Conformal transformations and the beginning of the Universe. Part II.

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- Two **spacetimes**[1] (M, g) and $(\hat{M}, \hat{g})$ are **conformally related** iff there exists a diffeomorphism $\phi : M \to \hat{M}$ such that $g = \mathrm{e}^{2\Upsilon} \cdot \phi^*(\hat{g})$, with Υ a differentiable function on M.
- In the **index notation**:
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- In this way the geodesic equation for a curve $x^{\mu} = x^{\mu}(t)$ is:

$$\boxed{\frac{\mathrm{d}\dot{x}^{\mu}}{\mathrm{d}t} + \Gamma^{\mu}{}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = \lambda \dot{x}^{\mu}}$$

, or if we replace Γ by $\hat{\Gamma}$, is:

$$\boxed{\frac{\mathrm{d}\dot{x}^{\mu}}{\mathrm{d}t} + \hat{\Gamma}^{\mu}{}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = (\lambda - 2\Upsilon_{\rho}\dot{x}^{\rho})\dot{x}^{\mu}} + g(\dot{x}, \dot{x})\Upsilon^{\mu}.$$

- This shows that a **null**, i.e. satisfying $g(\dot{x}, \dot{x}) = 0$, **geodesic** in metric g is also a **null geodesic** in the metric $\hat{g}$.

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Conformal compactification

The **conformal compactifications** of spacetimes were introduced by **Roger Penrose** in the process of making mathematically correct theory of **gravitational radiation.**

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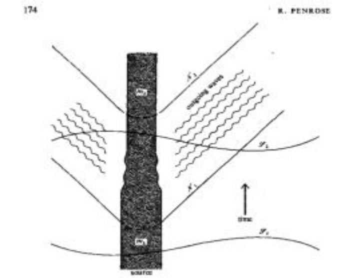

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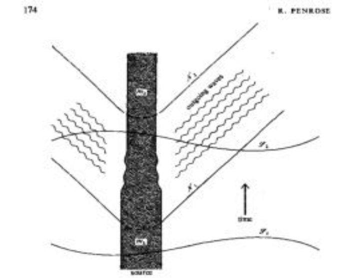

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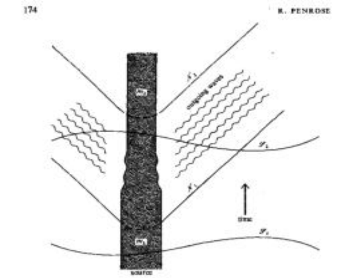

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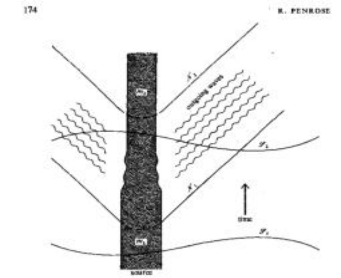

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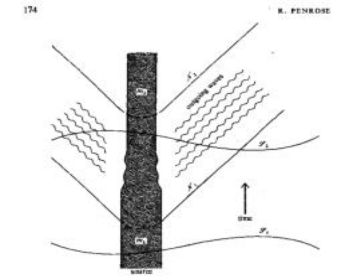

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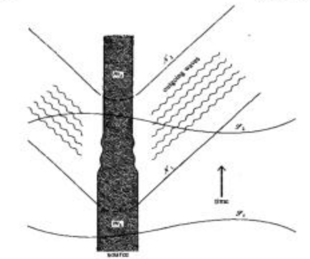

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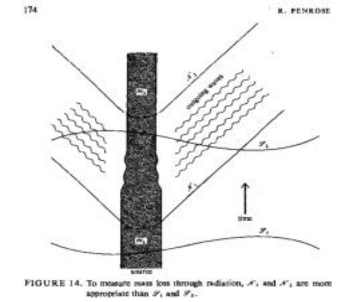
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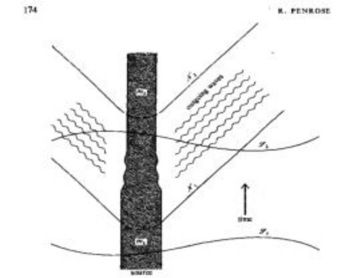

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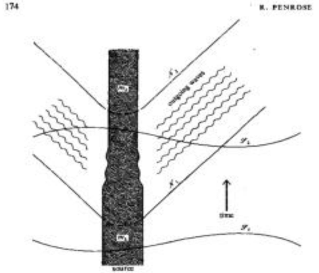

