F of the second order

$$\sigma_F = \sum_{i\leq j} \frac{\partial F}{\partial u_{ij}} \partial_i\partial_j = \sum_{i,j} \sigma_{ij}(u)\partial_i \otimes \partial_j,$$

where $\sigma_{ij}(u) = \frac{1}{2}(1+\delta_{ij})\partial_{u_{ij}} F$.

Thus, for second order PDEs the characteristic variety is a field of quadrics, which we assume nondegenerate, i.e. $\det(\sigma_{ij}(u)) \neq 0$.

The nondegeneracy of σ_F implies that is inverse

$$g_F = \sum_{ij} g_{ij}(u) dx^i dx^j, \qquad (g_{ij}(u)) = (\sigma_{ij}(u))^{-1},$$

defines a symmetric bilinear form on $T_{\boldsymbol{x}} M_u$. The canonical conformal structure $c_F = [g_F]$ on solutions of $\mathcal{E}$ is a base for geometric approach to integrability.

Nondegeneracy condition: geometric interpretation

Freezing 1-jet of u, equation $F=0$ determines a hypersurface $\mathcal{E}'$ in the Lagrangian Grassmannian $\Lambda \overset{\text{loc}}{\simeq} \mathbb{R}^{\binom{d+1}{2}}(u_{ij})$, and $\forall\ L \in \mathcal{E}'$ the tangent $T_L\Lambda \simeq S^2L^* \simeq \mathbb{R}^{\binom{d+1}{2}}(v_{ij})$ contains two ingredients:

(a) the Veronese cone $V=\{p\odot p : p \in L^*\}$ of rank 1 matrices;
(b) tangent hyperplane $T_L\mathcal{E}'$ defined by the linearized equation

$$\sum_{i\leq j}\frac{\partial F}{\partial u_{ij}}\,v_{ij}=0.$$

The non-degeneracy of $\mathcal{E}$ means $T_L\mathcal{E}'$ is not tangential to V.

In fact, because $T_pV = p\odot L^*$ for $p\neq 0$, the kernel of the quadratic form

$$\sigma_F(p)=\sum_{i\leq j}\frac{\partial F}{\partial u_{ij}}\,p_ip_j$$

on $L^*\simeq T^*_{\boldsymbol{x}}M$ is

$$\operatorname{Ker}\sigma_F=\{p\in L^* : T_pV\subset T_L\mathcal{E}\}.$$

Dispersionless Lax pairs (dLp)

Definition

A dispersionless Lax pair of order k is a bundle $\hat{\pi} \colon \hat{M}_u \to M_u$ (correspondence space), whose fibres are connected curves, together with a rank 2 distribution $\hat{\Pi} \subseteq T\hat{M}_u$ such that:

- $\forall\ \hat{\boldsymbol{x}} \in \hat{M}_u$, $\hat{\Pi}(\hat{\boldsymbol{x}})$ depends only on $j^k_{\boldsymbol{x}} u$,
- $\hat{\Pi}$ is transversal to the fibres of $\hat{\pi}$,
- Frobenius integrability $[\Gamma(\hat{\Pi}), \Gamma(\hat{\Pi})] = \Gamma(\hat{\Pi})$ holds mod $\mathcal{E}$.

Thus we have the twistor fibration

$$\begin{array}{ccc} & \hat{M}_u^{d+1} & \\ {\scriptstyle \mathbb{P}^1} \swarrow & & \searrow {\scriptstyle \hat{\Pi}^2} \\ M_u^d & & \mathcal{T}w^{d-1} \end{array}$$

2-plane congruence $\Pi(\hat{\boldsymbol{x}}) := (d\hat{\pi})_{\hat{\boldsymbol{x}}}(\hat{\Pi}) \subset T_{\boldsymbol{x}} M$ is parametrized by a spectral parameter λ ($\mathbb{P}^1$ coordinate).

The characteristic condition

Theorem (D. Calderbank & BK 2016-2018)

A dLp $\hat{\Pi}$ *is characteristic for* $\mathcal{E}$*, i.e.* $\forall\ u \in \mathrm{Sol}(\mathcal{E})$, $\hat{\boldsymbol{x}} \in \hat{M}_u$ *and* $\theta \in \mathrm{Ann}(\Pi(\hat{\boldsymbol{x}})) \subseteq T^*_{\boldsymbol{x}} M_u$ *we have* $\sigma_F(\theta) = 0 \Leftrightarrow [\theta] \in \mathrm{Char}(\mathcal{E})$.

