$G(3)$ supergeometry and a supersymmetric extension of the Hilbert–Cartan equation

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Based on joint work with B. Kruglikov and D. The on Adv. Math. (2021)

Plan of the talk:

– Prelude on Lie superalgebras and supermanifolds
– Realizations of G_2 as symmetry algebra
– The Lie superalgebra $G(3)$: algebraic aspects and parabolic subalgebras
– Spencer cohomology of $G(3)$
– Geometric realizations of $G(3)$ as supersymmetry of differential equations
– Curved $G(3)$ supergeometries (time-dependent)

Prelude: Lie superalgebras

Def. A Lie superalgebra is a complex vector space of the form

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \bigoplus \mathfrak{g}_{\bar{1}}$$

endowed with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

- $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subset \mathfrak{g}_{\bar{0}}$, $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{1}}$, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{0}}$;
- for any homogeneous X, Y (i.e. with $X \in \mathfrak{g}_{\bar{i}}$, $Y \in \mathfrak{g}_{\bar{j}}$)

$$[X, Y] = -(-1)^{|X||Y|}[Y, X] \qquad \left(|X| = \text{ parity of } X = \begin{cases} 0 \\ 1 \end{cases} \right)$$

- for any homogeneous X, Y, Z

$$(-1)^{|Z||X|}[X, [Y, Z]] + (-1)^{|Y||Z|}[Z, [X, Y]] + (-1)^{|X||Y|}[Y, [Z, X]] = 0$$

Prelude: simple Lie superalgebras

Finite-dimensional simple (complex) Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ were classified by V. Kac in 1977 and split into two main families:

- classical, for which the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible;
- Cartan Lie superalgebras, analogs to simple Lie algebras of vector fields.

Classical LSA include in turn the LSA with a non-degenerate "Killing form":

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	$\mathfrak{g}_{\bar{1}}$
$\mathfrak{sl}(m\|n)$ $m, n \geqslant 1$	$\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbb{C}$	$\left(\mathbb{C}^m \otimes (\mathbb{C}^n)^*\right) \oplus \left((\mathbb{C}^m)^* \otimes \mathbb{C}^n\right)$
$\mathfrak{osp}(m\|2n)$ $m, n \geqslant 1$	$\mathfrak{so}(m) \oplus \mathfrak{sp}(2n)$	$\mathbb{C}^m \otimes \mathbb{C}^{2n}$
$\mathfrak{osp}(4\|2;\alpha)$ $\alpha \neq 0, \pm 1, \infty$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
$F(3\|1)$	$\mathfrak{so}(7) \oplus \mathfrak{sl}(2)$	$\mathbb{S} \otimes \mathbb{C}^2$
$G(3)$	$G_2 \oplus \mathfrak{sl}(2)$	$\mathbb{C}^7 \otimes \mathbb{C}^2$

Rem. The smallest non-trivial representation of $G(3)$ is the adjoint representation.

Prelude: supermanifolds

Def. A supermnfd of dimension $(m|n)$ is a pair $M = (M_o, \mathcal{A}_M)$, where

- M_o is an m-dimensional mnfd, called the body,
- $\mathcal{A}_M$ is a sheaf of superalgebras on M_o s.t. locally $\mathcal{A}_M|_\mathcal{U} \cong \Gamma(\Lambda^\bullet E^*)$, where $E \to \mathcal{U}$ is vector bundle of rank n over $\mathcal{U} \subset M_o$.

Rem I. Sections of $\mathcal{A}_M$ are called superfunctions. They are locally of the form

$$f = f_0(x) + f_{\alpha_1}(x)\theta^{\alpha_1} + \cdots + f_{\alpha_1 \ldots \alpha_n}(x)\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_n} \ ,$$

where $f_{\alpha_1 \ldots \alpha_k} \in \mathcal{C}^\infty(\mathcal{U})$ and the (θ^α) are generators of the bundle E^*. In other words, superfunctions admit an analytic expansion in the "odd coordinates" (θ^α).

Rem II. The odd coordinates anticommute $\theta^\alpha \wedge \theta^\beta = -\theta^\beta \wedge \theta^\alpha$, so they are nilpotent and the evaluation $\mathrm{ev}_x : f \mapsto f_0(x)$ at any point $x \in M_o$ send them to zero: superfunctions are not determined by their values at points.

Prelude: superdistributions

A superdistribution on $M = (M_o, \mathcal{A}_M)$ is a graded $\mathcal{A}_M$-subsheaf $\mathcal{D}$ of the tangent sheaf $\mathcal{T}M = \mathrm{Der}(\mathcal{A}_M)$ that is locally a direct factor.