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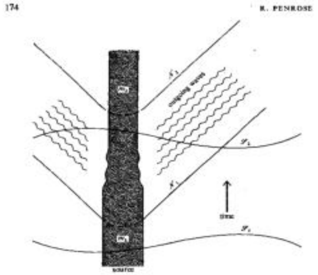

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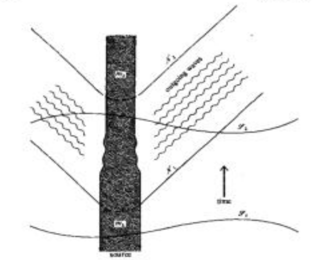

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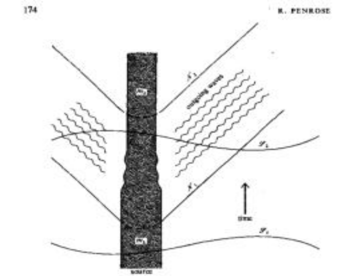

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Definition

We say that a 4-dimensional Lorentzian manifold $(\hat{M}, \hat{g})$ with **boundary** $\partial\hat{M}$ is a **conformal compactification** of a spacetime (M, g) iff there exists a diffeomorphism

$$\phi : M \to \mathrm{Int}\hat{M}$$

and a function Ω on $\hat{M}$, such that (i) $\hat{g} = \Omega^2 \phi_*(g)$, and (ii) $\Omega = 0$ on $\partial\hat{M}$, and (iii) $\mathrm{d}\Omega \neq 0$ at $\partial\hat{M}$.

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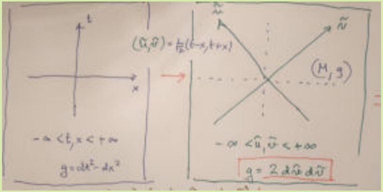
We say that a 4-dimensional Lorentzian manifold $(\hat{M}, \hat{g})$ with **boundary** $\partial\hat{M}$ is a **conformal compactification** of a spacetime (M, g) iff there exists a diffeomorphism

$$\phi : M \to \mathrm{Int}\hat{M}$$

and a function Ω on $\hat{M}$, such that (i) $\hat{g} = \Omega^2 \phi_*(g)$, and (ii) $\Omega = 0$ on $\partial\hat{M}$, and (iii) $\mathrm{d}\Omega \neq 0$ at $\partial\hat{M}$.

- In $M = \mathbb{R}^2$ with the Minkowski metric $g = \mathrm{d}t^2 - \mathrm{d}x^2$, change coordinates to $\tilde{u} = (t - x)/\sqrt{2}$ and $\tilde{v} = (t + x)/\sqrt{2}$. This parametrizes M by $-\infty < \tilde{u}, \tilde{v} < +\infty$, and the Minkwski metric is $g = 2\mathrm{d}\tilde{u}\mathrm{d}\tilde{v}$.
- Change coordinates in M from $(\tilde{u}, \tilde{v})$ to (u, v) such that $\tilde{u} = \operatorname{tg} u$ and $\tilde{v} = \operatorname{tg} v$. This transforms the entire $M = \mathbb{R}^2$, in a one-to-one fashion, onto the **interior** of a **diamond** $\mathrm{Int}\hat{M} = \{(u, v) \in \mathbb{R}^2 \ : \ -\pi/2 < u, v < \pi/2\}$.



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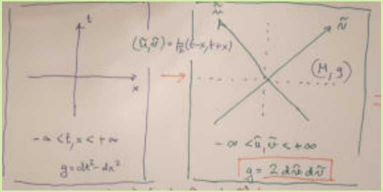

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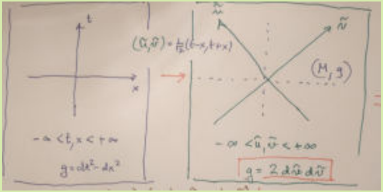

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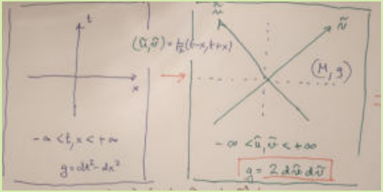

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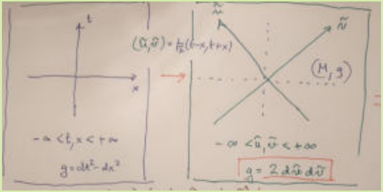

