Tutorial: caclulating Cartan tensor and deeper invariants. Theory, algorithm A, algorithm B

Algorithm A

input: vector fields V_1, V_2, given in any coordinates;
a point p at which the distribution D described by V_1, V_2 is (2,3,5).

output: the Cartan tensor of D at p.

Algorithm B

input: a 3×5 characteristic matrix of an endowed 5-dim algebra $(\mathcal{A}, \mathcal{P})$.

output: the Cartan tensor of the homogeneous left-invariant distribution D induced by $(\mathcal{A}, \mathcal{P})$.

• In algorithm B we realize by vector fields neither $(\mathcal{A}, \mathcal{P})$ nor $\mathcal{D}$.

• Algorithm B given an explicit formula for Cartan tensor in terms of 15 parameters of the characteistic matrix.

L_N-equivalence of pairs of vector fields

Take any symbol (nilpotent approximation) $N = (N_1, N_2)$ meaning (here) two vector fields, expressed in a local coordinate system $x_1, \dots, x_5$ such that N_1, N_2 are quasi-homogeneous of degree -1 wrt the weights 1,1,2,3,3 and the vector fields

$$N_1,\ N_2,\ N_3 = [N_1, N_2],\ N_4 = [N_1, N_3],\ N_5 = [N_2, N_3]$$

are linearly independent at $0 \in \mathbb{R}^5$. Recall the linear operator

$$L_N :\ (Z, H) \to \left[Z, \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \right] + H \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

Z is a vector field

H is a 2×2 matrix whose entries are functions

Definition. Two pairs of vector fields are L_N-eqivalent if their difference belongs to the image of L_N.

Preliminary normal form 1 wrt L_N-equivalence

Any pair of vector fields can be expressed in the form

$$\begin{aligned} U_1 &= F_{11}N_1 + F_{12}N_2 + F_{13}N_3 + F_{14}N_4 + F_{15}N_5 \\ U_2 &= F_{21}N_1 + F_{22}N_2 + F_{23}N_3 + F_{24}N_4 + F_{25}N_5 \end{aligned} \tag{1}$$

Claim 1. The pair of vector fields (1) is L_N-equivalent to

Preliminary normal form 1

$$\begin{aligned} U_1 &= A_{11}N_4 + A_{12}N_4 \\ U_2 &= A_{21}N_4 + A_{22}N_5 \end{aligned}$$

where $A_{11} = F_{14},\ A_{12} = F_{15}, A_{21} = F_{24},\ A_{22} = F_{25}$

Proof. Take $Z = f_1 N_1 + f_2 N_2$. Then
$[Z, N_1] = -f_2 N_3 \text{ mod } N_1, N_2, \quad [Z, N_2] = f_1 N_3 \text{ mod } N_1, N_2$
and the claim follows.

(Z, H) preserving preliminary normal form 1

Saying that (Z, H) preserves a normal form we mean that the operator L_N brings (Z, H) to a pair of vector fields in this normal form.

Claim 2. A pair (Z, H) preserves the preliminary normal form 1 if and only if

$$\begin{gathered}\text{(Z,H) preserving preliminary normal form 1}\\ Z = f_1 N_1 + f_2 N_2 + f_3 N_3 + f_4 N_4 + f_5 N_5 \\ f_1 = -N_2(f_3),\ f_2 = N_1(f_3),\ H = \begin{pmatrix} N_1(f_1) & N_1(f_2) \\ N_2(f_1) & N_2(f_2) \end{pmatrix}\end{gathered}$$

Proof. We have

$$\begin{gathered}Z = f_1 N_1 + f_2 N_2 + f_3 N_3 + f_4 N_4 + f_5 N_5 \implies \\ [Z, N_1] = -N_1(f_1)N_1 - N_1(f_2)N_2 - (f_2 + N_1(f_3))N_3 \mod N_4, N_5 \\ [Z, N_2] = -N_2(f_1)N_1 - N_2(f_2)N_2 + (f_1 - N_2(f_3))N_3 \mod N_4, N_5\end{gathered}$$

and the claim follows.

Equivalence problem for 2×2 matrices (whose entries are functions)

For any pair (Z, H) preserving the preliminary normal form 1 (see the formulas in the previous slide) we have

$$L_N(Z,H) = \begin{array}{l} -(f_3 + N_1(f_4))N_4 - N_1(f_5)N_5 \\ -N_2(f_4)N_4 - (f_3 + N_2(f_5))N_5 \end{array}$$

Introduce the following operator, from the space of triples of functions to the space of 2×2 matrices with functional entries:

$$M_N(f_3, f_4, f_5) = \begin{pmatrix} f_3 + N_1(f_4) & N_1(f_5) \\ N_2(f_4) & f_3 + N_2(f_5) \end{pmatrix}$$

Therefore the L_N-equivalence for pairs of vector fields reduces to the following equivalence for 2×2 matrices with functional entries:

Definition. Two 2×2 matrices, with functional entries, are equivalent if their difference belongs to the image of the operator M_N.

Good choice of N_1, N_2

My choice of N_1, N_2 is, as I said in the lectures, as follows:

$$N_1 = \partial_{x_1} + x_2(\partial_z + x_1\partial_{y_1} + x_2\partial_{y_2})$$
$$N_2 = \partial_{x_2} - x_1(\partial_z + x_1\partial_{y_1} + x_2\partial_{y_2})$$

The main advantage, that will be used throughout constructing an exact normal form is as follows:

$$x_1 N_1 + x_2 N_2 = \text{Euler vector field} = E = x_1\partial_{x_1} + x_2\partial_{x_2}$$

Preliminary normal form 2

In what follows N_1, N_2 is the nilpotent approximation in the previous slide.

Claim 3. Any 2×2 matrix with functional entries
is equivalent to a matrix Q such that

preliminary normal form 2

$$Q = (Q_{ij}) : \quad Q^{\mathrm{tr}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \;\Leftrightarrow Q = \begin{pmatrix} x_2 S_1 & x_2 S_2 \\ -x_1 S_1 & -x_1 S_2 \end{pmatrix}$$

Proof. We have $M_N(f_3 = 0, f_4, f_5) = \begin{pmatrix} N_1(f_4) & N_1(f_5) \\ N_2(f_4) & N_2(f_5) \end{pmatrix}$, thereore
$\Big(M_N(f_3 = 0, f_4, f_5)\Big)^{\mathrm{tr}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} E(f_4) \\ E(f_5) \end{pmatrix}$, and the claim follows from the simple fact that the equation $E(f) = g$ has a solution f for any g in the ideal (x_1, x_2).

