Non-holonomic generalization(s) of Cayley's ruled cubic

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Outline

1. Cayley's ruled cubic
2. Non-holonomic version of Cayley's ruled cubic
3. Deformations of rational homogeneous varieties

Basic properties

1. **Cayley's ruled cubic** is a surface in P^3 given by the following equation (in affine coordinates):

$$z = xy - x^3/3.$$

2. Ruled by lines:

$$\{x = a, \quad z = ay - a^3/3\}.$$

3. Not smooth at infinity. Equation in projective coordinates is:

$$ZW^2 - XYW + X^3/3 = 0.$$

   Singular locus is $\{X = W = 0\}$.

4. Hyperbolic outsize the singularity locus, that is has non-degenerate second fundamental form of signature $(1, 1)$.

Advanced properties

1. As any hyperbolic surface, has two asymptotic directions at each (non-singular) point. The first family of asymptotic curves consists of lines. The second family of asymptotic curves consists of rational normal curves:

$$[1 : x : b + x^2/2 : bx + x^3/6].$$

2. Has 3-dim symmetry algebra of projective transformations:

$$\langle \partial_x + x\partial_y + y\partial_z, \quad \partial_y + x\partial_z, \quad x\partial_x + 2y\partial_y + 3z\partial_z \rangle.$$

3. In fact, any hyperbolic hypersurface in P^3 with 3-dim infinitesimal symmetry is locally equivalent Cayley's ruled cubic via projective transformations.

4. Note that a hyperbolic quardic has 6-dim symmetry isomorphic to $\mathfrak{so}(2,2)$. So, Cayley's ruled surface is *a (locally) unique submaximal model of hyperbolic surfaces in* P^3.

A bit of a history

VII. *A Memoir on Cubic Surfaces.* *By Professor* Cayley, *F.R.S.*

Received November 12, 1868,—Read January 14, 1869.

The twenty-three Cases of Cubic Surfaces—Explanations and Table of Singularities.
Article Nos. 1 to 13.

1. I designate as follows the twenty-three cases of cubic surfaces, adding to each of them its equation:

I	$=12,$	$(X, Y, Z, W)^3=0,$
II	$=12-C_2,$	$W(a, b, c, f, g, h)(X, Y, Z)^2+2kXYZ=0,$

XXI	$=12-3B_3,$	$WXZ+Y^3=0,$
XXII	$=3, S(1, 1),$	$WX^2+ZY^2=0,$
XXIII	$=3, S(\overline{1, 1}),$	$X(WX+YZ)+Y^3=0;$

PDEs approach to Cayley's ruled cubic

1. Consider a system of 2nd order linear PDEs:

$$u_{xx} = A_1 u_x + B_1 u_y + C_1 u$$
$$u_{yy} = A_2 u_x + B_2 u_y + C_2 u,$$

where A_i, B_i, C_i are functions of x and y.

2. Assume that the compatibility conditions are satisfied. Then this system has 4-dimensional solution space. Each solution is uniquely determined by u, u_x, u_y, u_{xy} at a point.
3. If $\{u_0, u_1, u_2, u_3\}$ is a basis in the solution space, then the surface $[u_0 : u_1 : u_2 : u_3]$ is a hyperbolic surface in P^3, whose asymptotic curves are given by $x = \text{const}$ and $y = \text{const}$.
4. Conversely, any hyperbolic surface in P^3 can be represented this way.
5. The trivial system $u_{xx} = u_{yy} = 0$ corresponds to a quadric. The system that corresponds to Cayley's ruled cubic is:

$$u_{xx} = u_y, \quad u_{yy} = 0.$$

Non-holonomic version of the above PDEs

1. replace ∂x and ∂_y in the above equations by non-commuting vector fields $X = \partial_x$ and $Y = \partial_y + x\partial_z$, so that $[X, Y] = \partial_z$.
2. Consider now linear systems of PDEs of the form:

$$X^2 u = A_1 X u + B_1 Y u + C_1 u$$
$$Y^2 u = A_2 X u + B_2 Y u + C_2 u,$$

   where $u = u(x, y, z)$ and A_i, B_i, C_i are functions of x, y and z.
3. Assume that the compatibility conditions are satisfied. Then this system has **8-dimensional** solution space. Each solution is uniquely determined by u, Xu, Yu, XYu, Zu, XZu, YZu, Z^2u at a point.
4. If $\{u_0, u_1, \dots, u_7\}$ is a basis in the solution space, then we get a 3-dim submanifold in P^7 given by $[u_0 : u_1 : \cdots : u_7]$.
5. How to characterize these submanifolds?

Flat case

1. Consider first the trivial system $X^2u = Y^2u = 0$.
2. It has the following solutions:

$$1, \quad x, \quad y, \quad z, \quad xy, \quad xz, \quad y(z-xy), \quad z(z-xy).$$

3. Computing the symmetry, one finds that it is 17-dimensional and has the following structure $\mathfrak{gl}(3,\mathbb{R}) \leftthreetimes \mathbb{R}^8$, where the second summand corresponds to the shifts along the solutions, and $\mathfrak{gl}(3,\mathbb{R})$ acts irreducibly on it $\Rightarrow$ equivalent to the adjoint action on $\mathfrak{sl}(3,\mathbb{R})$ (possibly with some weight to account for the action for the center).
4. In fact, this 3-dim submanifold in P^7 is nothing more than the highest weight (root) orbit of the adjoint action of $SL(3,R)$ on $P^7 = P(\mathfrak{sl}(3,\mathbb{R}))$.
5. Note that in the "holonomic" case the trivial system $u_{xx} = u_{yy} = 0$ also corresponds to the highest weight orbit of $SO(2,2)$ (or $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$) on $\mathbb{R}^4 = \mathbb{R}^2 \otimes \mathbb{R}^2$.

Deforming the flat case

1. Now modify the trivial system as $X^2 u = \epsilon Y u$, $Y^2 u = 0$.
2. Again, its solution space is 8-dim with a basis:

$$1, x, y + \epsilon x^2/2, z + \epsilon x^3/6, z - xy, xz + \epsilon x^4/12,$$
$$y(z - xy) + \epsilon x^2 (z/2 - xy/6) + \epsilon^2 x^5/60,$$
$$z(z - xy) + \epsilon x^3 (z/6 - xy/12) + \epsilon^2 x^6/360.$$

3. It gives us a certain deformation of the highest weight orbit of the adjoint representation of $SL(3, \mathbb{R})$.
4. The deformation is non-trivial: the symmetry algebra of the above equation is 13-dimensional. So, if we account for "trivial" symmetries (shifts along the solutions and scaling), then we get a 4-**dim symmetry algebra** of the 3-dim submanifold that corresponds to this system.

Characterizing the deformations

1. How to characterize the class of 3-dim submanifolds in P^7 that corresponds to the systems
$$X^2u = A_1Xu + B_1Yu + C_1u$$
$$Y^2u = A_2Xu + B_2Yu + C_2u$$
2. First, these are all embeddings of a 3-dim contact manifold M^3 with the contact distribution spanned by X and Y.
3. At each point $p \in M^3$ consider a (weak) osculating flag $\mathcal{O}_p$:
$$\mathcal{O}_p^{-1} = \langle p \rangle, \mathcal{O}_p^{-2} = T^{-1}M,$$
$$\mathcal{O}_p^{-i-1} = \mathcal{O}_p^{-i} + \underline{T^{-1}M}\left(\mathcal{O}_p^{-i}\right), \quad i \geq 2.$$
4. The Lie algebra $\operatorname{gr} T_pM^3$ naturally acts on $\operatorname{gr}\mathcal{O}_p$. Moreover, this action is equivalent to $\mathfrak{g}_-$ action on $\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ with the "Borel" grading shifted by -3.
5. So, all such $M^3 \subset P^7$ (a) carry a contact structure, (b) have constant embedding symbol ($\operatorname{gr} T_pM^3$-module $\operatorname{gr}\mathcal{O}_p$ is equivalent to $\mathfrak{g}_-$-module $\mathfrak{g}$).