2-dimensional Minkowski space

- In $M = \mathbb{R}^2$ with the Minkowski metric $g = \mathrm{d}t^2 - \mathrm{d}x^2$, change coordinates to $\tilde{u} = (t - x)/\sqrt{2}$ and $\tilde{v} = (t + x)/\sqrt{2}$. This parametrizes M by $-\infty < \tilde{u}, \tilde{v} < +\infty$, and the Minkwski metric is $g = 2\mathrm{d}\tilde{u}\mathrm{d}\tilde{v}$.
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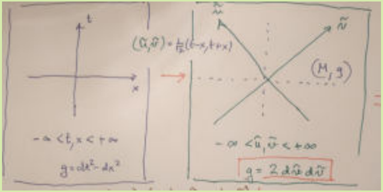

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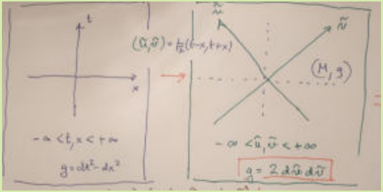

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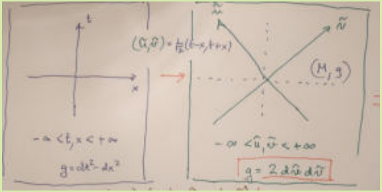
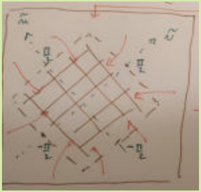
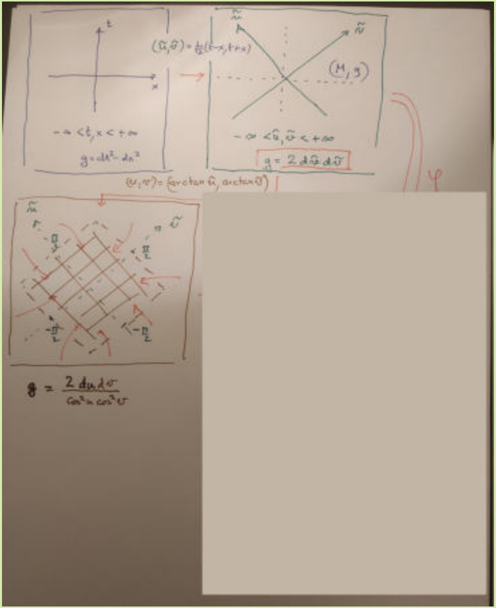
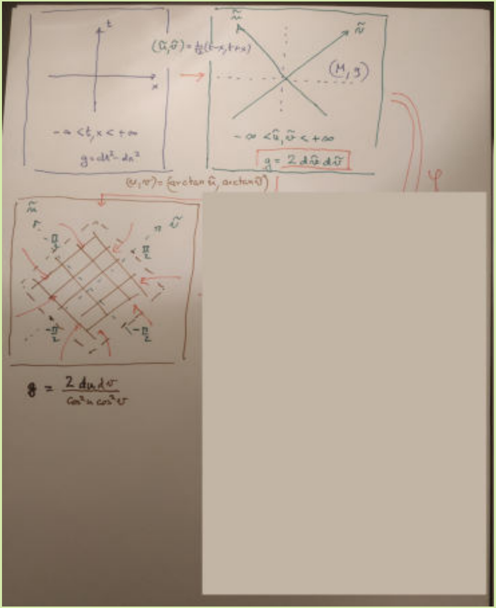





(M,g)

2-dimensional Minkowski space



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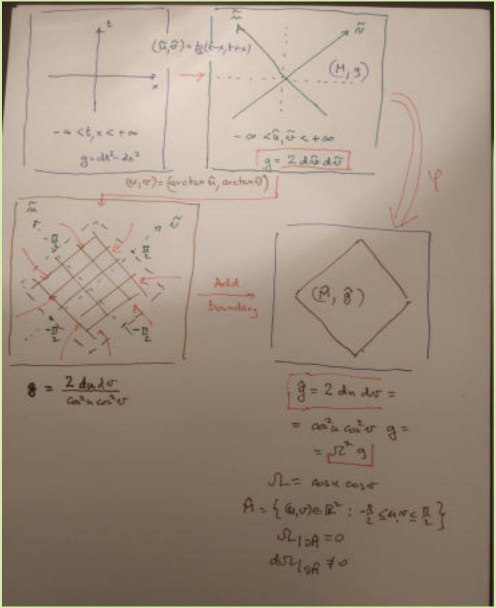