Recall that E is the Euler vector field $x_1\partial_{x_1} + x_2\partial_{x_2}$.

Reduction to preliminary normal form 2

In order to bring an arbitrary 2×2 matrix A to preliminary normal form 2, we need the following operators. Introduce the operator T:

$$\mathrm{T}(f) = \text{a unique function } \widetilde{f} \text{ such that } \widetilde{f} + E(\widetilde{f}) = f$$

$$f = \sum_{i \geq 0} f^{(i)}, \quad f^{(i)} = \sum_{i_1+i_2=i} c_{i_1,i_2}(z, y_1, y_2) x_1^{i_1} x_2^{i_2} \implies \widetilde{f} = \sum_{i \geq 0} \frac{f^{(i)}}{i+1}.$$

We will need the following operators $\mathrm{T}_1, \mathrm{T}_2$:

$$\mathrm{T}_1(f) = N_1(\mathrm{T}(f)), \qquad \mathrm{T}_2(f) = N_2(\mathrm{T}(f))$$

Claim 4. A matrix $A = (A_{ij})$ is equivalent to preliminary normal form 2 with

$$S_1 = \mathrm{T}_2(A_{11}) - \mathrm{T}_1(A_{21}), \quad S_2 = \mathrm{T}_2(A_{12}) - \mathrm{T}_1(A_{22}).$$

Proof. Exercise.

(f_3, f_4, f_5) preserving preliminary normal form 2

Like above, we will say that (f_3, f_4, f_5) preserves preliminary normal form 2 in the operator M_N brings (f_3, f_4, f_5) to this normal form.

Claim 5. (f_3, f_4, f_5) preserves preliminary normal form 2 if and only if

f_3, f_4, f_5 preserving preliminary normal form 2

$$f_4 = x_1 h + \alpha_1(z, y_1, y_2),\ \ f_5 = x_2 h + \alpha_2(z, y_1, y_2),\ \ f_3 = -h - E(h)$$

Proof. We have the equations

$$E(f_4) + x_1 f_3 = 0, \quad E(f_5) + x_2 f_3 = 0$$

It follows $\frac{\partial f_4}{\partial x_2} \in (x_1)$ and $\frac{\partial f_5}{\partial x_1} \in (x_2)$. Consequently $f_4 = x_1 h_1 + \alpha_1(z, y_1, y_2)$ and $f_5 = x_1 h_2 + \alpha_2(z, y_1, y_2)$. The two equations take the form $h_1 + E(h_1) + f_3 = 0,\ h_2 + E(h_2) + f_3 = 0$. It follows $h_1 = h_2 = h$ and $f_3 = -h - E(h)$. Q.E.D.

Preliminary normal form 3

Claim 6. Any 2×2 matrix with functional entries is equivalent to a matrix Q such that

preliminary normal form 3

$$Q = (Q_{ij}): \quad Q^{\mathrm{tr}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \ \mathrm{trace}\ Q = 0 \ \Leftrightarrow Q = F \cdot \begin{pmatrix} x_1 x_2 & x_2^2 \\ -x_1^2 & -x_1 x_2 \end{pmatrix}$$

Proof. Take f_3, f_4, f_5 that preserve te preliminary normal form 2 (see the previous slide) with $\alpha_1 = \alpha_2 = 0$. Then

$$M_N(f_3, f_4, f_5) = \begin{pmatrix} -E(h) + x_1 N_1(h) & x_2 N_1(h) \\ x_1 N_2(h) & -E(h) + x_2 N_2(h) \end{pmatrix} = \\ = \begin{pmatrix} -x_2 N_2(h) & x_2 N_1(h) \\ x_1 N_2(h) & -x_1 N_1(h) \end{pmatrix}$$

Therefore trace $M_N(f_3, f_4, f_5) = -E(h)$ and preliminary normal form 3 follows from preliminary normal form 2.

Reduction of preliminary normal form 2 to preliminary normal form 3

Claim 7. One has

$$\begin{pmatrix} x_2 S_1 & x_2 S_2 \\ -x_1 S_1 & -x_1 S_2 \end{pmatrix} \sim F \cdot \begin{pmatrix} x_1 x_2 & x_2^2 \\ -x_1^2 & -x_1 x_2 \end{pmatrix}$$

with

$$F = \mathrm{T}_1(S_1) + \mathrm{T}_2(S_2).$$

Proof. Exercise.

f_3, f_4, f_5 preserving preliminary normal form 3

Claim 8. f_3, f_4, f_5 preserve preliminary normal form 3
if and only if for some functions

$$\alpha_1 = \alpha_1(z, y_1, y_2),\ \alpha_2 = \alpha_2(z, y_1, y_2),\ \beta = \beta(z, y_1, y_2)$$

one has

$$\begin{gathered} f_3 = -2R_1(\alpha_1, \alpha_2) - 3R_2(\alpha_1, \alpha_2) - \beta \\ f_4 = x_1\Big(R_1(\alpha_1, \alpha_2) + R_2(\alpha_1, \alpha_2) + \beta\Big) + \alpha_1 \\ f_5 = x_2\Big(R_1(\alpha_1, \alpha_2) + R_2(\alpha_1, \alpha_2) + \beta\Big) + \alpha_2 \end{gathered}$$

where

$$\begin{gathered} R_1(\alpha_1, \alpha_2) = x_2 \tfrac{\partial \alpha_1}{\partial z} - x_1 \tfrac{\partial \alpha_2}{\partial z} \\ R_2(\alpha_1, \alpha_2) = \tfrac{1}{2}\Big(- x_1^2 \tfrac{\partial \alpha_2}{\partial y_1} + x_1 x_2 \big(\tfrac{\partial \alpha_1}{\partial y_1} - \tfrac{\partial \alpha_2}{\partial y_2}\big) + x_2^2 \tfrac{\partial \alpha_2}{\partial y_2}\Big) \end{gathered}$$