Example I. $M = \mathbb{C}^{5|2}$ with even coordinates x, u, p, q, z and odd coordinates θ, ν. The subsheaf generated by the supervector fields

$$
\begin{aligned}
D_x &= \partial_x + p\partial_u + q\partial_p + q^2\partial_z \ , \quad \partial_q \ , \\
D_\theta &= \partial_\theta + q\partial_\nu + \theta\partial_p + 2\nu\partial_z \ ,
\end{aligned}
$$

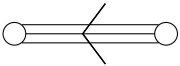
is a superdistribution of rank $(2|1)$.

Example II. A subsheaf that is not a superdistribution (even upon localization) is the derived sheaf of the superdistribution of Example I. Indeed

$$
\begin{aligned}
&[\partial_q, D_x] = \tfrac{1}{2}[D_\theta, D_\theta] = \partial_p + 2q\partial_z \ , \\
&[\partial_q, D_\theta] = \partial_\nu \ , \quad [D_x, D_\theta] = -\theta\partial_u \ ,
\end{aligned}
$$

hence the sheaf is generated by linearly dependent supervector fields: $\theta(\theta\partial_u) = 0$.

Some geometric realizations of G_2

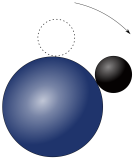


This is an abstract description via Dynkin diagrams. What about realizations as symmetries?

– $GL_7(\mathbb{C})$ acts with open orbit on 3-forms on $\mathbb{C}^7$ and $G_2 = \mathrm{Stab}_{GL_7(\mathbb{C})}(\phi)$ for generic $\phi \in \bigwedge^3(\mathbb{C}^7)^*$ (Engel, 1900);

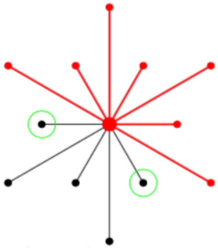
– Compact form $G_2 = \mathrm{Aut}(\mathbb{O})$ (Cartan, 1914);

– Configuration space M of a 2-sphere rolling on another w/o twisting or slipping is 5-dimensional, with the constraints given by a rank 2 distribution $\mathcal{D} \subset \mathcal{T}M$ of filtered growth $(2, 3, 5)$. If the ratio of the radii of spheres is 3, then split $G_2 = \mathrm{Aut}(M, \mathcal{D})$ (Bryant, Zelenko, Bor–Montgomery, Baez–Huerta).



$(2, 3, 5)$-geometry from the G_2 root diagram

G_2/P_1



Fundamental invariant of $(2, 3, 5)$-distributions: binary quartic field (Cartan 1910). Modern perspective: the quartic arises from $H^{4,2}(\mathfrak{m}, \mathfrak{g}) \cong S^4(\mathbb{C}^2)$, where $\mathfrak{g} = G_2$ has $|3|$-grading $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \cdots \oplus \mathfrak{g}_3$ with negative part

$$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} = \langle e_1, e_2 \rangle \oplus \langle e_3 \rangle \oplus \langle e_4, e_5 \rangle$$

$$[e_1, e_2] = e_3 \ , \qquad [e_1, e_3] = e_4 \ , \qquad [e_2, e_3] = e_5 \ ,$$

and 0-degree component $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{m}) \cong \mathfrak{gl}(2)$.

Some geometric realizations of G_2

– Engel (1893): G_2 as symmetry of contact distribution $\mathcal{C}$ on 5-dim. mnfd with field of twisted cubics $\mathcal{V} \subset \mathbb{P}(\mathcal{C})$;

– Cartan (1893, 1910): G_2 as symmetry of

Dim	Geometric structure	Model
5	ODE with flat $(2,3,5)$-distribution	$du - u' dx$, $du' - u'' dx$, $dz - (u'')^2 dx$, *Hilbert–Cartan equation* $z' = (u'')^2$
6	Involutive pair of PDE	$u_{xx} = \frac{1}{3}(u_{yy})^3, \quad u_{xy} = \frac{1}{2}(u_{yy})^2$

D. The in 2018 gave explicit generalizations of the Cartan–Engel models to all exceptional Lie algebras.

Main Motivations and Goals

General motivations.

- Give geometric realizations of Lie superalgebras $G(3)$, $F(3|1)$, $\mathfrak{osp}(4|2;\alpha)$ as symmetry superalgebras of simple objects;
- We are interested in geometries that have high symmetry, a lot of solns to BGG eqns, Killing spinor eqns, etc. For example, we plan to understand relationship between symmetries of superdistributions and solutions of PDE;
- Here is another suggestion: does any given classical geometry admit a non-trivial supersymmetric extension?

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Goals achieved so far.