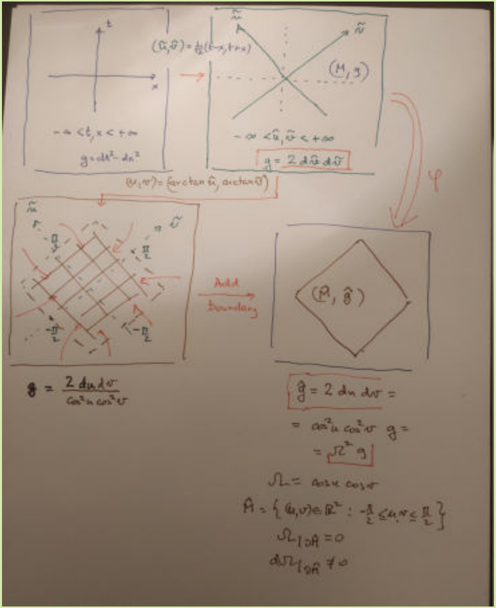

-∞ < t,x < +∞
g = dt² - dx²
(M, g)
-∞ < ũ, ṽ < +∞
g = 2 dũ dṽ
(u, v) = (arctan ũ, arctan ṽ)
φ
Add boundary
(M̂, ĝ)
g = 2 du dv / cos²u cos²v
ĝ = 2 du dv =
= cos²u cos²v g =
= Ω² g
Ω = cos u cos v
Ω|∂M̂ = 0
dΩ|∂M̂ ≠ 0

- The compactified 2D Minkowski space $\hat{M} = \{(u, v) \ : -\frac{\pi}{2} \leq u, v \leq \frac{\pi}{2}\}$ has a boundary $\partial\hat{M}$ with the following components:
  - $\mathscr{I}^{+} = \{(u, v) \ : \ u = \frac{\pi}{2}, -\frac{\pi}{2} < v < \frac{\pi}{2}\}$ or $\mathscr{I}^{+} = \{(u, v) \ : \ -\frac{\pi}{2} < u < \frac{\pi}{2}, v = \frac{\pi}{2}, \}$- **null infinity in the future**;
  - $\mathscr{I}^{-} = \{(u, v) \ : \ u = -\frac{\pi}{2}, -\frac{\pi}{2} < v < \frac{\pi}{2}\}$ or $\mathscr{I}^{-} = \{(u, v) \ : \ -\frac{\pi}{2} < u < \frac{\pi}{2}, v = -\frac{\pi}{2}, \}$- **null infinity in the past**;
  - $i^{0} = \{(u, v) \ : \ u = \frac{\pi}{2}, v = -\frac{\pi}{2}\}$ or $i^{0} = \{(u, v) \ : \ u = -\frac{\pi}{2}, v = \frac{\pi}{2}, \}$- **spacelike infinity**;
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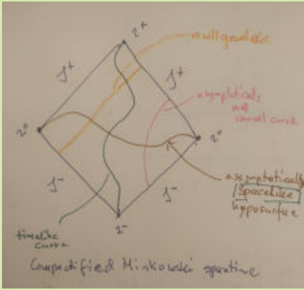

In particular i^{0} is a point in which every **spacelike** hypersurfcae ends, similarly i^{-} is a point where every **initially timelike curve starts**, and i^{+} is a point where every **finally timelike curve ends**.

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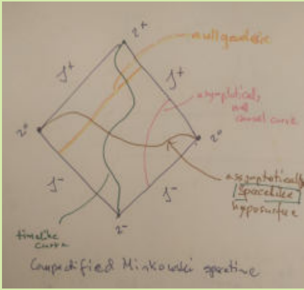

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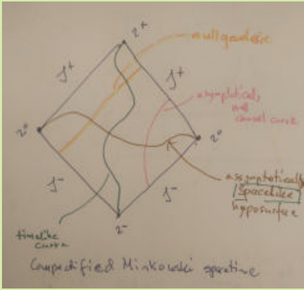

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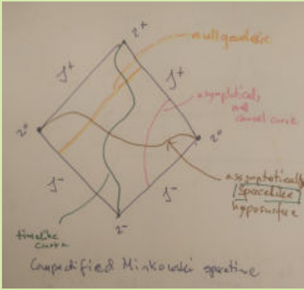

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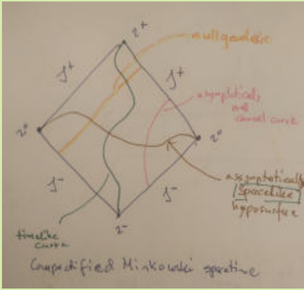