Proof of claim 8

Recall that f_3, f_4, f_5 preserve preliminary form 3 if and only if

$$f_4 = x_1 h + \alpha_1(z, y_1, y_2),\ \ f_5 = x_2 h + \alpha_2(z, y_1, y_2),\ \ f_3 = -h - E(h)$$

For such f_3, f_4, f_5 the matrix $M_N(f_3, f_4, f_5)$ is in preliminary normal form 2 and it is in preliminary form 3 if and only if it has the zero trace. It gives the equation

$$E(h) = N_1(\alpha_1(z, y_1, y_2)) + N_2(\alpha_1(z, y_1, y_2))$$

whose general solution is $h = R_1(\alpha_1, \alpha_2) + R_2(\alpha_1, \alpha_2) + \beta$, where $\beta = \beta(z, y_1, y_2)$ is an arbitrary function. Note that $R_1(\alpha_1, \alpha_2)$ and $R_2(\alpha_1, \alpha_2)$ are homogeneous polynomials in x_1, x_2 of degrees 1 and 2 respectively. Therefore

$$f_3 = -h - E(h) = -2R_1(\alpha_1, \alpha_2) - 3R_2(\alpha_1, \alpha_2) - \beta$$

Q.E.D.

How f_3, f_4, f_5 that preserve preliminary normal form 3 changes the function F in this normal form?

We have $F \to F + \Delta F$ where $M_N(f_3, f_4, f_5) = \Delta F \cdot \begin{pmatrix} x_1x_2 & x_2^2 \\ -x_1^2 & -x_1x_2 \end{pmatrix}$

The function f_3, f_4, f_5 are determined by
$\alpha_1 = \alpha_1(z, y_1, y_2), \alpha_2 = \alpha_2(z, y_1, y_2), \beta = \beta(z, y_1, y_2)$.

Claim 9 (proof: straightforward calculation). The ΔF above is the following degree 3 polynomial in x_1, x_2:

$$\Delta F = \sum_{i_1+i_2 \leq 3} Q_{ij} x_1^{i_1} x_2^{i_2}, \quad Q_{ij} = Q_{ij}(z, y_1, y_2)$$

$$Q_{00} = \frac{\partial\beta}{\partial z} + \frac{1}{2}\frac{\partial\alpha_1}{\partial y_1} + \frac{1}{2}\frac{\partial\alpha_2}{\partial y_2}, \quad Q_{10} = \frac{\partial\beta}{\partial y_1} - \frac{\partial^2\alpha_2}{\partial z^2}, \quad Q_{01} = \frac{\partial\beta}{\partial y_2} + \frac{\partial^2\alpha_1}{\partial z^2}$$

$$Q_{20} = -\frac{3}{2}\frac{\partial^2\alpha_2}{\partial z\partial y_1}, \quad Q_{11} = \frac{3}{2}\frac{\partial^2\alpha_1}{\partial z\partial y_1} - \frac{3}{2}\frac{\partial^2\alpha_2}{\partial z\partial y_2}, \quad Q_{02} = \frac{3}{2}\frac{\partial^2\alpha_1}{\partial z\partial y_2}$$

$$Q_{30} = -\frac{1}{2}\frac{\partial^2\alpha_2}{\partial y_1^2}, \quad Q_{21} = \frac{1}{2}\frac{\partial^2\alpha_1}{\partial y_1^2} - \frac{\partial^2\alpha_2}{\partial y_1\partial y_2}$$

$$Q_{12} = -\frac{1}{2}\frac{\partial^2\alpha_2}{\partial y_2^2} + \frac{\partial^2\alpha_1}{\partial y_1\partial y_2}, \quad Q_{03} = \frac{1}{2}\frac{\partial^2\alpha_1}{\partial y_2^2}$$

Exact normal form

The L_N-equivalence of pairs of vector fields has been reduced to the following equivalence of functions:

• Two functions $F(x_1, x_2, z, y_1, y_2), \widetilde{F}(x_1, x_2, z, y_1, y_2)$ are equivalent if their difference has the form ΔF in the previous slide, with some functions $\alpha_1(z, y_1, y_2), \alpha_2(z, y_1, y_2), \beta(z, y_1, y_2)$.

Proposition 10. With respect to this equivalence, an exact normal form is the ideal I that I defined in the first lecture.

The proof requires some work,
but it is not difficult if you know certain techniques.

We have constructed an exact normal form for pairs of vector fields with respect to the L_N-equivalence:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + F \cdot \begin{pmatrix} x_1 x_2 & x_2^2 \\ -x_1^2 & -x_1 x_2 \end{pmatrix} \begin{pmatrix} [N_1, [N_1, N_2]] \\ [N_2, [N_1, N_2]] \end{pmatrix}, \quad F \in \text{ideal I}$$

If Cartan tensor only, not deeper invariants: much simpler

Note that if a pair of vector fields in preliminary normal form 3 is quasi-homogeneous of degree i then the function F in this normal form is quasi-homogeneous of degree $i+1$.

The Cartan tensor is an invariant in the classification of quasi-3-jets. Therefore for finding Cartan tensor we have to normalize

$$F = F^{[1]} + F^{[2]} + F^{[3]} + F^{[4]}.$$

Here and in what follows $[i]$ denotes the an object (function, vector field) is quasi-homogeneous of degree i.

It is easy to prove

$$F^{[1]} \sim 0, \;\; F^{[2]} \sim 0, \;\; F^{[3]} \sim 0.$$

Claim 11.
An exact normal form for $F \in [4]$ is the space of homogeneous degree 4 polynomials $c_{4,0}x_1^4 + \cdots + c_{0,4}x_2^4$.
Any $F(x_1, x_2, z, y_1, y_2) \in [4]$ is equivalent to $F(x_1, x_2, 0, 0, 0)$

Proof. It is a direct corollary of the fact that ΔF in claim 9 does not contain monomials $x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4 \in [4]$ and

Claim 12. All other monomials in $[4]$ are in ΔF, with suitable $\alpha_1(y_1, y_2, z), \alpha_2(y_1, y_2, z), \beta(y_1, y_2, z)$.

