# Conformal transformations and the beginning of the Universe. Part II. 

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## Null geodesics as conformal objects

- Two spacetimes ${ }^{1}(M, g)$ and ( $\left.\hat{M}, \hat{g}\right)$ are conformally related iff there exists a diffeomorphism $\phi: M \rightarrow \hat{M}$ such that $g=\mathrm{e}^{2 \Upsilon} \cdot \phi^{*}(\hat{g})$, with $\Upsilon$ a differentiable function on $M$.
- In the index notation:
- the metric is $\hat{g}_{\mu \nu}=e^{-2 \gamma} g_{\mu \nu}$, the inverse metric is $\hat{g}^{\mu \nu}=\mathrm{e}^{2 \gamma} \mathrm{~g}^{\mu \mu \nu}$, and the Levi-Civita connection coefficients are related by $\Gamma \mu_{\nu \rho}=\Gamma \mu_{\nu \rho}-\delta_{\mu} \gamma_{\rho}-\delta_{\mu} \rho_{\nu}+g_{\nu \rho} \Upsilon^{\prime} \mu$, where $\Upsilon_{\mu}=\Upsilon_{, \mu}$ and $\Upsilon^{\mu}=g^{\mu \nu} \Upsilon_{\nu}$.
- In this way the geodesic equation for a curve $x /=x(t)$ is:

| $\frac{\mathrm{d} \dot{x}^{\mu}}{\mathrm{dt}}+\Gamma^{\mu} \nu_{\rho} \dot{x}^{\prime} \dot{x}^{\rho}=\lambda \dot{x}^{\mu}$ |
| :--- |
| $\frac{\mathrm{d} \dot{x}^{\mu}}{\mathrm{d} t}+\hat{\Gamma}^{\mu}{ }_{\nu \rho} \dot{x}^{\prime} \dot{x}^{\rho}=\left(\lambda-2 \Upsilon_{\rho} \dot{x}^{\rho}\right) \dot{x}^{\mu}+g(\dot{x}, \dot{x}) \Upsilon^{\mu}$. |

- This shows that a null, i.e. satisfying $g(\dot{x}, \dot{x})=0$, geodesic in metric $g$ is also a null geodesic in the metric $\hat{g}$.

[^0]
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- Two spacetimes (M.g) and (M.g) are conformally related iff
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[^10]
## Conformal compactification

The conformal compactifications of spacetimes were introduced by Roger Penrose in the process of making mathematically correct theory of gravitational radiation.

- It was motivated by Trautman-Bondi way of associating energy to gravitational waves. In the Einstein's theory gravitational field is described in terms of the Riemann tensor, Riemann, which decomposes on its trace, Ricci, known as the Ricci tensor, and its totally traceless part, Weyl, known as the Weyl tensor. Schematically Riemann $=$ Weyl + Ricci. It is Ricci which is totally determined by the Einstein's equations, schematically Ricci $=T$. The rest of the curvature, namely the Weyl tensor, is totally undetermined by the energy momentum tensor $T$; one may think about Weyl as the free gravitational part of the curvature. It is remarkable that this 'free part of the curvature' is conformally invariant.

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## Need for null infinity



- To define an amount of energy radiated, one may try to associate energy $m_{1}$ to a spacelike hypersurface $S_{1}$, and then energy $m_{2}$ to a later spacelike hypersurface S2. Simply integrate some expression of mass density
- The difference $m_{1}-m_{2}$ could be then the amount of energy radiated. But $S_{2}$ as going to infinity intercepts all the waves emitted from $S_{1}$; Therefore $m_{2}=m_{1}$.
- It is why one should associate 'mass' to null or asymptotically null hypersurfaces $N_{1}$ and $N_{2}$. The difference of these masses would be the energy carried by waves. For waves, what is important, is this what they carry along null geodesics to infinity, to the place in spacetime where null geodesics end.
- Penrose's idea then, is to introduce boundary to spavetime $M$, whose points constitute future and pasr end-points to each null geodesic in $M$. It follws that only conformal properttes matter here. over $S_{1}$ and then $S_{2}$.


FIGURE 14 To mesarte mass lons throag
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- It is why one should associate 'mass' to null or asymptotically null hypersurfaces $N_{1}$ and $N_{2}$. The difference of these masses would be the energy carried by waves.
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## Conformal compactification

> Definition
> We say that a 4-dimensional Lorentzian manifold ( $\hat{M}, \hat{g}$ ) with
> boundary $\partial \hat{M}$ is a conformal compactification of a
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> $M \rightarrow \operatorname{Int} \hat{M}$
> and a function $\Omega$ on $\hat{M}$, such that (i) $\hat{g}=\Omega^{2} \phi_{*}(g)$, and (ii) $\Omega=0$
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## 2-dimensional Minkowski space

- In $M=\mathbb{R}^{2}$ with the Minkowski metric $g=\mathrm{d} t^{2}-\mathrm{d} x^{2}$, change coordinates to $\tilde{u}=(t-x) / \sqrt{2}$ and $\tilde{v}=(t+x), \sqrt{2}$. This parametrizes $M$ by $-\infty<\tilde{u}, \tilde{v}<+\infty$, and the Minkwski metric is $g=2 \mathrm{~d} \tilde{u} \mathrm{~d} \tilde{v}$.
- Change coordinates in $M$ from ( $\tilde{u}$ v) to (u.v) such that $\tilde{u}=\operatorname{tg} u$ and $\tilde{v}=\operatorname{tg} v$. This transforms the entire $M=\mathbb{R}^{2}$, in a one-to-one fashion, onto the interior of a diamond Int $\hat{M}=\left\{(u, v) \in \mathbb{R}^{2}:-\pi / 2<u, v<\pi / 2\right\}$.

