Differential invariants of curves in G_2 flag varieties

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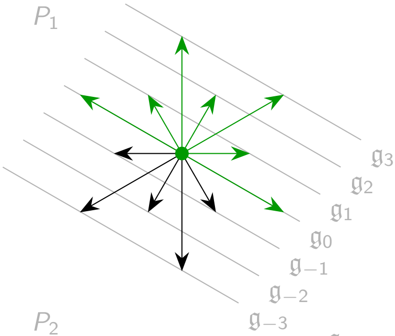
Introduction

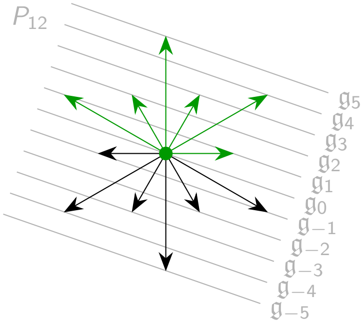
- The goal is to find differential invariants of regular unparametrized curves in G_2/P, where P is a parabolic subgroup.
- We compute the Hilbert function that counts the number of differential invariants for curves of constant type.
- We focus on integral curves and generic curves.
- Integral curves correspond to minimal orbits of the action

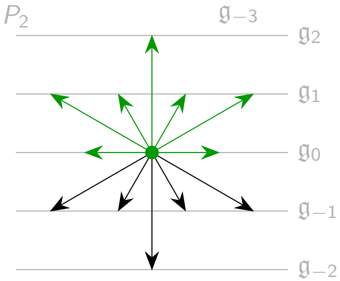
  $$G_2 \circlearrowright J^1(G_2/P, 1)\,.$$

  Differential invariants of such curves were computed by Doubrov and Zelenko, and we revisit them differently.
- The twistor correspondence of G_2/P_1 and G_2/P_2 via G_2/P_{12} gives us a correspondence between curves that allows us to relate the equivalence problems for all 3 choices of the parabolic subgroup P.

Invariants of curves in G_2/P







The filtration
$$\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$$
is invariant with respect to $\mathfrak{p} := \mathfrak{g}^0$.

G_2 invariant structures in G_2/P

- $M := G_2/P_1$. Coordinates (x, y, p, q, z) on M.
  - Rank 2 distribution $\Pi = \langle \partial_x + p\partial_y + q\partial_p + q^2\partial_z \, , \, \partial_q \rangle$.
  - Nurowski conformal structure $[g]$ with null cone N,

$$g = q^2 dx^2 - 2q\, dx\, dp + 6p\, dx\, dq - 3\, dx\, dz - 6\, dy\, dq + 4\, dp^2 \, .$$

- $\hat{M} := G_2/P_{12}$. Coordinates (x, y, p, q, z, r) on $\hat{M}$.
  - Rank 2 distribution $\Delta = \langle \partial_x + p\partial_y + q\partial_p + q^2\partial_z + r\partial_q \, , \, \partial_r \rangle$.
  - Induced degenerate conformal structure from M with null cone $\hat{N} \cong N \times \mathbb{R}^1$.

- $K := G_2/P_2$. Coordinates (x, y, p, q, z) on K.
  - Contact structure $D = \mathrm{Ann}(\langle dz - pdx - qdy \rangle)$.
  - Cone field $\Gamma \subset D$ of rational normal curves given by the ideal

$$\langle 3\, dx\, dp - dy\, dq, \sqrt{3}\, dx\, dy - dq^2, \sqrt{3}\, dp\, dq - dy^2 \rangle$$

Invariants of curves in $M := G_2/P_1$

- P_1 is the stabilizer of a point $o \in M$ under the action of G_2.
- $J^k(M,1)$ space of k-jets of unparametrized regular curves.
- The fiber of the bundle $J^1(M,1) = \mathbb{P}TM$ over $o \in M$ is identified with $(T_oM \backslash \{0\})/\mathbb{R}_\times \cong \mathbb{P}\mathfrak{m}$, where $\mathfrak{m} = \mathfrak{g}/\mathfrak{p}$.
- Bijection between the orbits of $G_2 \circlearrowright J^k(M,1)$ and $P_1 \circlearrowright J^k_o(M,1)$.
- Assume that at any point of the curve its 1-jet belongs always to the same P_1 orbit in $\mathbb{P}\mathfrak{m}$: the type $\mathtt{t}$ of the curve is constant.
- The dimension of this orbit is $d_\mathtt{t}$. A curve of type $\mathtt{t}$ is given by $4 - d_\mathtt{t}$ 1st order equations.

This gives a submanifold $\mathcal{E}_\mathtt{t} \subset J^1(M,1)$ of codimension $4 - d_\mathtt{t}$.
We construct a tower of bundles by prolonging $\mathcal{E}_\mathtt{t}$

$$\begin{array}{rl} & \mathcal{E}^k_\mathtt{t} \subset J^k(M,1) \\ \pi_{k,k-1} & \downarrow \\ & \mathcal{E}^{k-1}_\mathtt{t} \subset J^{k-1}(M,1) \end{array}$$

Number of invariants of curves in $M := G_2/P_1$

- s_k is the codimension of the orbit of the action of G_2 on $J^k(M,1)$. This coincides with the number of invariants of order k.
- $h_k = s_k - s_{k-1}$ is the number of invariants of pure order k.