The simplest way to prove Claim 12 is using the dimensional counting; it reduces Claim 12 to

Claim 13. Let $\alpha_1(z, y_1, y_2), \alpha_2(z, y_1, y_2) \in [7], \beta(z, y_1, y_2) \in [6]$, so that $\Delta F \in [4]$. The equation $\Delta F = 0$ holds if a 14-dim vector space of tuples $\alpha_1, \alpha_2, \beta$.

We can easily calculate a basis of this 14-dim vector space.
It gives a parameterization of $\mathfrak{g}_2 = \ker L_N$.
Along with the calculations above it gives
a representation of $\mathfrak{g}_2$ by vector fields.

The Cartan invariant is the invariant in the classification of quasi-3-jets, with respect to the weights 1,1,2,3,3.

It is a part of the invariants in the classification of usual 5-jets, with respect to the weights 1,1,1,1,1.
(Attn: for the usual jets $x_1^2\partial_{x_1}$ has degree 1, not 0).
In the clasification of usual 4-jets there are no invariants.

The calculations above and the ideal I (first lecture) lead to the following normal form for the usual 5-jets:

$$V_1 = N_1 + x_2 G(x_1\partial_{y_1} + x_2\partial_{y_2})$$

$$V_2 = N_2 - x_1 G(x_1\partial_{y_1} + x_2\partial_{y_2})$$

$$G = F^{(4)}(x_1, x_2) + zF^{(3)}(x_1, x_2) + z^2 F^{(2)}(x_1, x_2)+$$
$$+r_1 x_1 z(x_1 y_2 - x_2 y_1) + r_2 x_2 z(x_1 y_2 - x_2 y_1) + w(x_1 y_2 - x_2 y_1)^2$$

(i) = homogeneous degree i

$F^{(4)}$=Cartan tensor, up to a non-singular linear transformation of x_1, x_2.

Here, except Cartan tensor, we have 10 more parameters:
the coefficients of $F^{(3)}(x_1, x_2), F^{(2)}(x_1, x_2)$ and r_1, r_2, w.

But the tuple of these 10 parameters is not an invariant,
bacause of 2-dim ${\mathfrak{g}_2}^{[1]}$ and 1-dim ${\mathfrak{g}_2}^{[2]}$, and 2-dim ${\mathfrak{g}_2}^{[3]}$.

Since $zF^{(3)}(x_1, x_2) \in [5], z^2 F^{(2)}(x_1, x_2) \in [6]$,
$x_1 z(x_1 y_2 - x_2 y_1), x_2 z(x_1 y_2 - x_2 y_1) \in [7]$
there is a certain action of ${\mathfrak{g}_2}^{[1]}$ on $F^{(3)}(x_1, x_2)$, of ${\mathfrak{g}_2}^{[2]}$ on $F^{(2)}(x_1, x_2)$,
and of ${\mathfrak{g}_2}^{[3]}$ on (r_1, r_2) in the normal form in the previous slide.

This action depends on the Cartan tensor $F^{(4)}(x_1, x_2)$.

Claim 14. For a generic Cartan tensor
$(\sim \pm x_1^4 + \mathrm{c} x_1^2 x_2^2 \pm x_2^4,\ \mathrm{c} \neq \pm 2)$ we have the following reduction:

$F^{(3)}(x_1, x_2) \to w_1 x_1^3 + w_2 x_2^3$
$F^{(2)}(x_1, x_2) \to w_3 x_1^3 + w_4 x_2^3$
$r_1, r_2 \to 0$

After this reduction, the tuple c, w_1, w_2, w_3, w_4, w is a complete invariant in the classification of 5-jets of (2,3,5) distributions with respects to the weights 1,1,1,1,1.

Question. Probably there is a certain geometric object, that can be constructed in a canonical way (some curvature?) and can be identified with the equivalence class of 5-jets of (2,3,5) distributions with respects to the weights 1,1,1,1,1,
in the same way as the Cartan tensor and can be identified with the equivalence class of 5-jets of (2,3,5) distributions with respects to the weights 1,1,2,3,3.
I would be happy if Dennis can answer.

Can we do a similar work with another nilpotent approximation?

Conceptually: yes. Practically: NOT.

Take for example the "Monge symbol"

$N_1 = \partial_{x_1},\ N_2 = \partial_{x_2} + x_1\partial_{x_3} + x_3\partial_{x_4} + x_1^2\partial_{x_5}.$

It is easy to obtain the following preliminary normal form with respect to the L_N-equivalence:

$V_1 = N_1,\ V_2 = N_2 + x_1^2\Big(P^{(4)}(x_1, x_2) + [\geq 5]\Big)\partial_{x_5}$

and $P^{(4)}(x_1, x_2)$ in this normal form is another way to express the Cartan tensor.

BUT this normal form does not respect the group $GL(2)$=quasi-homogeneous degree [0] symmetries of N, i.e. in the terminology of my first lecture it is not good.

The Cartan invariant will be the equivalence class of the tuple of 5 coefficients of $P^{(4)}(x_1, x_2)$ with respect to a very involved action of $GL(2)$.

And normalization of $[\geq 5]$ to some ideal, like I did for "my" nilpotent approximation is, I guess, not doable.

The thing is that the infinitesimal symmetries of quasi-degree [0] of the Monge symbol are very involved (one can easily calculate them), whereas "my" symbol has the following advantage which is the reason why I can do simple calculations, without being blocked after few steps, and why I can effectively use the obtained normal form:

The quasi-homogeneous infinitesimal symmetries of "my" symbol of quasi-degree [0] with respect to the weights 1,1,2,3,3 are homogeneous of degree (0) with respect to the weights 1,1,1,1,1.

Algorithm A, first two steps

Algorithm A:

input: vector fields V_1, V_2, given in any coordinates;
a point p at which the distribution D described by V_1, V_2 is (2,3,5).

output: the Cartan tensor of D at p.

Step 1: shift the coordinates such that $p = (0, 0, 0, 0, 0)$.

Step 2: calculate the usual 5-jet of the vector fields (with respect to the weights 1,1,1,1,1; attn.: it is the 6-jet of the coefficients of the vector fields) and take away the higher order terms - they do not affect the Cartan tensor.

Algorithm A, steps 3 and 4

Step 3. Work with the usual 1-jet of the vector fields (with respect to the weights 1,1,1,1,1; attn.: it is the 2-jet of the coefficients of the vector fields) in order to change the coordinates such that the vector fields do not contain quasi-homogeneous parts, with respect to the weights 1,1,2,3,3, of degree [-3] and [-2], and the quasi-homogeneous degree [-1] part is "my" symbol.