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## 2-dimensional Minkowski space



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## Parts of a boundary

- The compactified 2D Minkowski space $\hat{M}=\left\{(u, v):-\frac{\pi}{2} \leq u, v \leq \frac{\pi}{2}\right\}$ has a boundary $\partial M$ with the following componenis:
- $\mathscr{I}+=\{(u, v)$

$$
\left.u=\frac{\pi}{2},-\frac{\pi}{2}<v<\frac{\pi}{2}\right\} \text { or }
$$ infinity in the future;

$\cdot \mathscr{I}^{-}=\{(U, V)\}$ $\left.u=-\frac{\pi}{2},-\frac{\pi}{2}<v<\frac{\pi}{2}\right\}$ or
$\left.-\frac{\pi}{2}<u<\frac{\pi}{2}, v=-\frac{\pi}{2},\right\}-$ null infinity in the past;

- $i^{0}=\{(u, v)$ $i=\left\{(U, V): U=-\frac{\pi}{2}, V=\frac{\pi}{2},\right\}$-spacelike infinity;
- $i^{-}=\{(u, v)$ $i=\{(u, v)$
infinity in the past or in the future.


In particular $i^{0}$ is a point in which every spacelike hypersurfcae ends, similarly $i^{-}$is a point where every initially timelike curve starts, and $i^{+}$is a point where every finally timelike curve ends.

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- $i^{-}=\left\{(u, v): u=-\frac{\pi}{2}, v=-\frac{\pi}{2}\right\}$ or $i^{+}=\left\{(u, v): u=\frac{\pi}{2}, v=\frac{\pi}{2},\right\}$ - timelike infinity in the past or in the future.


In particular $i^{0}$ is a point in
which every spacelike
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## Parts of a boundary

- The compactified 2D Minkowski space $\hat{M}=\left\{(u, v):-\frac{\pi}{2} \leq u, v \leq \frac{\pi}{2}\right\}$ has a boundary $\partial \hat{M}$ with the following components:
$\begin{aligned}-\mathscr{I}^{+} & =\left\{(u, v): u=\frac{\pi}{2},-\frac{\pi}{2}<v<\frac{\pi}{2}\right\} \text { or } \\ \mathscr{I}^{+} & =\left\{(u, v):-\frac{\pi}{2}<u<\frac{\pi}{2}, v=\frac{\pi}{2},\right\}-\text { null }\end{aligned}$ infinity in the future;
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## 4-dimensional Minkowski space

We start with Minkowski spacetime $(M, g)$ with
$g=\mathrm{d} t^{2}-\mathrm{d} r^{2}-r^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \chi^{2}\right)=\mathrm{d} t^{2}-\mathrm{d} r^{2}-r^{2} \mathrm{~d} s^{2}$, where $\mathrm{d} s^{2}$ is the standard metric on a round sphere $\mathbb{S}^{2}$ of radius 1. Here $-\infty<t<\infty, r \geq 0$, and $(\theta, \phi)$ are the usual
latitude-longitude coordinates on $\mathbb{S}^{2}$.

- Now the change of coordinates $t-r=\sqrt{2} \operatorname{tg} u$, $t+r=\sqrt{2} \operatorname{tg} v$ brings the Minkowski metric to $\Omega^{2} g=2 \mathrm{~d} u \mathrm{~d} v-\frac{1}{2} \sin ^{2}(v-u) \mathrm{ds}{ }^{2}$, where $\Omega=\cos u \cos v$.
- Now the range of coordinates $(v, u)$ is $-\pi / 2 \leq v, u \leq \pi / 2$ and $v-u \geq 0$, so that the resulting picture of the conformally compactified Minkowski space with the regular metric $\hat{g}=2 \mathrm{dud} v-\frac{1}{2} \sin ^{2}(v-u) \mathrm{ds}^{2}$ is as follows:

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## Penrose diagram for 4d Minkowski



Note that both $\mathscr{I}^{ \pm}$are null hypersurfaces.
Note also that Minkowski spacetime is a solution of vacuum Einstein equations with vanishing cosmological constant
$\Lambda=0$.