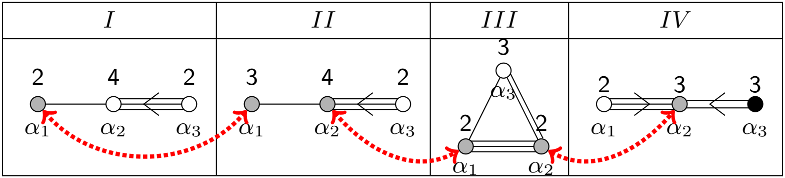
- Various geometric realizations of $G(3)$;
- Understanding of the deformations of these flat structures.

In particular, we will exhibit super-extensions of the flat as well as some non-flat $(2,3,5)$-geometries, and give bounds on supersymmetry dimension.

Simple root systems of $\mathfrak{g} = G(3)$

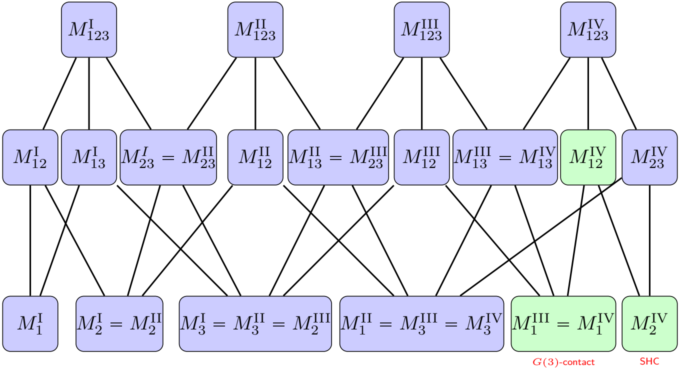
Fix Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}} = G_2 \oplus \mathfrak{sp}(2) \rightsquigarrow$ root system $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$. The Killing form of $G(3)$ is non-degenerate and induces non-degenerate $\langle -, - \rangle$ on $\mathfrak{h}^*$. If $\alpha \in \Delta_{\bar{0}}$ then $\langle \alpha, \alpha \rangle \neq 0$ and the even reflection $S_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$ preserves $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$. The Weyl group $W = \langle S_\alpha \mid \alpha \in \Delta_{\bar{0}} \rangle$ is generated by even reflections. Up to W-equivalence, there are four inequivalent simple systems $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$. They are related by odd reflections along isotropic $\alpha \in \Pi_{\bar{1}}$ as indicated below:

$$S_\alpha(\beta) = \begin{cases} \beta + \alpha, & \langle \alpha, \beta \rangle \neq 0; \\ \beta, & \langle \alpha, \beta \rangle = 0,\ \beta \neq \alpha; \\ -\alpha, & \beta = \alpha. \end{cases}$$

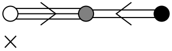
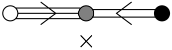


Roots: even ○, odd isotropic ◉ or odd nonisotropic ●.

Map of $G(3)$-supergeometries



We considered two $\mathbb{Z}$-gradings of $\mathfrak{g} = G(3)$ with (graded) growths:

	marked Dynkin diagram	$\dim(\mathfrak{g}_{-1})$	$\dim(\mathfrak{g}_{-2})$	$\dim(\mathfrak{g}_{-3})$
$G(3)$-contact		$4\|4$	$1\|0$	
SHC		$2\|4$	$1\|2$	$2\|0$

More details on gradings

Any $\mathbb{Z}$-grading of $\mathfrak{g} = G(3)$ has $\mathfrak{g}_{-k}^* \cong \mathfrak{g}_k$ (non-degenerate Killing form).

$G(3)$-contact. In this case $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{f}$, where $\mathfrak{f} \cong \mathfrak{osp}(3|2)$.

k	$(\mathfrak{g}_k)_{\bar{0}}$	$(\mathfrak{g}_k)_{\bar{1}}$	dim
0	$\mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sp}(2)$	$S^2\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$7\|6$
-1	$S^3\mathbb{C}^2 \boxtimes \mathbb{C}$	$\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$4\|4$
-2	$\mathbb{C} \boxtimes \mathbb{C}$		$1\|0$

SHC grading. In this case $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{osp}(1|2)$.

k	$(\mathfrak{g}_k)_{\bar{0}}$	$(\mathfrak{g}_k)_{\bar{1}}$	dim
0	$\mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sp}(2)$	$\mathbb{C} \boxtimes \mathbb{C}^2$	$7\|2$
-1	$\mathbb{C}^2 \boxtimes \mathbb{C}$	$\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$2\|4$
-2	$\mathbb{C} \boxtimes \mathbb{C}$	$\mathbb{C} \boxtimes \mathbb{C}^2$	$1\|2$
-3	$\mathbb{C}^2 \boxtimes \mathbb{C}$		$2\|0$

Geometric structures associated to M_1^{IV} and M_2^{IV}.