G_2 is finite dimensional, hence $h_k = d_{\mathtt{t}}$ for large enough k.

h_k		ⓚ 0	1	2	3	4	5	6	7	8	9	10	...
ⓣ	$TM\backslash(N \cup \Pi^2)$	0	0	1	2	4	4	4	4	4	4	4	4
	$N\backslash\Pi^2$	0	0	0	1	2	3	3	3	3	3	3	3
	$\Pi^2\backslash\Pi$	0	0	0	0	0	1	2	2	2	2	2	2
	$\Pi\backslash\{0\}$	0	0	0	0	0	0	0	0	0	0	1	1

Number of invariants of curves in $\hat{M}$ and K

	h_k	k: 0	1	2	3	4	5	6	7	8	9	10	...
G_2/P_{12}	$T\hat{M}\backslash(\Delta^4 \cup \hat{N})$	0	0	2	5	5	5	5	5	5	5	5	5
	$\hat{N}\backslash\Delta^4$	0	0	1	3	4	4	4	4	4	4	4	4
	$\Delta^4\backslash(\Delta^3 \cup \hat{N} \cup H_3)$	0	1	0	3	4	4	4	4	4	4	4	4
	$H_3\backslash(\Delta^3 \cup \hat{N})$	0	0	0	1	3	3	3	3	3	3	3	3
(t)	$(\Delta^4 \cap \hat{N})\backslash(\Delta^3 \cup H_3)$	0	0	0	1	3	3	3	3	3	3	3	3
	$\Delta^3\backslash(\Delta^2 \cup H_2)$	0	0	0	1	3	3	3	3	3	3	3	3
	$(\hat{N} \cap H_3)\backslash\Delta^3$	0	0	0	0	0	2	2	2	2	2	2	2
	$H_2\backslash\Delta^2$	0	0	0	0	0	2	2	2	2	2	2	2
	$\Delta^2\backslash\Delta$	0	0	0	0	0	2	2	2	2	2	2	2
	$\Delta\backslash\{0\}$	0	0	0	0	0	0	0	0	0	1	1	1
G_2/P_2	$TK\backslash D$	0	0	0	3	4	4	4	4	4	4	4	4
	$D\backslash T\Gamma$	0	0	0	1	2	3	3	3	3	3	3	3
	$T\Gamma\backslash\Gamma$	0	0	0	0	0	1	2	2	2	2	2	2
(t)	$\Gamma\backslash\{0\}$	0	0	0	0	0	0	0	0	0	0	1	1

Invariants of integral curves in M

There is an order 10 absolute differential invariant $I_{10} = \dfrac{{R_{10}}^3}{{R_8}^7}$, where

$$R_8 = 196\, q_2^5 q_8 - 2352\, q_2^4 q_3 q_7 - 5040\, q_2^4 q_4 q_6 - 3255 q_2^4 q_5^2$$
$$+16632\, q_2^3 q_3^2 q_6 + 59598\, q_2^3 q_3 q_4 q_5 + 13772\, q_2^3 q_4^3 - 83160\, q_2^2 q_3^3 q_5$$
$$-174735\, q_2^2 q_3^2 q_4^2 + 297000\, q_2 q_3^4 q_4 - 118800\, q_3^6\,,$$

$$R_{10} = 21 q_2 R_8 \mathcal{D}_x\big(q_2 \mathcal{D}_x R_8\big) - \frac{91}{4}\big(q_2 \mathcal{D}_x R_8\big)^2 + 9 {R_8}^2\big(13 q_3^2 - 19 q_2 q_4\big).$$

$\mathcal{D}_x$ is the operator of total derivative on $\mathcal{E}_\Pi$,

$$\mathcal{D}_x = \partial_x + p\partial_y + q\partial_p + q^2\partial_z + \sum_{i=0}^{\infty} q_{i+1}\partial_{q_i}\,.$$

Theorem 1

The algebra $\mathcal{A}_{int}$ of (micro-local) differential invariants of integral curves is generated in the Lie-Tresse sense by I_{10} and $\square_{int} = \dfrac{q_2}{{R_8}^{1/6}} \cdot \mathcal{D}_x$.

Invariants of generic curves in M

- 2nd order invariant $I_2 = \frac{{R_2}^2}{{R_1}^3}$, where

$$R_1 = (q + 2p_1)^2 + 6(q_1(p - y_1) - qp_1) - 3z_1\,,$$
$$R_2 = -\frac{(q + 2p_1)^3}{18} + (p - y_1)\left(qp_2 - \frac{z_2}{2}\right) + (q + y_2)\left(qp_1 - \frac{z_1}{2}\right) + p_1 z_1 - q^2\frac{y_2}{2}\,.$$

- There exists a canonical frame $\underbrace{Y\,, V}_{\in \Pi}, \underbrace{Z}_{\in \Pi^2/\Pi}, X\,, W$ along γ adapted to the filtration.
- X is tangent to γ. $\mathcal{L}_X I_2 = 1$ and $g(X, X) = 1$.
- The Gram matrix of the frame Y, V, Z, X, W with respect to g is expressed through I_2.
- The Levi-Civita connection of g on the canonical frame produces invariants I_{31}, I_{32} (order 3) and $I_{41}, I_{42}, I_{43}, I_{44}$ (order 4).