This means that for each solution u and $\hat{\boldsymbol{x}} \in \hat{M}_u$, $\Pi(\hat{\boldsymbol{x}})$ is a coisotropic 2-plane for the conformal structure c_F. Such 2-planes can only exist for $2 \leq d \leq 4$: for $d = 2$ the condition is vacuous; for $d = 3$ the coisotropic 2-planes at each point $\boldsymbol{x}$ form a rational conic $\mathbb{P}^1$; for $d = 4$ they form a disjoint union of two rational curves $2 \times \mathbb{P}^1$, the so-called α-planes and β-planes.

The passage from a 2-plane congruence $\Pi = \langle X, Y \rangle$ to a dLp can be understood as a lift, with respect to the projection $\hat{\pi}$:

$$\hat{X} = X + m\,\partial_\lambda, \quad \hat{Y} = Y + n\,\partial_\lambda.$$

The resulting rank 2 distribution $\hat{\Pi} = \langle \hat{X}, \hat{Y} \rangle$ on $\hat{M}_u$ is integrable mod $\mathcal{E}$ (on-shell), but not identically (off-shell).

Integrable background geometry for $d = 3$: Example

Einstein-Weyl structure on M^3 is a conformal structure $[g]$, 1-form ω and a torsion-free linear connection $\mathbb{D}$ that satisfy for some $\Lambda \in C^\infty(M)$:

$$\mathbb{D}g = \omega \otimes g, \quad \mathrm{Ric}^{\mathrm{sym}}_{\mathbb{D}} = \Lambda\, g.$$

Example (3D: EW from dKP)

The dispersionless Kadomtsev-Petviashvili equation:

$$u_{tx} = (uu_x)_x + u_{yy}.$$

EW structure on solutions and dLp $\hat{\Pi} \subseteq T\hat{M}_u$, $\hat{M}_u \simeq \mathbb{R}^4(x, y, t, \lambda)$:

$$g = 4\, dx\, dt - dy^2 + 4\, u\, dt^2, \quad \omega = -4u_x\, dt,$$

$$\hat{X} = \partial_y - \lambda\partial_x + u_x\partial_\lambda,\ \hat{Y} = \partial_t - (\lambda^2 + u)\partial_x + (u_x\lambda + u_y)\partial_\lambda.$$

This gives a large family of EW structures parametrized by solutions of the Gibbons-Tsarev system.

Integrable background geometry for $d = 4$: Example

In 4D, the key invariant of a conformal structure $[g]$ is its Weyl tensor W. Its self-dual and anti-self-dual parts are $W_\pm = \frac{1}{2}(W \pm *W)$, where $*$ is the Hodge star operator

$$*W^i_{jkl} = \frac{1}{2}\sqrt{\det g}\; g^{ia} g^{bc} \epsilon_{ajbd} W^d_{ckl}.$$

A conformal structure is said to be half-flat if W_- or W_+ vanishes. The SD/ASD conditions switch under change of orientation.

Example (4D: Self-dual gravity)

The second Plebanski equation:

$$u_{xz} + u_{yt} + u_{xx}u_{yy} - u_{xy}^2 = 0.$$

SD structure on solutions, $M_u \simeq \mathbb{R}^4(x, y, z, t)$:

$$g = dx\,dz + dy\,dt - u_{yy}dz^2 + 2u_{xy}dz\,dt - u_{xx}dt^2,$$

$$\hat{X} = \partial_t + u_{xx}\partial_y - (u_{xy} - \lambda)\partial_x,\ \hat{Y} = \partial_z - (u_{xy} + \lambda)\partial_y + u_{yy}\partial_x.$$

Two theorems on integrability

Let $\mathcal{E} : F = 0$ be a nondegenerate determined PDE of the second order with the corresponding conformal structure c_F.

Theorem (E. Ferapontov & BK 2014)

The integrability of $\mathcal{E}$ by the method of hydrodynamic reductions is equivalent to

3D: *the Einstein–Weyl property for c_F on any solution of the PDE*;

4D: *the self-duality property for c_F on any solution of the PDE.*

Theorem (D. Calderbank & BK 2018)

The integrability of $\mathcal{E}$ via a nondegenerate dispersionless Lax pair is equivalent to

3D: *the Einstein–Weyl property for c_F on any solution of the PDE*;

4D: *the self-duality property for c_F on any solution of the PDE.*

The Monge-Ampère property

Monge-Ampère equation $\mathcal{E}: F = 0$ is a second order PDE that is linear combinations of minors of the Hessian matrix $d^2u = (u_{ij})_{d\times d}$ with coefficients being arbitrary functions on J^1M. Freezing the 1-jet, the equation $\mathcal{E}$ can be written in the form

$$u_{00} = f(u_{01}, \dots, u_{0n}, u_{11}, u_{12}, \dots, u_{nn}), \quad d = n+1. \qquad (\dagger)$$

Theorem (E. Ferapontov, BK & V.Novikov 2019)