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Parts of a boundary

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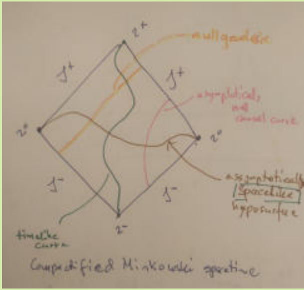

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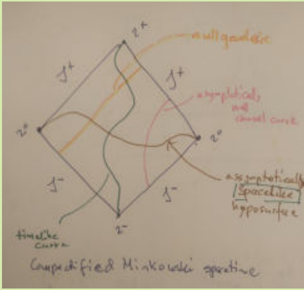

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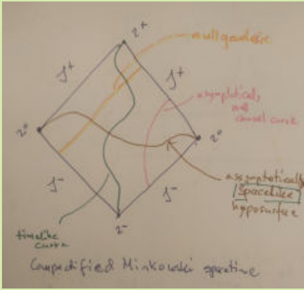

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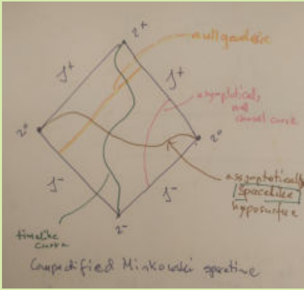

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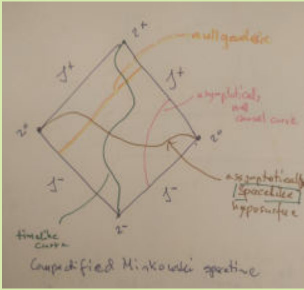

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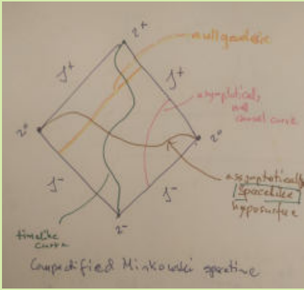

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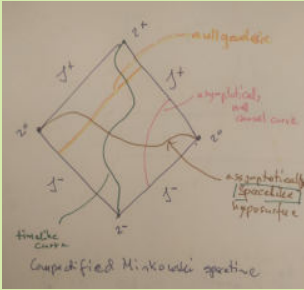

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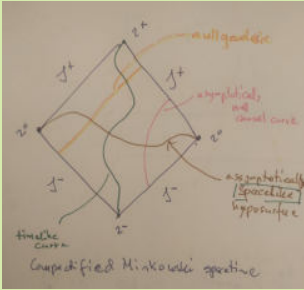

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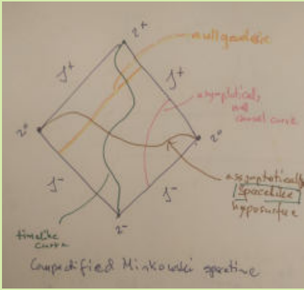

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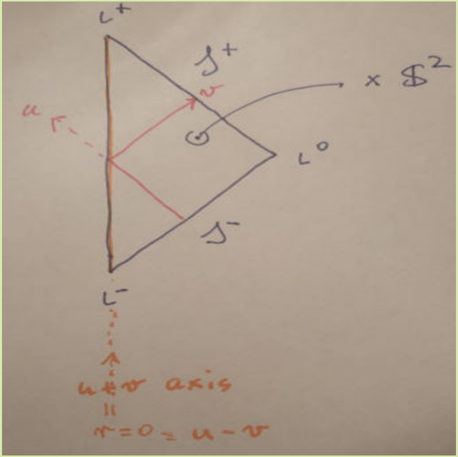

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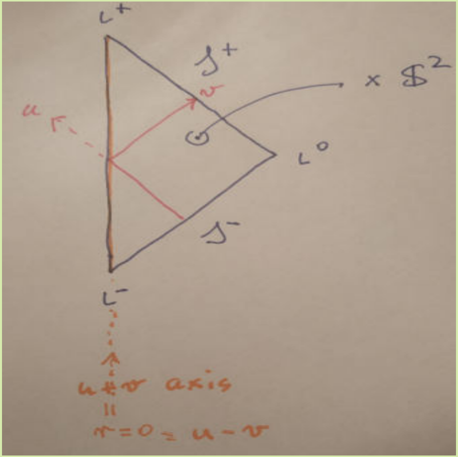
Penrose diagram for 4d Minkowski