G(3)-contact super-PDE:

$$u_{xx} = \frac{1}{3}(u_{yy})^3 + 2u_{yy}u_{y\nu}u_{y\tau}, \quad u_{xy} = \frac{1}{2}(u_{yy})^2 + u_{y\nu}u_{y\tau},$$

$$u_{x\nu} = u_{yy}u_{y\nu}, \quad u_{x\tau} = u_{yy}u_{y\tau}, \quad u_{\nu\tau} = -u_{yy}.$$

where $u = u(x, y, \nu, \tau) : \mathbb{C}^{2|2} \to \mathbb{C}^{1|0}$.

Super Hilbert-Cartan equation (SHC):

$$z_x = \frac{(u_{xx})^2}{2} + u_{x\nu}u_{x\tau}, \quad z_\nu = u_{xx}u_{x\nu}, \quad z_\tau = u_{xx}u_{x\tau}, \quad u_{\nu\tau} = -u_{xx},$$

where $(u, z) = (u(x, \nu, \tau), z(x, \nu, \tau)) : \mathbb{C}^{1|2} \to \mathbb{C}^{2|0}$.

Thm[Kruglikov, S., The] These super-PDE have symmetry superalgebras $G(3)$. Unlike the Hilbert-Cartan eqn, whose general solution depends on one arbitrary function of one variable, solutions of SHC depend only on five constants.

Tanaka–Weisfeiler prolongation and cohomology.

Given negatively graded Lie superalgebra $\mathfrak{m} = \mathfrak{m}_{-\mu} \oplus \cdots \oplus \mathfrak{m}_{-1}$ and $\mathfrak{g}_0 \subset \mathfrak{der}_{gr}(\mathfrak{m})$, we let the Tanaka–Weisfeiler prolongation $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ be graded Lie superalgebra s.t.:

(i) $\mathrm{pr}_{\leqslant 0}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}_0$;

(ii) $[X, \mathfrak{g}_{-1}] = 0$ for $X \in \mathrm{pr}_+(\mathfrak{m}, \mathfrak{g}_0)$ implies $X = 0$;

(iii) $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ is maximal with these properties.

If $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{m})$, we simply write $\mathrm{pr}(\mathfrak{m})$.

Relevance: $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ is the symmetry superalgebra of the left-invariant geometric structure on the ''standard model" $(\exp(\mathfrak{m}), \mathfrak{g}_{-1})$.

Rem I. Although $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ can be obtained via an iterative process, one can test a candidate $\mathfrak{g}$ that extends $\mathfrak{m} \oplus \mathfrak{g}_0$ via the criteria:

- $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$ if and only if $H^1_{\geqslant 0}(\mathfrak{m}, \mathfrak{g}) = 0$;
- $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ if and only if $H^1_+(\mathfrak{m}, \mathfrak{g}) = 0$.

Rem II. Kostant Thm efficiently computes these cohomology groups in the classical setting but in super-setting his "harmonic cohomology" is usually bigger.

Spencer cohomology

There is a very useful relationship between Spencer cohomology groups of $\mathfrak{g} = G(3)$ and classical Kostant cohomology. Let

$$0 \longrightarrow K^n \longrightarrow \Lambda^n \mathfrak{m}^* \xrightarrow{\text{res}} \Lambda^n \mathfrak{m}^*_{\bar{0}} \longrightarrow 0$$

be the short exact sequence given by the natural restriction $\text{res} : \Lambda^n \mathfrak{m}^* \to \Lambda^n \mathfrak{m}^*_{\bar{0}}$ with kernel

$$K^0 = 0 \,, \qquad K^n = \sum_{1 \leqslant i \leqslant n} \Lambda^{n-i} \mathfrak{m}^*_{\bar{0}} \otimes \Lambda^i \mathfrak{m}^*_{\bar{1}} \quad \text{for} \quad n > 0 \,,$$

and, by tensoring with $\mathfrak{g}$, we may consider the short exact sequence of differential complexes

$$0 \longrightarrow \underbrace{C^\bullet(\mathfrak{m}_{\bar{1}}, \mathfrak{g}) = \mathfrak{g} \otimes K^\bullet}_{} \longrightarrow C^\bullet(\mathfrak{m}, \mathfrak{g}) \xrightarrow{\text{res}} C^\bullet(\mathfrak{m}_{\bar{0}}, \mathfrak{g}) \longrightarrow 0$$

Cochains that vanish when

all entries are in $\mathfrak{m}_{\bar{0}}$

Every morphism in the sequence is $(\mathfrak{g}_0)_{\bar{0}}$-equivariant, with $(\mathfrak{g}_0)_{\bar{0}} \simeq \mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sp}(2)$.