It is simple but a bit technical; can be easily algoritmized.
I have no time to explain.

Step 4. Calculate the quasi-homogeneous parts of the vector fields, with respect to the weights 1,1,2,3,3, of degrees $[0], [1], [2], [3]$ and take away the terms of higher degrees with respect to these weights - they do not affect the Cartan tensor. Now we have

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + W^{[0]} + W^{[1]} + W^{[2]} + W^{[3]}$$

where $W^{[i]}$ are certain pairs of quasi-homogeneous vector fields of degree $[i]$.

Algorithm A, steps 5 and 6

Step 5. Find $(Z, H) \in [1]$ such that $L_N(Z, H) = -W^{[0]}$.
We know that such (Z, H) exists and unique
up to the 2-dim vector space ${\mathfrak{g}_2}^{[1]}$.
Does not make difference which (Z, H) to take.

A straightforward way is to solve a system
of 48 equations with respect to 50 unknowns.
A better way is to use step-by-step reduction formulas given above.

Step 6. Make a change of coordinates $\exp(Z, H)$ with a properly defined exponential map for (Z, H).

We obtain

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \widetilde{W}^{[1]} + \widetilde{W}^{[2]} + \widetilde{W}^{[3]} + \dots$$

with new quasi-homogeneous parts of degrees [1], [2], [3].
The h.o.t. can be taken away.

The calculation of $\widetilde{W}^{[1]}, \widetilde{W}^{[2]}, \widetilde{W}^{[3]}$ is immediate using the formula

$$\exp(Z, H)_* V = V + L_V(Z, H) + \frac{1}{2!} L_V^2(Z, H) + \frac{1}{3!} L_V^3(Z, H) + \cdots$$

where $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ and $L_V(Z, H) = [Z, V] + HV$.

Remark. No need to calculate $\widetilde{W}^{[2]}$ - will be explained below.

Algorithm A, steps 7,8

Step 7. Find $(Z, H) \in [2]$ such that $L_N(Z, H) = -\widetilde{W}^{[1]}$.
We know that such (Z, H) exists and unique
up to the 1-dim vector space ${\mathfrak{g}_2}^{[2]}$.
Does not make difference which (Z, H) to take.

A straightforward way is to solve a system
of many p (around 100) equations with respect to $p + 1$ unknown.
A better way is to use step-by-step reduction formulas given above.

Step 8. Make a change of coordinates $\exp(Z, H)$. We obtain

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \widehat{W}^{[2]} + \widehat{W}^{[3]} + \dots$$

with new quasi-homogeneous parts of degrees [2], [3].
The h.o.t. can be taken away.

Algorithm A, steps 9,10

Step 9. Use the above formula for $\exp(Z,H)_* V$ to calculate $\widehat{W}^{[3]}$.
No need to calculate $\widehat{W}^{[2]}$.

Step 10. We know that $\widehat{W}^{[2]}$ can be killed by a suitable $(Z,H) \in [3]$.
It will change the quasi-homogeneous parts of degrees ≥ 4 but not $\widehat{W}^{[3]}$, so that up to equivalence

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \widehat{W}^{[3]} + [\geq 4]$$

The part $[\geq 4]$ can be taken away.

Algorithm A, final step 11

Express $\widehat{W}^{[3]}$ in the form

$$\widehat{W}^{[3]} = \begin{pmatrix} A_{11}N_4 + A_{12}N_5 \\ A_{21}N_4 + A_{22}N_5 \end{pmatrix} \mod N_1, N_2, [N_1, N_2]$$

Find Cartan tensor from A_{ij} by the explicit formulas above.

Algorithm B

input: a 3×5 characteristic matrix of an endowed 5-dim algebra $(\mathcal{A}, \mathcal{P})$.

output: the Cartan tensor of the homogeneous left-invariant distribution D induced by $(\mathcal{A}, \mathcal{P})$.

• In algorithm B we realize by vector fields neither $(\mathcal{A}, \mathcal{P})$ nor $\mathcal{D}$.

• Algorithm B given an explicit formula for Cartan tensor in terms of 15 parameters of the characteistic matrix.

$\mathfrak{g}_2 = \ker L_N$

$$\mathfrak{g}_2 = {\mathfrak{g}_2}^{[-3]} + {\mathfrak{g}_2}^{[-2]} + {\mathfrak{g}_2}^{[-1]} + {\mathfrak{g}_2}^{[0]} + {\mathfrak{g}_2}^{[1]} + {\mathfrak{g}_2}^{[2]} + {\mathfrak{g}_2}^{[3]}$$

$${\mathfrak{g}_2}^{[i]} = Z \text{ in } \ker L_N^{[i]}$$

$$L_N^{[i]} : \ (Z, H) \in [i] \ \to \ \left[Z, \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \right] + H \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

$${\mathfrak{g}_2}^{[\pm 1]} = span\left(\xi_1^{[\pm 1]}, \xi_2^{[\pm 1]}\right), \quad {\mathfrak{g}_2}^{[\pm 3]} = span\left(\xi_1^{[\pm 3]}, \xi_2^{[\pm 3]}\right)$$

$${\mathfrak{g}_2}^{[\pm 2]} = span\left(\xi^{[\pm 2]}\right)$$

$${\mathfrak{g}_2}^{[0]} = span\left(\xi_A^{[0]}, \ A \in \text{basis of } \mathfrak{gl}(2)\right)$$

$$\xi_A^{[0]} = \left\langle A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} \right\rangle + \operatorname{trace} A \cdot z\partial_z +$$
$$+ \left\langle (A + \operatorname{trace} A \cdot I) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \end{pmatrix} \right\rangle$$