Theorem 2

The algebra $\mathcal{A}_{gen}$ of differential invariants of generic curves is generated by 7 invariants I_2, I_{3i}, I_{4j} and invariant derivation $\square_{gen} = \frac{1}{\mathcal{D}_x I_2} \cdot \mathcal{D}_x$, where $\mathcal{D}_x = \partial_x + \sum_{i=0}^{\infty}\left(y_{i+1}\partial_{y_i} + p_{i+1}\partial_{p_i} + q_{i+1}\partial_{q_i} + z_{i+1}\partial_{z_i}\right)$.

Twistor correspondence

G_2/P_{12} can be viewed both as the geometric prolongation $\hat{M}$ of M and the geometric prolongation $\hat{K}$ of K.

- $\hat{M}$ is a collection of pairs (a, p_a) with $a \in M$ and $p_a \in \mathbb{P}\Pi$.
- $\hat{K}$ is a collection of pairs (b, p_b) with $b \in K$ and $p_b \in [\Gamma]$.

$\exists$ diffeomorphism $\varphi : \hat{M} \longrightarrow \hat{K}$, given by $\varphi(x, y, p, q, z, r) =$

$$\left(-\frac{1}{r}, \sqrt{3}\left(2p - \frac{q^2}{r}\right), 3z - \frac{q^3}{r}, \sqrt{3}\left(x - \frac{q}{r}\right), 6(xp - y) - \frac{3}{r}\left(z + xq^2\right) + \frac{2q^3}{r^2}, q\right)$$

$$\begin{array}{ccccc} & G_2/P_{12} : (\hat{M}, \Delta) & \xrightarrow{\varphi} & (\hat{K}, \tilde{\Delta}) & \\ \swarrow^{\pi_l} & & & & \searrow^{\pi_r} \\ G_2/P_1 : (M, \Pi) & & & & G_2/P_2 : (K, D, \Gamma) \end{array}$$

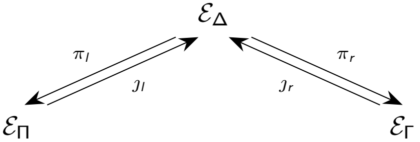
- φ interchanges the generators of the rank 2 distributions Δ and $\tilde{\Delta}$.
- We have the natural projections $\pi_l(a, p_a) = a$, $\pi_r(b, p_b) = b$.

Integral curves correspondence

Integral curves in G_2/P_1 and in G_2/P_2 are uniquely lifted to G_2/P_{12} given a point in the fiber, so we have

$$\mathcal{E}_\Pi \times \mathbb{P}^1 \simeq \mathcal{E}_\Delta \simeq \mathcal{E}_\Gamma \times \mathbb{P}^1$$

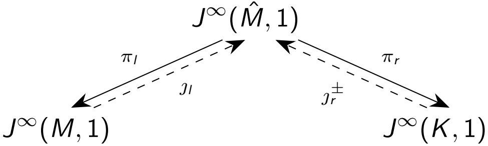
which gives an isomorphism between algebras of differential invariants of G_2/P_1 and G_2/P_2 for integral curves



- On left side, prolongation is given by $r = q_1$.
- On right side, prolongation is given by $r = q_1/\sqrt{3}$.
- $\jmath_l$ and $\jmath_r$ are the right inverses of π_l and π_r respectively.
- π_l and π_r just discard the coordinate r.

Generic curves correspondence

For generic curves, the lift $\jmath_l$ is 1:1 and the lift $\jmath_r^{\pm}$ is 1:2.



- On left side, lift is given by r such that ℓ_r is aligned to Y, where $\ell_r = \langle \partial_x + p\partial_y + q\partial_p + q^2\partial_z + r\partial_q \rangle \subset \Pi$.
- On right side, take distinguished curve δ tangent to γ at $b \in K$.
  - $\delta - \gamma$ determines 2-plane in T_bK intersecting D_b on one line Υ_X.
  - A point in $\mathbb{P}D$ gives secant line intersecting $[\Gamma]$ at points $\lambda_X^{\pm}$.

  The extra coordinate of the lift in $\hat{K}$ is given by $\lambda_X^{\pm} \circ [\Upsilon_X]$.

Thank you for your attention

B. Kruglikov, A. L, *Differential invariants of curves in G(2) flag varieties*, arXiv:2107.03664 (2021).

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B. Doubrov, I. Zelenko, *Geometry of curves in generalized flag varieties*, Transformation Groups **18**, no.2, 361-383 (2013).