Contact double fibrations

1. Similar to hyperbolic hypersurfaces in P^3, the contact submanifolds M^3 in P^7 with the above symbol (equivalent to $\mathfrak{g}_-$-module $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ carry a natural decomposition of the contact bundle $T^{-1}M$ into two lines. We call them asymptotic directions and define two families of *asymptotic curves* as integral curves of these directions.
2. For the flat case of the highest root adjoint orbit of $SL(3, \mathbb{R})$ both families of asymptotic curves turn out to be lines. So, we get the (non-holonomic) double ruling generalizing the double ruling of the hyperbolic quadrics in P^3.
3. In case of the submanifold M^3 that corresponds to the system $X^2u = Yu$, $Y^2u = 0$ one family still consists of lines, while the other family consists of rations curves in linear hypersurfaces of P^7. This generalizes the double fibration by lines and by rational curves for Caylay's ruled cubic.

Osculating embeddings of filtered manifolds into flag varieties

- Let M be a filtered manifold with a filtration $\{T^i M\}_{i<0}$. Let $\mathrm{Flag}_\alpha(V)$ be a flag manifold, α is a fixed multi-index:

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r < \dim V.$$

  It is naturally equipped with a structure of a filtered manifold.
- We say that a map $\varphi\colon M \to \mathrm{Flag}_\alpha(V)$ is *osculating*, if it preserves the filtrations. This is equivalent to the condition:

$$T^i \varphi(x)^j \subset \varphi(x)^{i+j} \text{ for all } i, j < 0.$$

- For each $x \in M$ this turns $\mathrm{gr}\,\varphi(x)$ into a graded module over $\mathrm{gr}\, TM$. We call it a *symbol of an osculating embedding*.
- In the following we assume that all these modules are equivalent to some fixed $\mathfrak{g}_-$-module V, where $\mathfrak{g}_-$ is then the (constant) symbol of M.

Rational homogeneous varieties

1. Let G be a semisimple Lie group and let V be its irreducible representation. Let $\mathfrak{g}$ be the Lie algebra of G. Fix a parabolic $P \subset G$. Let $\mathfrak{g} = \sum_{i=-\mu}^{\mu}$ be the corresponding grading of $\mathfrak{g}$, and let $e \in \mathfrak{g}_0$ be the grading element. Its action induces the compatible grading on V.
2. The action of G on the standard flag of V induces the osculating embedding $\varphi_0 \colon G/P \to \mathrm{Flag}_\alpha(V)$.
3. **Example:** the unique closed orbit (along with its osculating flag) of G acting irreducibly on $P(V)$. Such embeddings are know as *rational homogeneous varieties*.
4. We would like to consider *symbol-preserving* deformations of such rational homogeneous varieties with symmetry algebra of submaximal dimension.

Submaximal deformations

1. Cayley's ruled cubic is exactly such (unique up to projective equivalence) deformation of the highest weight orbit of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ acting on $\mathbb{R}^2 \otimes \mathbb{R}^2$.
2. The 3-dim contact manifold in P^7 that is given by the solution space of $X^2u = Yu$, $Y^2u = 0$ is exactly such (unique?) deformation of the highest root orbit of the adjoint representation of $SL(3,\mathbb{R})$.
3. (Hwang–Yamaguchi, Landsberg–Robles, D.–Machida–Morimoto) Invariants of such deformations are governed $H^1_+(\mathfrak{g}_-, \mathfrak{sl}(V)/\mathfrak{g})$, which can be computed via Kostant's theorem.
4. If G is complex simple, it is known that such cohomology may be non-zero only if G/P is one of P^ℓ, Q^ℓ, $F_{1,\ell}(\mathbb{C}^{\ell+1})$.
5. **The complete classification of submaximal deformations is open!**

References

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