Spencer cohomology

The associated long exact sequence in cohomology gives:

Proposition. For all $d \geqslant 0$, there exists a long exact sequence of $(\mathfrak{g}_0)_{\bar{0}}$-modules

$$0 \longrightarrow \xi^d_{\mathfrak{g}}(\mathfrak{m}_{\bar{0}}) \longrightarrow H^{d,1}(\mathfrak{m}_{\bar{1}}, \mathfrak{g}) \longrightarrow H^{d,1}(\mathfrak{m}, \mathfrak{g}) \longrightarrow H^{d,1}(\mathfrak{m}_{\bar{0}}, \mathfrak{g}) \longrightarrow$$
$$\longrightarrow H^{d,2}(\mathfrak{m}_{\bar{1}}, \mathfrak{g}) \longrightarrow H^{d,2}(\mathfrak{m}, \mathfrak{g}) \longrightarrow H^{d,2}(\mathfrak{m}_{\bar{0}}, \mathfrak{g})$$

where $\xi^d_{\mathfrak{g}}(\mathfrak{m}_{\bar{0}})$ is the component of degree d of the centralizer of $\mathfrak{m}_{\bar{0}}$ in $\mathfrak{g}$.

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where $\xi^d_{\mathfrak{g}}(\mathfrak{m}_{\bar{0}})$ is the component of degree d of the centralizer of $\mathfrak{m}_{\bar{0}}$ in $\mathfrak{g}$.

General strategy.

(i) Describe $H^{d,n}(\mathfrak{m}_{\bar{0}}, \mathfrak{g})$ for $n = 1, 2$ using Kostant's version of the BBW Theorem for G_2 (for us: $\mathfrak{m}_{\bar{0}}$ is negative part of $(2, 3, 5)$-grading of G_2),

(ii) Explicitly compute $H^{d,n}(\mathfrak{m}_{\bar{1}}, \mathfrak{g})$ for $n = 1, 2$ using $(\mathfrak{g}_0)_{\bar{0}}$-equivariance of the coboundary operator (rather involved for $n = 2$, but yet doable),

(iii) Get $H^{d,n}(\mathfrak{m}, \mathfrak{g})$ for $n = 1, 2$ via the above long exact sequence and the fact that this cohomology is a module for $\mathfrak{g}_0 \cong \mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{osp}(1|2)$.

The cohomology groups for the SHC grading

Thm[Kruglikov, S., The] Let $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \cdots \oplus \mathfrak{g}_3$ be $\mathbb{Z}$-grading of $\mathfrak{g} = \mathrm{G}(3)$ with parabolic subalgebra $\mathfrak{p}_2^{\mathrm{IV}}$. Then $H^{d,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for all $d \geqslant 0$, so that $\mathfrak{g} \cong \mathrm{pr}(\mathfrak{m})$. Moreover $H^{d,2}(\mathfrak{m}, \mathfrak{g})_{\bar{1}} = 0$ for all $d > 0$ while

$$H^{d,2}(\mathfrak{m}, \mathfrak{g})_{\bar{0}} \cong \begin{cases} 0 & \text{for all } d > 0, d \neq 2, \\ S^2\mathbb{C}^2 \boxtimes \Lambda^2\mathbb{C}^2 & \text{if } d = 2, \end{cases}$$

Rem I. As a $(\mathfrak{g}_0)_{\bar{0}}$-module, the space $C^{4,2}(\mathfrak{m}, \mathfrak{g})$ has a unique submodule $S^4\mathbb{C}^2 \boxtimes \mathbb{C}$, which is the space of Cartan's classical binary quartic invariants. Its elements are not closed in the complex $C^\bullet(\mathfrak{m}, \mathfrak{g})$.

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Rem II. This suggests the Cartan quartic of underlying generic rank 2 distribution on 5-dim. mnfd should admit a square root, hence it must be of Petrov type D (pair of double roots), N (quadruple root) or O (identically zero).

Finding models with desired symmetry

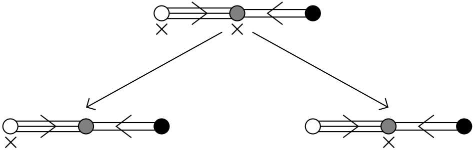
Two steps:

1. Find an explicit description of an invariant geometric structure. E.g. start with the $(2, 3, 5)$ symbol algebra and integrate structure eqns or use BCH to arrive at a local model (equivalent to the Hilbert–Cartan eqn).
   Rem. We obtained SHC eqn also in this way but it is too involved.

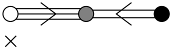
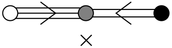
2. Prove that this homogeneous model has the expected symmetry dimension. Tanaka–Weisfeiler prolongation, via results on $H^1(\mathfrak{m}, \mathfrak{g})$, gives upper bound.
   Rem. In classical setting we have harmonic curvature as a test for flatness but this is unavailable in the super-setting.