Equation $(\dagger)$ is of Monge-Ampère type if and only if d^2f is a linear combination of the 2nd fundamental forms of the Plücker embedding of the Lagrangian Grassmannian Λ restricted to the hypersurface defined by $(\dagger)$. This property is characterized by $N(n) = \frac{1}{24}n(n+1)(n+2)(n+7)$ relations, forming an involutive second-order quasilinear PDE system for f.

$$\mathcal{E}^{\binom{d+1}{2}-1} \subset \Lambda \subset \mathbb{P}^{p(d)-1}, \quad p(n) = \frac{2(2n+1)!}{n!(n+2)!}.$$

Hirota type integrable systems in 4D

For Hirota type PDEs of the second order $F(u_{ij})=0$ in 4D integrability implies the Monge-Ampère property as proved by Ferapontov-BK-Novikov (2019).

Integrabile Monge-Ampère equations of Hirota type were investigated by Doubrov-Ferapontov (2010). The classification over $\mathbb{C}$ consists of 1 linear ultra-wave PDE and 5 versions of the Plebanski equation, obtained by deformations of the general heavenly equation

$$\alpha u_{12}u_{34}+\beta u_{13}u_{24}+\gamma u_{14}u_{23}=0, \quad \alpha+\beta+\gamma=0.$$

Here integrability is understood both in hydrodynamic and Lax sense. Note that for $d=4$ no additional ingredient (connection) is required for the lift, so the Lax pair (dLp) is uniquely obtained from the conformal structure and the equation.

General integrable systems in 4D

For general (translationally non-invariant) PDE integrability in hydrodynamic sense is not yet understood, hence integrability is considered only as the existence of a dispersionless Lax pair.

For general (translation non-invariant) PDEs in 4D we have:

Theorem (S.Berjawi, E.Ferapontov, BK, V.Novikov 2020)

Every nondegenerate equation of the second order such that c_F is half-flat on every solution is of Monge-Ampère type.

Freezing 1-jet of a solution yields a PDE that is linearizable or contact equivalent to one of five heavenly type equations.

Classification of the latter PDEs is out of reach at present.

Example of integrable deformation: 4D

There are several contact non-equivalent deformations of the heavenly equations, preserving integrability. Consider, for instance, the following family ($f = 1$ yields the 1st Plebansky equation)

$$u_{xz}u_{yt} - u_{xt}u_{yz} = f(\boldsymbol{x}, u, u_{\boldsymbol{x}}).$$

The corresponding conformal structure is

$$g = u_{xz}dx\,dz + u_{xt}dx\,dt + u_{yz}dy\,dz + u_{yt}dy\,dt.$$

Here $\det g = f^2/16$, the choice $\sqrt{\det g} = f/4$ has implications: $W_+ = 0 \Rightarrow f = 0$; $W_- = 0 \Rightarrow$ [consistent system of PDEs]. Solving/simplifying it mod equivalence group, the general branch:

$$f = \frac{u_x u_z}{(y-t)^2}.$$

The corresp dLp: $$\begin{cases} \hat{X} = u_{xz}\partial_t - u_{xt}\partial_z + \frac{u_z(\lambda-y)}{(\lambda-t)(y-t)}\partial_x, \\ \hat{Y} = u_{yz}\partial_t - u_{yt}\partial_z + \frac{u_z(\lambda-y)}{(\lambda-t)(y-t)}\partial_y. \end{cases}$$

Translation non-invariance

The above deformations explicitly involve independent variables, but can be made translation-invariant by a contact transformation. The necessary and sufficient condition for this in general $\dim = d$ is the existence of rank d comm algebra of contact symmetries.

Proposition

There exist nondegenerate integrable PDEs in dimensions $d = 3, 4$ that are not contact equivalent to any translationally invariant eqn.

For $d = 3$ such is an integrable deformation of the Veronese web equation (BK, A.Panasyuk 2017):

$$(x_1 - x_2)u_3u_{12} + (x_2 - x_3)u_1u_{23} + (x_3 - x_1)u_2u_{13} = 0. \qquad (\dagger)$$

It possesses dLp and its contact symmetry algebra

$$X_h = h(u)\partial_u,\ Y_0 = \partial_{x_1} + \partial_{x_2} + \partial_{x_3},\ Y_1 = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}$$

does not contain any three-dimensional Abelian subalgebra.
A generic combination of the LHS of $(\dagger)$ extends the phenomenon to $d = 4$, but for higher dimension this becomes impossible.

Integrable systems in higher dimensions $d > 4$

Attempts to generalise this result to higher dimensions meet an immediate obstacle: all known multi-dimensional $(d > 4)$ PDEs possessing a dispersionless Lax pair are degenerate.