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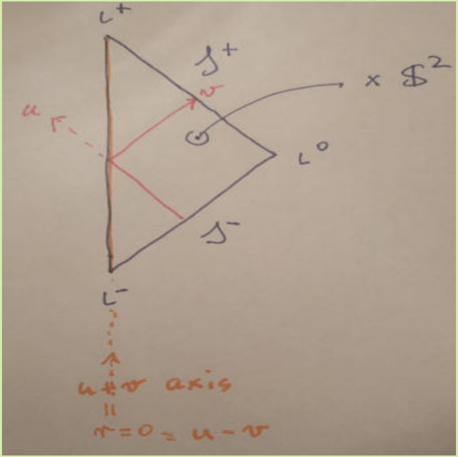

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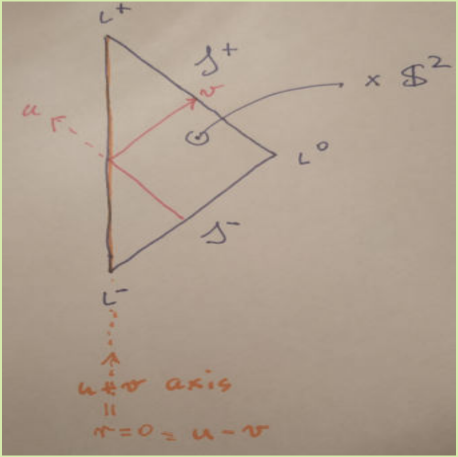

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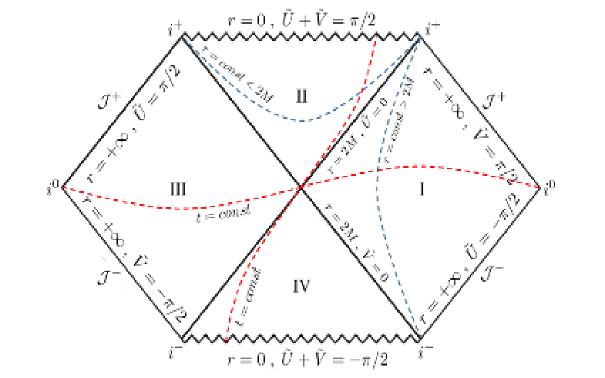

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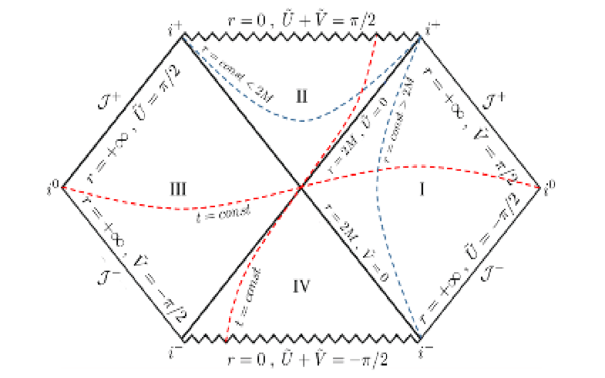

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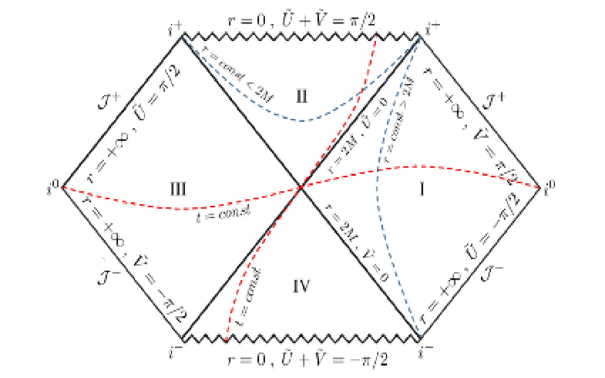

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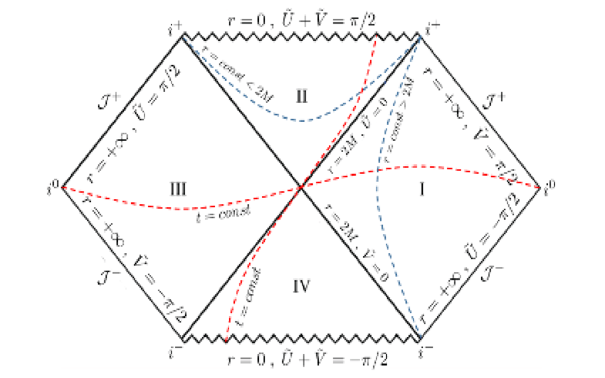