$\mathfrak{g}_2 = \ker L_N$: structure equations, part 1

$$\left[\xi_1^{[\pm 1]}, \xi_2^{[\pm 1]}\right] = \xi^{\pm 2}, \quad \left[\xi_1^{[\pm 1]}, \xi^{[\pm 2]}\right] = \xi_1^{[\pm 3]}, \quad \left[\xi_2^{[\pm 1]}, \xi^{[\pm 2]}\right] = \xi_2^{[\pm 3]}$$

$$\left[\xi_A^{[0]}, \begin{pmatrix} \xi_1^{[\pm 1]} \\ \xi_2^{[\pm 1]} \end{pmatrix}\right] = Q \begin{pmatrix} \xi_1^{[\pm 1]} \\ \xi_2^{[\pm 1]} \end{pmatrix}, \quad \left[\xi_A^{[0]}, \xi^{[\pm 2]}\right] = \pm \operatorname{trace} \xi^{[\pm 2]}$$

$$\left[\xi_A^{[0]}, \begin{pmatrix} \xi_1^{[\pm 3]} \\ \xi_2^{[\pm 3]} \end{pmatrix}\right] = (Q \pm \operatorname{trace} A \cdot I) \begin{pmatrix} \xi_1^{[\pm 1]} \\ \xi_2^{[\pm 1]} \end{pmatrix}$$

$$+ : \; Q = A, \qquad - : \; Q = -A^{\mathrm{tr}}$$

$\mathfrak{g}_2 = \ker L_N$: structure equations, part 2

	$\xi_1^{[1]}$	$\xi_2^{[1]}$	$\xi^{[2]}$	$\xi_1^{[3]}$	$\xi_2^{[3]}$
$\xi_1^{[-1]}$	$\xi^{[0]}\begin{pmatrix}1 & 0\\ 0 & -\frac{1}{2}\end{pmatrix}$	$\xi^{[0]}\begin{pmatrix}0 & \frac{3}{2}\\ 0 & 0\end{pmatrix}$	$-2\xi_2^{[1]}$	$-\frac{3}{2}\xi^{[2]}$	0
$\xi_2^{[-1]}$	$\xi^{[0]}\begin{pmatrix}0 & 0\\ \frac{3}{2} & 0\end{pmatrix}$	$\xi^{[0]}\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & 1\end{pmatrix}$	$2\xi_1^{[1]}$	0	$-\frac{3}{2}\xi^{[2]}$
$\xi^{[-2]}$	$2\xi_2^{[-1]}$	$2\xi_1^{[-1]}$	$\xi^{[0]}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$3\xi_1^{[1]}$	$3\xi_2^{[1]}$
$\xi_1^{[-3]}$	$\frac{3}{2}\xi^{[-2]}$	0	$-3\xi_1^{[-1]}$	$\xi^{[0]}\begin{pmatrix}\frac{9}{2} & 0\\ 0 & 0\end{pmatrix}$	$\xi^{[0]}\begin{pmatrix}0 & \frac{9}{2}\\ 0 & 0\end{pmatrix}$
$\xi_2^{[-3]}$	0	$\frac{3}{2}\xi^{[-2]}$	$-3\xi_2^{[-1]}$	$\xi^{[0]}\begin{pmatrix}0 & 0\\ \frac{9}{2} & 0\end{pmatrix}$	$\xi^{[0]}\begin{pmatrix}0 & 0\\ 0 & \frac{9}{2}\end{pmatrix}$

Normal form and symmetries

We have a homogeneous (2,3,5) distribution dscribed by vector fields

$$V_1 = N_1 + V_1^{[3]} + [\geq 4], \quad V_1^{[3]} = x_2 F(x_1 \partial_{y_1} + x_2 \partial_{y_2})$$
$$V_2 = N_2 + V_2^{[3]} + [\geq 4], \quad V_2^{[3]} = x_2 F(x_1 \partial_{y_1} + x_2 \partial_{y_2})$$

$$F = c_{40} x_1^4 + c_{31} x_1^3 x_2 + c_{22} x_1^2 x_2^2 + c_{13} x_1 x_2^3 + c_{04} x_2^4 = \mathrm{Cartan\ tensor}$$

We have 5 unfinitesimal symmetries

$$a_1, \; a_2, \; a_3 = [a_1, a_2], \; a_4 = [a_1, a_3], \; a_5 = [a_2, a_3]$$

where $a_1(0) = V_1(0), \; a_2(0) = V_2(0)$.

The generating 2-plane in the Lie algebra $span(a_1, ..., a_5)$ is

$$\mathcal{P} = span(a_1, a_2).$$

Since $[a_i, V_j] = 0 \mod V_1, V_2$ it follows that for the decompisition of a_1, a_2 by quasi-homogeneous parts $[i]$ we have

$$a_1 = a_1^{[-1]} + a_1^{[0]} + a_1^{[1]} + a_1^{[2]} + [\geq 3]$$
$$a_2 = a_2^{[-1]} + a_2^{[0]} + a_2^{[1]} + a_2^{[2]} + [\geq 3]$$

$$a_i^{[-1]} \in \mathfrak{g}_2{}^{[-1]},\ a_i^{[0]} \in \mathfrak{g}_2{}^{[0]},\ a_i^{[1]} \in \mathfrak{g}_2{}^{[1]},\ a_i^{[2]} \in \mathfrak{g}_2{}^{[2]}$$

Therefore we have the following normal form for (distribution; infinitesimal symmetries a_1, a_2):

$$V_1 = N_1 + V_1^{[3]} + [\geq 4]$$
$$V_2 = N_2 + V_2^{[3]} + [\geq 4]$$
$$a_1 = \xi_1^{[-1]} + \xi_A^{[0]} + r_{11}\xi_1^{[1]} + r_{12}\xi_2^{[1]} + r_{13}\xi^{[2]} + [\geq 3]$$
$$a_2 = \xi_2^{[-1]} + \xi_B^{[0]} + r_{21}\xi_1^{[1]} + r_{22}\xi_2^{[1]} + r_{23}\xi^{[2]} + [\geq 3]$$

parameterized by r_{ij} and 2×2 matrices A, B.

Simplification of the normal form

Apply a local diffeo (a change of coordinates) of the form

$$\Phi = \exp\left(w_1\xi_1^{[1]} + w_2\xi_2^{[1]} + w_3\xi^{[2]} + w_4\xi_1^{[3]} + w_5\xi_2^{[3]}\right)$$

Whatever are $w_1, \ldots, w_5$, we have

$$\begin{aligned} \Phi_* V_1 &= V_1 \mod V_1, V_2 \; + [\geq 4] \\ \Phi_* V_2 &= V_2 \mod V_1, V_2 \; + [\geq 4] \end{aligned}$$

Therefore this change of coordinates, along with multiplication of V_1, V_2 by a suitable 2×2 matrix, preserves the normal form for V_1, V_2 in the previous slide.