$G(3)$-double fibration

We investigated the $G(3)$-twistor correspondence



Recall:

	marked Dynkin diagram	$\dim(\mathfrak{g}_{-1})$	$\dim(\mathfrak{g}_{-2})$	$\dim(\mathfrak{g}_{-3})$
$G(3)$-contact		$4\|4$	$1\|0$	
SHC		$2\|4$	$1\|2$	$2\|0$

Strategy: flag supermnfd $G(3)/P_1$ is contact supermnfd $(M, \mathcal{C})$ with the additional reduction of structure group $COSp(3|2) \subset CSpO(4|4)$, which we realize as $(1|2)$-twisted cubic $\mathcal{V} \subset \mathbb{P}(\mathcal{C})$. Osculate $\mathcal{V}$ to get PDE $\mathcal{E} \cong G(3)/P_{12}$. Cartan superdistrib. of $\mathcal{E}$ has a "Cauchy characteristic", we quotient by it to get SHC eqn $\overline{\mathcal{E}} \cong G(3)/P_2$.

$G(3)$-contact case

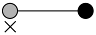
Idea: contact supermnfd + additional geometric structure.

k	$(\mathfrak{g}_k)_{\bar 0}$	$(\mathfrak{g}_k)_{\bar 1}$	dim
0	$\mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sp}(2)$	$S^2\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$7\|6$
-1	$S^3\mathbb{C}^2 \boxtimes \mathbb{C}$	$\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$4\|4$
-2	$\mathbb{C} \boxtimes \mathbb{C}$		$1\|0$

Prop. $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{osp}(3|2) \subset \mathbb{C} \oplus \mathfrak{spo}(\mathfrak{g}_{-1}) = \mathbb{C} \oplus \mathfrak{spo}(4|4)$ is a maximal subalgebra.

The basis of $V = \mathfrak{g}_{-1}$ given by $\{x^3, x^2y, xy^2, y^3 | xe, xf, ye, yf\}$ allows to make explicit the invariant $CSpO$-structure on V. The topological point $[x^3] \in \mathbb{P}(V_{\bar 0})$ has isotropy $\mathfrak{q} \subset \mathfrak{f} = \mathfrak{osp}(3|2)$ that is a parabolic subalgebra:

$$\mathfrak{f} = \mathfrak{f}_{-1} \oplus \overbrace{\mathfrak{f}_0 \oplus \mathfrak{f}_1}^{\mathfrak{q}}$$



k	$(\mathfrak{f}_k)_{\bar 0}$	$(\mathfrak{f}_k)_{\bar 1}$
1	X_1	A_1, A_2
0	H_1, H_2, X_2, Y_2	A_3, A_4
-1	Y_1	A_5, A_6

The $(1|2)$-twisted cubic $\mathcal{V}$

Def. The G_0-orbit $\mathcal{V} \subset \mathbb{P}(V)$ through $[x^3]$ is called the $(1|2)$-twisted cubic.

We describe $\mathcal{V}$ locally by exponentiating the action of $\mathfrak{f}_{-1} = \mathrm{span}\{Y_1, A_5, A_6\}$ through $[x^3]$:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xmapsto{\exp(\lambda Y_1)} \begin{pmatrix} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} \\ -\frac{\lambda^2}{2} \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xmapsto{\exp(\theta A_5)} \begin{pmatrix} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} \\ -\frac{\lambda^2}{2} \\ \hline \theta \\ 0 \\ 0 \\ -\theta\lambda \end{pmatrix} \xmapsto{\exp(\phi A_6)} \begin{pmatrix} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} + \phi\theta\lambda \\ -\frac{\lambda^2}{2} + \phi\theta \\ \hline \theta \\ \phi \\ \phi\lambda \\ -\theta\lambda \end{pmatrix},$$

with λ even parameter and θ, ϕ odd. By maximality, this supervariety $\mathcal{V} \subset \mathbb{P}(V)$ characterizes the reduction of the structure group $COSp(3|2) \subset CSpO(4|4)$.

Rem. Rigorously, we should appeal to the functor of points $\mathbb{A} \mapsto V(\mathbb{A})$: for any finite dimensional super-commutative superalgebra $\mathbb{A} = \mathbb{A}_{\bar{0}} \oplus \mathbb{A}_{\bar{1}}$, the above gives a free $\mathbb{A}$-module of rank $(1|0)$ in $V(\mathbb{A}) := (V \otimes \mathbb{A})_{\bar{0}}$.

Osculations of $\mathcal{V}$

Repeatedly applying $\mathfrak{f}_{-1}$ to $[x^3]$ yields the so-called osculating sequence

$$0 \subset V^0 \subset V^1 \subset V^2 \subset V^3 = V$$

of (higher order) affine tangent spaces of $\mathcal{V}$ at $[x^3]$.