For instance, both the 6-dimensional version of the second heavenly equation (Takasaki, Przanovski)

$$u_{15} + u_{26} + u_{13}u_{24} - u_{14}u_{23} = 0,$$

as well as the 8-dimensional generalisation of the general heavenly equation (Konopelchenko, Schief)

$$(u_{16}-u_{25})(u_{38}-u_{47})+(u_{27}-u_{36})(u_{18}-u_{45})+(u_{35}-u_{17})(u_{28}-u_{46}) = 0$$

have symbols σ_F of rank 4. We however have the following

Theorem (S.Berjawi, E.Ferapontov, BK, V.Novikov 2020)

Integrability of a rank 4 second order PDE in any dimension $d \geq 4$ via a non-trivial dLp implies the Monge-Ampère property.

3D: Weyl potential $\rightsquigarrow$ dLp

Let $\mathcal{E}$ be a second order PDE in 3D such that its conformal structure c_F has EW property, with Weyl covector $\omega = \omega_i \theta^i$. Then $\mathcal{E}$ is integrable and the corresponding dispersionless Lax pair can be calculated explicitly (no integration).

Let us introduce the so-called null coframe $\theta^0, \theta^1, \theta^2$ (it depends on a finite jet of a solution $u \in \mathrm{Sol}(\mathcal{E})$) such that

$$g_F = 4\theta^0\theta^2 - (\theta^1)^2.$$

Let V_0, V_1, V_2 be the dual frame, and let c_{ij}^k be the structure functions defined by commutator expansions $[V_i, V_j] = c_{ij}^k V_k$. The Lax pair is given by the vector fields

$$\hat{X} = V_0 + \lambda V_1 + m\partial_\lambda, \quad \hat{Y} = V_1 + \lambda V_2 + n\partial_\lambda,$$

where

$$m = (\tfrac{1}{2}c_{12}^1 - \tfrac{1}{4}\omega_2)\lambda^3 + (\tfrac{1}{2}c_{02}^1 - c_{12}^2 - \tfrac{1}{2}\omega_1)\lambda^2 + (\tfrac{1}{2}c_{01}^1 - c_{02}^2 - \tfrac{1}{4}\omega_0)\lambda - c_{01}^2,$$

$$n = -c_{12}^0\lambda^3 + (\tfrac{1}{2}c_{12}^1 - c_{02}^0 + \tfrac{1}{4}\omega_2)\lambda^2 + (\tfrac{1}{2}c_{02}^1 - c_{01}^0 + \tfrac{1}{2}\omega_1)\lambda + (\tfrac{1}{2}c_{01}^1 + \tfrac{1}{4}\omega_0)$$

General integrable systems in 3D

For Hirota type PDEs of the second order $F(u_{ij}) = 0$ in 3D integrability and Monge-Ampère property imply linearizability by a contact transformation. The general integrable equation is a modular form. The EW background structure is given by g_F and the following components of the Weyl covector

$$\omega_k = 2g_{kj}\mathcal{D}_{x^s}(g^{js}) + \mathcal{D}_{x^k}(\ln\det g_{ij}).$$

For general PDEs of second order $F(x^i, u, u_i, u_{ij}) = 0$ this formula is not applicable. Yet the EW structure can be determined.

Theorem (S.Berjawi, E.Ferapontov, BK, V.Novikov)

For nondegenerate non-Monge-Ampère equations of second order with EW property, the Weyl covector ω is algebraically determined.

Corollary

Under the above condition, the dispersionless Lax pair is algebraically determined by the equation.

Example of integrable deformation: 3D

Consider Monge-Ampère equations of the form

$$(u_{tt}-u)u_{xy}-(u_{xt}-u_x)(u_{yt}+u_y)=f(x,y,t,u,u_x,u_y,u_t).$$

For $f=4e^{2\rho t}$ this equation was derived by Dunajski and Tod in the context of hyper-Kähler metrics with conformal symmetry. Its conformal structure c_F and Weyl covector are:

$$g=(u\,dt+u_x dx-u_y dy-du_t)^2+4f\,dxdy;$$

$$\omega=2\Big(\frac{u_{xt}-u_x}{u_{tt}-u}dx-\frac{u_{yt}+u_y}{u_{tt}-u}dy\Big)+2R\Big(dt+\frac{u_{xt}-u_x}{u_{tt}-u}dx+\frac{u_{yt}+u_y}{u_{tt}-u}dy\Big),$$

where $R=\frac{\mathcal{D}_t f}{f}$. The EW requirement is a PDE system with solutions mod the equivalence giving 6 cases in addition to DT. The most general is:

$$f=c^2\frac{(u_x+u_t+u)(u_y+u_t-u)}{\cosh^2 c(x+y-t)}.$$

The generalised DT equation is quasi-linearisable: via a contact transformation it is a deformation of the Bogdanov equation.

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