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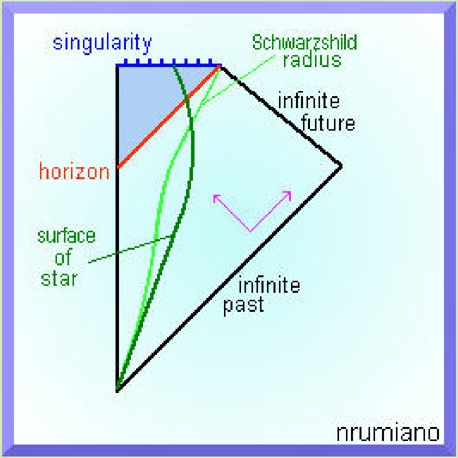

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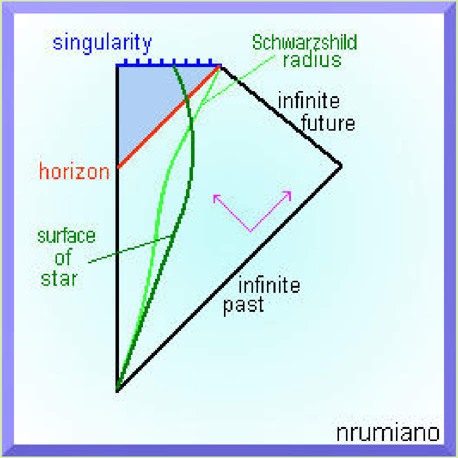
Penrose diagram for Schwarzschild black hole



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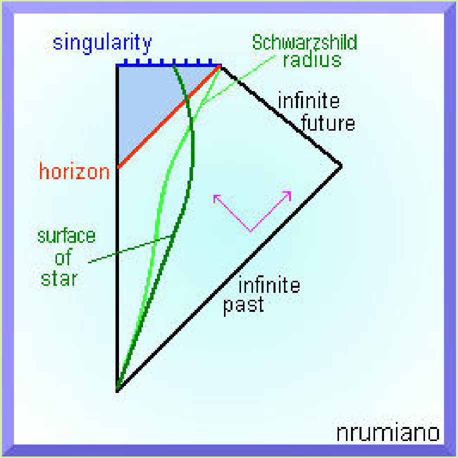

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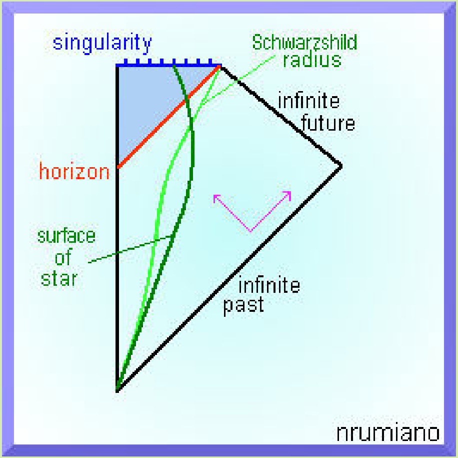

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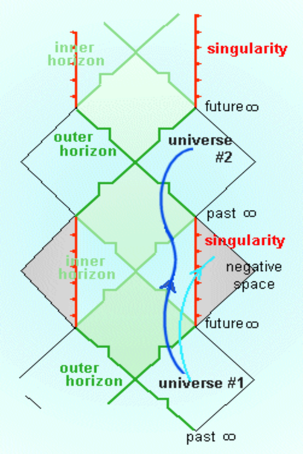

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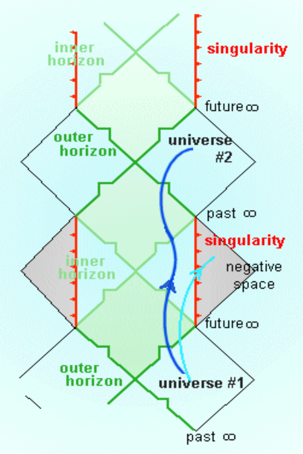

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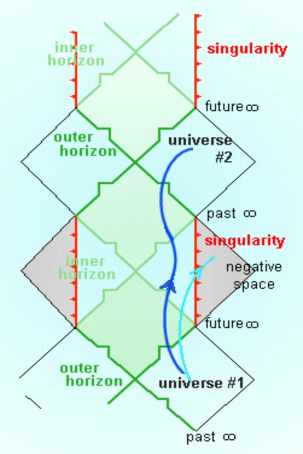

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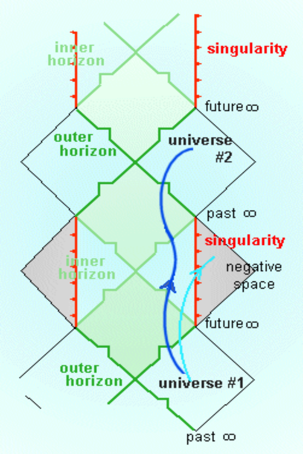