Important fact I: The affine tangent space $V^1 \subset V \cong \mathbb{C}^{4|4}$ is Lagrangian w.r.t. $CSpO$-structure on V (in particular $\dim V^1 = (2|2)$).

Important fact II: The associated graded v.s. $\mathrm{gr}(V) = N_0 \oplus \cdots \oplus N_3$ has natural $\mathfrak{osp}(1|2)$-equivariant $\mathbb{Z}$-graded superalgebra structure and $N_1 \otimes N_1 \to N_2 \cong N_1^*$ is a supersymmetric cubic form $\mathfrak{C} \in S^3 N_1^*$ on $N_1 \cong \mathbb{C}^{1|2}$. (It is cubic form of simple Jordan superalgebra structure on N_1 called the Kaplansky superalgebra.)

General framework for 2nd order super-PDE

Global	Local
Contact supermfld $(M^{5\|4}, \mathcal{C}) \cong J^1(\mathbb{C}^{2\|2}, \mathbb{C}^{1\|0})$	(x^i, u, u_i), $\sigma = du - \sum_{i=1}^4 u_i dx^i$ $\mathcal{C} = \langle \sigma = 0 \rangle = \langle \partial_{x^i} + u_i \partial_u, \partial_{u_i} \rangle$
$\mathcal{C}$ has frames of conformal symplectic-orthogonal supervector fields	$d\sigma\|_{\mathcal{C}} = \left(\begin{array}{cccc\|cccc} &&&&1&&& \\ &&&&&1&& \\ &&&&&&1& \\ &&&&&&&1 \\ \hline -1&&&&&&& \\ &-1&&&&&& \\ &&1&&&&& \\ &&&1&&&& \end{array}\right)$ $\partial_{x^i} + u_i \partial_u,\ \partial_{u_i}$ is adapted frame
Lagrangian subspace of $\mathcal{C}$ at $m \in M$	$\langle \partial_{x^i} + u_i \partial_u + u_{ij} \partial_{u_j} \rangle$
Lagrange–Grassmann bundle $(\widetilde{M}^{9\|8}, \widetilde{\mathcal{C}}) \cong J^2(\mathbb{C}^{2\|2}, \mathbb{C}^{1\|0})$	$(x^i, u, u_i, u_{ij} = \pm u_{ji})$ $\widetilde{\mathcal{C}} = \langle \partial_{x^i} + u_i \partial_u + u_{ij} \partial_{u_j}, \partial_{u_{ij}} \rangle$

A 2nd order super-PDE is a submanifold of 2nd order jet-space $\widetilde{M}$ and an external symmetry is a symmetry of $(\widetilde{M}, \widetilde{\mathcal{C}})$ that preserves the submanifold.

Key steps

- Lagrangian lift. At any "point" of $(M, \mathcal{C})$ we have $(1|2)$-parametric family of Lagrangian subspaces of $\mathcal{C}$: the affine tangent spaces along $\mathcal{V}$. It gives $(6|6)$-dimensional submanifold $\mathcal{E} \subset \widetilde{M}$, i.e., the $G(3)$-contact super-PDE;
- Cubic form. The $G(3)$-contact super-PDE can be parametrically written as

$$\begin{pmatrix} u_{00} & u_{0a} \\ u_{a0} & u_{ab} \end{pmatrix} = \begin{pmatrix} \mathfrak{C}(T^3) & \frac{3}{2}\mathfrak{C}_a(T^2) \\ \frac{3}{2}\mathfrak{C}_a(T^2) & 3\mathfrak{C}_{ab}(T) \end{pmatrix} \qquad (a, b = 1, 2, 3) \; .$$

  This extends to $G(3)$ a formula giving geometric realizations of exceptional Lie algebras – for different cubic forms – obtained by D. The in 2018.
- Symmetries. External symmetries of $G(3)$-contact super-PDE are derived explicitly by a hand computation using expression of generating functions on $(M, \mathcal{C})$ via the cubic form;

Key steps

- Spencer cohomology. The previous computation tells that supersymmetry dimension is $(17|14)$, i.e., the upper bound coming from Tanaka–Weisfeiler prolongation is attained. Moreover, the grading element guarantees that the symmetry superalgebra is exactly $G(3)$.
- Cauchy characteristic reduction. On $\mathcal{E} \cong G(3)/P_{12}$ we have the Cartan superdistribution $\mathcal{H} \subset \mathcal{T}\mathcal{E}$ of rank $(3|4)$. The Cauchy characteristic space

$$\mathrm{Ch}(\mathcal{H}) = \{X \in \Gamma(\mathcal{H}) \mid \mathcal{L}_X \mathcal{H} \subset \mathcal{H}\}$$

  is a module for the space of superfunctions of $\mathcal{E}$ and it is generated by a nowhere-vanishing even supervector field. The quotient $\overline{\mathcal{E}} = \mathcal{E}/\mathrm{Ch}(\mathcal{H})$ is then $(5|6)$-dimensional and is endowed with superdistribution of rank $(2|4)$.
- SHC-equation. We have $\overline{\mathcal{E}} \cong G(3)/P_2$ endowed with the Cartan superdistribution associated to SHC-eqn.