Claim. Taking suitable $w_1, \ldots, w_5$, we can change the parameters in the normal form for a_1, a_2 in the previous slide such that

A and B are traceless matrices (by w_1, w_2)

$r_{12} = r_{21}$ (by w_3)

$r_{13} = r_{23} = 0$ (by w_4, w_5)

Working with the normal form for a_1, a_2

We obtain the following normal form for a_1, a_2:

$$\begin{aligned} a_1 &= \xi_1^{[-1]} + \xi_A^{[0]} + r_1\xi_1^{[1]} + s\xi_2^{[1]} + [\geq 3], \quad \operatorname{trace} A = 0 \\ a_2 &= \xi_2^{[-1]} + \xi_B^{[0]} + s\xi_1^{[1]} + r_2\xi_2^{[1]} + [\geq 3], \quad \operatorname{trace} B = 0 \end{aligned}$$

We have

$$\begin{aligned} a_3 &= \xi^{[-2]} + a_3^{[-1]} + a_3^{[0]} + a_3^{[1]} + [\geq 2] \\ a_4 &= \xi_1^{[-3]} + a_4^{[-2]} + a_4^{[-1]} + a_4^{[0]} + [\geq 1] \\ a_5 &= \xi_2^{[-3]} + a_5^{[-2]} + a_5^{[-1]} + a_5^{[0]} + [\geq 1] \end{aligned}$$

$$\begin{aligned} [a_1, a_4] &= [a_1, a_4]^{[-3]} + [a_1, a_4]^{[-2]} + [a_1, a_4]^{[-1]} + [\geq 0] \\ [a_1, a_5] &= [a_1, a_5]^{[-3]} + [a_1, a_5]^{[-2]} + [a_1, a_5]^{[-1]} + [\geq 0] \\ [a_2, a_5] &= [a_2, a_5]^{[-3]} + [a_2, a_5]^{[-2]} + [a_2, a_5]^{[-1]} + [\geq 0] \end{aligned}$$

All blue quasi-homogeneous parts are uniquely determined by the tracelsss matrices A, B and r_1, r_2, s.

The traceless matrices A, B and r_1, r_2, s are determined by the 3×5 characteristic matrix (t_{ij})

We have

$a_3^{[-1]} = (-A_{12} + B_{11})\xi_1^{[-1]} + (A_{11} + B_{21})\xi_2^{[-3]}$

It follows

$a_4^{[-2]} = (A_{11} + B_{21})\xi^{[-2]}, \quad a_5^{[-2]} = (A_{12} - B_{11})\xi^{[-2]}$

It follows

$[a_1, a_4]^{[-3]} = B_{21}\xi_3^{[-1]} - A_{21}\xi_3^{[-2]}$

$[a_1, a_5]^{[-3]} = -B_{11}\xi_3^{[-1]} + A_{11}\xi_3^{[-2]}$

$[a_2, a_5]^{[-3]} = -B_{12}\xi_3^{[-1]} + A_{12}\xi_3^{[-2]}$

On the other hand, from the characteristic matrix:

$[a_1, a_4]^{[-3]} = t_{14}\xi_1^{[-3]} + t_{15}\xi_2^{[-3]}$

$[a_1, a_5]^{[-3]} = t_{24}\xi_1^{[-3]} + t_{25}\xi_2^{[-3]}$

$[a_2, a_5]^{[-3]} = t_{34}\xi_1^{[-3]} + t_{35}\xi_2^{[-3]}$

It follows:

$$A = \begin{pmatrix} t_{25} & t_{35} \\ -t_{35} & -t_{25} \end{pmatrix}, \quad B = \begin{pmatrix} -t_{24} & -t_{34} \\ -t_{14} & t_{24} \end{pmatrix}$$

A similar calculation of $[a_1, a_4]^{[-2]}, [a_1, a_5]^{[-2]}, [a_2, a_5]^{[-2]}$,
from the structure equations for $\mathfrak{g}_2$
versus from the characteristic matrix leads to:

$$r_1 = \tfrac{1}{5}(-t_{13} - t_{14}^2 - 4t_{15}t_{24} + 2t_{14}t_{25} + t_{25}^2 - 2t_{15}t_{35})$$
$$r_2 = \tfrac{1}{5}(-t_{33} + t_{24}^2 - 4t_{25}t_{34} - 2t_{14}t_{34} - t_{35}^2 + 2t_{24}t_{35})$$
$$s = \tfrac{1}{5}(-t_{23} - t_{14}t_{24} - 3t_{24}t_{25} - t_{15}t_{34} + 2t_{14}t_{35} - t_{25}t_{35})$$

What does it give?

Now we have a normal form for V_1, V_2 describing the distribution

$$V_1 = N_1 + V_1^{[3]} + [\geq 4], \quad V_1^{[3]} = x_2 F(x_1 \partial_{y_1} + x_2 \partial_{y_2})$$
$$V_2 = N_2 + V_2^{[3]} + [\geq 4], \quad V_2^{[3]} = x_2 F(x_1 \partial_{y_1} + x_2 \partial_{y_2})$$

$$F = c_{40} x_1^4 + c_{31} x_1^3 x_2 + c_{22} x_1^2 x_2^2 + c_{13} x_1 x_2^3 + c_{04} x_2^4 = \text{Cartan tensor}$$

and a normal form <u>in the same coordinates</u> for a_1, a_2:

$$a_1 = \xi_1^{[-1]} + \xi_A^{[0]} + r_1 \xi_1^{[1]} + s \xi_2^{[1]} + \phi_1^{[3]} + [\geq 3]$$
$$a_2 = \xi_2^{[-1]} + \xi_B^{[0]} + s \xi_1^{[1]} + r_2 \xi_2^{[1]} + \phi_2^{[3]} + [\geq 3]$$

and we know everything blue, in terms of the entries of the characteristic matrix, but we do not know $\phi_1^{[3]}, \phi_2^{[3]}$.