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## Penrose diagram for Schwarzschild



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## Penrose diagram for Schwarzschild black hole



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## Penrose diagram for Kerr



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## Compactify deSitter space

- The deSitter space as a global manifold can be identified with a quadric $Q$ in $\mathbb{R}^{5}$ given by the equation

$$
-T^{2}+X^{2}+Y^{2}+Z^{2}+W^{2}=H^{2}=\text { const. }
$$

- It acquires a Lorentzian metric from the 5D Minkowski metric $g=-\mathrm{d} T^{2}+\mathrm{d} X^{2}+\mathrm{d} Y^{2}+\mathrm{d} Z^{2}+\mathrm{d} W^{2}$ in $\mathbb{R}^{5}$.
- Parametrizing $Q$ by $T=\frac{\sinh H t}{H}, X=\frac{\cosh H t}{H} \sin r \sin \theta \cos \phi$, $Y=\frac{\cosh H t}{H} \sin r \sin \theta \sin \phi, Z=\frac{\cosh H t}{H} \sin r \cos \theta$, $W=\frac{\cosh H t}{H} \cos r$, one shows that the metric $g$ on $Q$ is $g=-\mathrm{d} t^{2}+\left(\frac{\cosh H t}{H}\right)^{2}\left(\mathrm{~d} r^{2}+\sin ^{2} r\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}\right)\right)$.
Note that the spacial part is conformal to the standard metric on a 3 -sphere $\mathbb{S}^{3}$.
- Introduce new coordinate $\tau$ such that $\mathrm{d} \tau=\frac{H \mathrm{Ht}}{\cosh H t}$, then
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## Compactified deSitter space

$g=\left(\frac{\cosh H t}{H}\right)^{2}\left(-\mathrm{d} \tau^{2}+\left(\mathrm{d} r^{2}+\sin ^{2} r\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}\right)\right)\right)=\left(\frac{\cosh H t}{H}\right)^{2} \hat{g}$,

- where

$$
\hat{g}=-d^{2}+\left(a-2+\sin ^{2}+\left(a n^{2}+\sin ^{2} n a \theta^{2}\right)\right)
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is the Minkowski metric.

- Now, $\tau=2 \operatorname{arctg}\left(\operatorname{tgh} \frac{\mathrm{Ht}}{2}\right)$, so since $\operatorname{tgh} \frac{\mathrm{Ht}}{2} \rightarrow \pm 1$ as $t \rightarrow \pm \infty$, then if $t \rightarrow \pm \infty$ the new time variable $\tau \rightarrow \pm \pi$.
- Introducing $\Omega=\frac{H}{\cosh H t}$, we see that $\Omega \rightarrow 0$ when $\tau \rightarrow \pm \pi$.
- We thus have a compactification of the deSitter spacetime $Q$ to $\hat{Q}=[-\pi, \pi] \times \mathbb{S}^{3}$, but now the boundary $\partial \hat{Q}$ corresponding to $\tau= \pm \pi$ is spacelike!


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- Now, $\tau=2 \operatorname{arctg}\left(\operatorname{tgh} \frac{\mathrm{Ht}}{2}\right)$, so since $\operatorname{tgh} \frac{\mathrm{Ht}}{2} \rightarrow \pm 1$ as $t \rightarrow \pm \infty$, then if $t \rightarrow \pm \infty$ the new time variable $\tau \rightarrow \pm \pi$.
- Introducing $\Omega=\frac{H}{\cosh H t}$, we see that $\Omega \rightarrow 0$ when $\tau \rightarrow \pm \pi$.
- We thus have a compactification of the deSitter spacetime $Q$ to $\hat{Q}=[-\pi, \pi] \times \mathbb{S}^{3}$, but now the boundary $\partial \hat{Q}$ corresponding to $\tau= \pm \pi$ is spacelike!

Compactified deSitter space


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## Why scri of Minkowski is null and scri of deSitter is spacelike?

One can check that the conformally flat deSitter spacetime satisfies vacuum Einstein's equations with positive cosmological constant $\wedge$. Actually the deSitter metric

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## Assymptotic simplicity

## Definition

A smooth spacetime $M$ with metric $g$ is asymptotically simple if there is a smooth manifold $\hat{M}$ with boundary $\mathscr{I}$ and a metric $\hat{g}$ and a smooth scalar function $\Omega$ such that

- $M=\operatorname{Int} \hat{M}$,
- $\hat{g}=\Omega^{2} g$,
- $\Omega>0$ in $M ; \Omega=0$ and $\mathrm{d} \Omega \neq 0$ on $\mathscr{I}$,
- every null geodesic in $M$ has a future and a past endpoint on $\mathscr{I}$.
The last condition is too strong to include spacetimes with black holes. One releases it introducing a notion of a weakly assymptotically simple spacetime (WAS). We will not consider it here, but of course assymptotically simple spacetime is WAS. We have


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R=\Omega^{2} \hat{R}-6 \Omega \hat{\square} \Omega+12 \hat{g}^{\mu \nu} \Omega_{\mu} \Omega_{\nu},
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where $\bar{\square}$ is the D'Alambertian operator in the metric $\hat{g}$.
(Perhaps this formula has some sign errors, because I screwed up the signature conventions; but I believe that it is right.)

- On the other hand, using the Einstein's equastions, we can relate the Ricci sclar curvature $R$ to the trace of the energy momentum tensor $T=T_{\mu}^{\mu}$ and the cosmological constant $\wedge$. This gives

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R=4 \Lambda-\kappa T
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