Curved $G(3)$-supergeometries

Let $M = (M_o, \mathcal{A}_M)$ be a $(5|6)$-dimensional supermnfd with a bracket-generating superdistribution $\mathcal{D}$ of growth $(2|4, 1|2, 2|0)$ and w/o Cauchy characteristics. Then its symbol superalgebra $\mathfrak{m} = \mathfrak{g}_-$ is *fundamental* (i.e., it is generated by $\mathfrak{g}_{-1}$) and *non-degenerate* (i.e., it has no central elements in $\mathfrak{g}_{-1}$). The Lie brackets on the even part $\mathfrak{m}_{\bar{0}}$ of $\mathfrak{m}$ consist of

$$\omega : \Lambda^2(\mathfrak{g}_{-1})_{\bar{0}} \to (\mathfrak{g}_{-2})_{\bar{0}} \qquad \text{and} \qquad \beta : (\mathfrak{g}_{-1})_{\bar{0}} \otimes (\mathfrak{g}_{-2})_{\bar{0}} \to (\mathfrak{g}_{-3})_{\bar{0}} \ .$$

The remaining Lie brackets are

$$q : \Lambda^2(\mathfrak{g}_{-1})_{\bar{1}} \to (\mathfrak{g}_{-2})_{\bar{0}}, \qquad \Xi : (\mathfrak{g}_{-1})_{\bar{0}} \otimes (\mathfrak{g}_{-1})_{\bar{1}} \to (\mathfrak{g}_{-2})_{\bar{1}},$$

$$\Theta : (\mathfrak{g}_{-1})_{\bar{1}} \otimes (\mathfrak{g}_{-2})_{\bar{1}} \to (\mathfrak{g}_{-3})_{\bar{0}}.$$

Rem: As usual $\Lambda^\bullet$ is meant in the super-sense, in particular q is a quadratic form.

Symbol superalgebras

Fundamental, non-degenerate symbols with growth $(2|4, 1|2, 2|0)$:

(M1) SHC symbol algebra;

(M2) $\mathrm{rank}(\beta) = 1$;

(M3) $q = 0$ and $\Theta = 0$;

(M4) $\omega = 0$ and q, β, Ξ, Θ are the same as for the SHC symbol algebra.

Thm[Kruglikov, S., The] Any fundamental, non-degenerate symbol superalgebra $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}$ of growth $(2|4, 1|2, 2|0)$ is isomorphic to one of (M1)–(M4). The even part $\mathfrak{m}_{\bar{0}}$ of $\mathfrak{m}$ is the HC symbol for (M1) and (M3).

Superdistribution $\mathcal{D}$ is called of SHC type if its symbol superalgebra is (M1). It follows that symbol superalgebra of any superdistribution of SHC type is rigid w.r.t. small deformations of the superdistribution preserving the growth vector.

A gap phenomenon

Thm[Kruglikov, S., The] Let $\mathcal{D}$ be superdistribution of SHC-type on $(5|6)$-dimens. supermnfd M such that symmetry superalgebra $\mathfrak{inf}(M, \mathcal{D})$ acts locally transitively. If $\mathfrak{inf}(M, \mathcal{D}) \neq G(3)$, then $\dim \mathfrak{inf}(M, \mathcal{D}) \leqslant (10|8)$ (in the strong sense: we have both $\dim \mathfrak{inf}(M, \mathcal{D})_{\bar{0}} \leqslant 10$ and $\dim \mathfrak{inf}(M, \mathcal{D})_{\bar{1}} \leqslant 8$.)

System of PDE involving one ordinary function f of 1 variable:

$$z_x = f(u_{xx}) + u_{x\nu} u_{x\tau}, \quad z_\nu = f'(u_{xx}) u_{x\nu},$$
$$z_\tau = f'(u_{xx}) u_{x\tau}, \quad u_{\nu\tau} = -f'(u_{xx}).$$

The associated Cartan superdistribution $\mathcal{D}$ is of SHC type when $f'' \neq 0$. In this case it shall be considered as a super-extension of the classical family of rank 2 distributions with the Monge normal form $z_x = f(u_{xx})$.

Thm[Kruglikov, S., The] The super-extensions of classical submaximally symmetric models $f(u) = \frac{1}{m} u^m$ $(m \neq -1, 0, \frac{1}{3}, \frac{2}{3}, 1, 2)$ satisfy $\dim \mathfrak{inf}(M, \mathcal{D}) = (10|8)$.

Thanks!