$\phi_1^{[3]}, \phi_2^{[3]}$ and Cartan tensor

The fact that a_1 and a_2 are infinitesimal symmetries implies the equations

$$L_N\left(\phi_1^{[3]}, H_1\right) + \left[\xi_1^{[-1]}, \begin{pmatrix} V_1^{[3]} \\ V_2^{[3]} \end{pmatrix}\right] = 0$$

$$L_N\left(\phi_2^{[3]}, H_2\right) + \left[\xi_2^{[-1]}, \begin{pmatrix} V_1^{[3]} \\ V_2^{[3]} \end{pmatrix}\right] = 0$$

where $H_1, H_2 \in [3]$ are some 2×2 matrices. We know that these equations are solvable wrt $\phi_1^{[3]}, H_1$ and $\phi_2^{[3]}, H_2$ for any $V_1^{[3]}$, $V_2^{[3]}$. The solutions are unique up to linear combinations, with numerical coefficients, of $\xi_1^{[3]}, \xi_2^{[3]} \in \mathfrak{g}_2{}^{[3]}$. Therefore

$$\phi_1^{[3]} = c_{40} R_{11}^{[3]} + \cdots + c_{04} R_{15}^{[3]} + q_{11}\xi_1^{[3]} + q_{12}\xi_2^{[3]}$$

$$\phi_2^{[3]} = c_{40} R_{21}^{[3]} + \cdots + c_{04} R_{25}^{[3]} + q_{21}\xi_1^{[3]} + q_{22}\xi_2^{[3]}$$

where $c_{40}, \ldots, c_{04}$ are the coefficients of the cartan tensor and $R_{ij}^{[3]}$ are fixed functions, we can express them by a formula.

Formulas for the coefficients $c_{40}, \ldots, c_{04}$ of the Cartan tensor

Calculating $[a_1, a_4]^{[-1]}, [a_1, a_5]^{[-1]}, [a_2, a_5]^{[-1]}$,
from the structure equations for $\mathfrak{g}_2$
versus from the characteristic matrix
gives us certain equations where $c_1, \ldots, c_5$ and q_{ij} are not involved.
These equations give us the relations between the entries of the characteristic matrix that follow from Jacobi identity.

But calculating $[a_1, a_4]^{[0]}, [a_1, a_5]^{[0]}, [a_2, a_5]^{[0]}$,
from the structure equations for $\mathfrak{g}_2$
versus from the characteristic matrix
leads to a system of 12 linear equations wrt the 9 unknowns
$c_1, \ldots, c_5, q_{11}, q_{12}, q_{21}, q_{22}$.
We know that this system is solvable.
It has a unique solution, and we obtain formulas for $c_{40}, \ldots, c_{04}$.

Are these formulas involved?

No, here they are:

$$c_{40} = \frac{1}{100}(9t_{13}^2 + 100t_{12}t_{14} + 18t_{13}t_{14}^2 + 9t_{14}^4 - 100t_{11}t_{15} + 60t_{14}t_{15}t_{23} - 188t_{13}t_{15}t_{24} + 72t_{14}^2t_{15}t_{24} + 364t_{15}^2t_{24}^2 + 164t_{13}t_{14}t_{25} - 36t_{14}^3t_{25} + 180t_{15}t_{23}t_{25} - 464t_{14}t_{15}t_{24}t_{25} - 198t_{13}t_{25}^2 + 238t_{14}^2t_{25}^2 + 608t_{15}t_{24}t_{25}^2 - 404t_{14}t_{25}^3 + 189t_{25}^4 - 60t_{15}^2t_{33} - 60t_{14}t_{15}^2t_{34} - 60t_{15}^2t_{25}t_{34} + 96t_{13}t_{15}t_{35} - 24t_{14}^2t_{15}t_{35} - 416t_{15}^2t_{24}t_{35} + 308t_{14}t_{15}t_{25}t_{35} - 276t_{15}t_{25}^2t_{35} + 96t_{15}^2t_{35}^2)$$

and not more involved formulas for $c_{31}, c_{22}, c_{13}, c_{04}$. You want to have these formulas? E-mail to me and you will have them.

Recall that the parameters t_{ij} of a characteristic matrix is not an arbitrary tuple of 15 real numbers, there are certain relations because of Jacobi identity. But these relations do not simplify the formulas substantialy.

The case of endowed (5,3,0) algebra

Recall (lecture 1, lecture 3) that the characteristic matrix of a (5,3,0) endowed algebra is determined by its last 2-columns - reduced characteristic matrix.
It is a 3×2 matrix, and its 6 entries are arbitrary numbers.

Any reduced characteristic matrix is equivalent to

$$\begin{pmatrix} a & b \\ c & d \\ 0 & e \end{pmatrix}$$

In this case the coefficients of Cartan tensor are as follows:

$c_{40} = 9a^4 + 90a^2bc + 125b^2c^2 - 54a^3d - 190abcd + 101a^2d^2 + 60bcd^2 - 60ad^3 - 42a^2be - 90b^2ce + 66abde + 9b^2e^2$

$c_{31} = 4(3c - e)(3a^3 + 25abc - 13a^2d + 12ad^2 - 12abe - 9bde)$

$c_{22} = 2(3c - e)(6a^2c + 50bc^2 - 26acd + 24cd^2 - a^2e - 15bce + 9ade - 18d^2e - 9be^2)$

$c_{13} = -4(a - 3d)(4c - 3e)(3c - e)e$

$c_{04} = -(4c - 3e)(5c - 3e)(3c - e)e$

We see that in the case $e = 3c$ the Cartan tensor is either $\pm x_1^4$ or 0. Note that we know that without computing Cartan tensor, because in the case $e = 3c$ the reduced characteristic matrix is special (see lecture 3) and then the endowed 5-dim algebra is an endowed subalgebra of one of the 7-dim endowed algebras.

And we see a number of cases when the Cartan tensor is 0, i.e. the distribution is flat. Example: $e = \frac{4c}{3}, a = \frac{4d}{3}, 27bc = 20d^2$.

All what was explained in this minicourse, including the tutorial, is published in:

Proceedings of the GRIEG seminar
B. Kruglikov, O. Makhmali, P. Nurowski Eds
Warsaw-Oslo, 2021

For futher tutorials please e-mail to me
what you are interested in,
and we will do it by Zoom hosted by me.

THANKS TO ALL THE LISTENERS!