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Compactify deSitter space

- The deSitter space as a global manifold can be identified with a **quadric** Q in $\mathbb{R}^5$ given by the equation
$$-T^2 + X^2 + Y^2 + Z^2 + W^2 = \frac{1}{H}^2 = \mathrm{const}.$$
- It acquires a Lorentzian metric from the 5D Minkowski metric $g = -\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 + \mathrm{d}W^2$ in $\mathbb{R}^5$.
- Parametrizing Q by $T = \frac{\sinh Ht}{H}$, $X = \frac{\cosh Ht}{H} \sin r \sin\theta \cos\phi$, $Y = \frac{\cosh Ht}{H} \sin r \sin\theta \sin\phi$, $Z = \frac{\cosh Ht}{H} \sin r \cos\theta$, $W = \frac{\cosh Ht}{H} \cos r$, one shows that the metric g on Q is
$$g = -\mathrm{d}t^2 + \left(\frac{\cosh Ht}{H}\right)^2 (\mathrm{d}r^2 + \sin^2 r(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)).$$
Note that the spacial part is conformal to the standard metric on a 3-sphere $\mathbb{S}^3$.
- Introduce new coordinate τ such that $\mathrm{d}\tau = \frac{H\mathrm{d}t}{\cosh Ht}$, then

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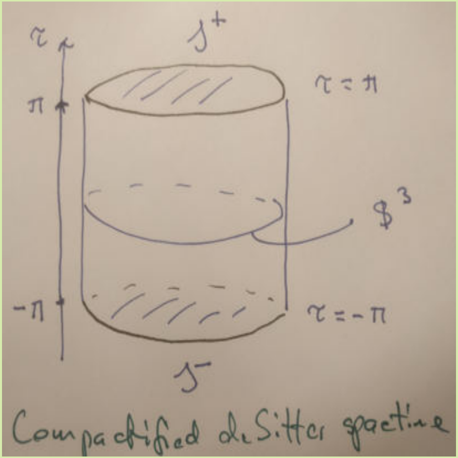

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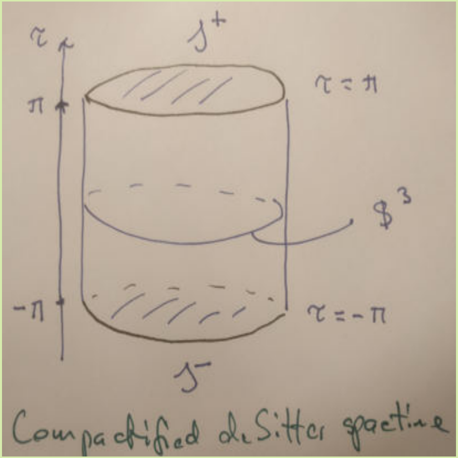

Why scri of Minkowski is null and scri of deSitter is spacelike?

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The last condition is too strong to include spacetimes with **black holes.** One releases it introducing a notion of a **weakly assymptotically simple spacetime** (WAS). We will not consider it here, but of course assymptotically simple spacetime is WAS. We have

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The last condition is too strong to include spacetimes with **black holes.** One releases it introducing a notion of a **weakly assymptotically simple spacetime** (WAS). We will not consider it here, but of course assymptotically simple spacetime is WAS. We have

Definition

A smooth spacetime M with metric g is **asymptotically simple** if there is a smooth manifold $\hat{M}$ with boundary $\mathscr{I}$ and a metric $\hat{g}$ and a smooth scalar function Ω such that

- $M = \mathrm{Int}\hat{M}$,
- $\hat{g} = \Omega^2 g$,
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Theorem

The boundary $\mathscr{I}$ of a **(weakly) assymptotically simple spacetime**satisfying Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

with $T^{\mu}{}_{\mu} = 0$ in the vicinity of $\mathscr{I}$, is

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- Using the transformation for the Levi-Civita connection coefficients for the metric $\hat{g}$ and g of a WAS spacetime, one gets the following relation between the Ricci scalars $\hat{R}$ and R:
$$R = \Omega^2 \hat{R} - 6\Omega \hat{\Box}\Omega + 12\hat{g}^{\mu\nu}\Omega_\mu \Omega_\nu,$$
where $\hat{\Box}$ is the D'Alambertian operator in the metric $\hat{g}$. (Perhaps this formula has some sign errors, because I screwed up the signature conventions; but I believe that it is right.)
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