Normal forms and symmetries for (2,3,5) and (3,5) distributions

111 years after E.Cartan's 5 variables paper

Michail Zhitomirskii

Technion

GRIEG SEMINAR, March-April 2021

Do we need Cartan tensor?

Yes. By many reasons, probably the most important is as follows:

without Cartan tensor there is no way to answer the following important question: if a given (2,3,5) distribution is flat or not?

I will give one of equivalent definitions of a flat (2,3,5) distributins using one of my tools: characteristic matrix of an endowed 5-dim Lie algebra.

Endowed 5-dim algebras and their characteristic matrices

By definition, an endowed 5-dim algebra is a 5-dim Lie algebra $\mathcal{A}$ endowed with a generating 2-plane $\mathcal{P} = span(a_1, a_2)$ meaning that the vectors

$$a_1, a_2,\ a_3 = [a_1, a_2],\ a_4 = [a_1, a_3],\ a_5 = [a_2, a_3]$$

are linearly independent.

Any endowed 5-dim algebra can be described by the following 3×5 matrix $(r_{i,j})$ that I call a characteristic matrix:

$$\begin{array}{ccc} [a_1, a_4] & = & r_{1,1}a_1 + \cdots + r_{1,5}a_5 \\ [a_1, a_5] = [a_2, a_4] & = & r_{2,1}a_1 + \cdots + r_{2,5}a_5 \\ [a_2, a_5] = & = & r_{3,1}a_1 + \cdots + r_{3,5}a_5 \end{array}$$

All other structure equations follow from the Jacobi identity.

Homogeneous (2,3,5) distribution induced by an andowed 5-dim algebra

Let $(\mathcal{A}, \mathcal{P})$ be an endowed 5-dim algebra.

Let (G, id) be a nbhd of id of the Lie group G of $\mathcal{A}$,
so that $\mathcal{P}$ is a 2-plane in $T_{\mathrm{id}}G$.

Push $\mathcal{P}$ to (G, id) by the flows of left invariant vector fields on G.

We obtain a local (germ at id) (2,3,5) distribution on (G, id).

It is homogeneous becuse its symmetry algebra either is $\mathcal{A}$ or contains $\mathcal{A}$.

The most known classical theorem

Theorem 1. (Cartan, Tanaka). Let D be the germ at $0 \in \mathbb{R}^5$ of a (2,3,5) distribution. The following are equivalent:

1. D is equivalent to a distribution induced by the endowed 5-dim algebra with the zero characteristic matrix.

2. The symmetry algebra of D is $\mathfrak{g}_2$

3. The Cartan tensor of D vanishes at any point near $0 \in \mathbb{R}^5$

Definition. If the conditions of Theorem 1. hold, the (2,3,5) distribution is called flat.

Question. What can be said about the (2,3,5) distribution induced by the endowed 5-dim algebra with non-zero characteristic matrix?
a. Can it be flat? **b**. Its symmetry algebra?

I will give complete answers. Quick answer to **a**: yes

Less known very important theorem

Theorem 2. (Cartan) If D is not flat, its symmetry algebra has dim ≤ 7.

My proof of Theorem 2 is as follows.

Theorem 3. (M. Zh) Let D be the germ at $0 \in \mathbb{R}^5$ of a non-flat (2,3,5) distribution. The isotropy subalgebra at 0 of the symmetry algebra of D is, as an abstract Lie algebra, one of the following:

a. 1-dim; **b**. 2-dim non-Abelian; **c.** $\mathfrak{sl}(2)$.

Each of these cases is realizable.
In case **c**, the Cartan tensor of D at 0 is equal to 0.

Theorem 2. is a logical corollary of Theorems 3. and 1.

How I prove Theorem 3? My base theorem for non-flat distributions

My proof of Theorem 3, and of many other theorems, uses the following statement that for me plays the key role.

Theorem 4 (the base theorem for non-flat distributions (M. Zh.)
The isotropy subalgebra of the symmetry algebra any non-flat (2,3,5) distribution germ at $0 \in \mathbb{R}^5$ does not contain vector fields with the zero linear approximation at 0.

Theorem 5 (logical corollary of Theorem 4)
The isotropy subalgebra $\mathcal{I}$ of the symmetry algebra any non-flat (2,3,5) distribution germ D at $0 \in \mathbb{R}^5$ is isomorphic (as an abstract Lie alegbra) to the Lie algebra $j_0^1\mathcal{I}$ (the linear approximation of $\mathcal{I}$).

Theorem 6 (M.Zh). Let D and $\mathcal{I}$ be as in Corollary 5. Then $j_0^1\mathcal{I}$ is isomorphic to one of **a, b, c** in Theorem 3.

Theorem 3. is a logical corollary of Theorems 5 and 6

My approach is based on:

In order to prove Cartan-Tanaka classical Theorem 1, my base Theorem 4, and Theorem 6 (Theorems 2,3,5 are, as I explained, their corollaries), and many other statements that will be formulated in this mini-course, I use the following, for (2,3,5) distributions D:

A. A theorem explaining the Cartan tensor of D at a fixed point

B. A theorem on characteristic polynomial of D :
a generalization of Cartan tensor at a fixed point

C. A theorem on almost exact normal form for all possible D

Theorems **A,B,C** for local Riemannian metrics

Given the germ at $0 \in \mathbb{R}^n$ of a Riemannian metric on $\mathbb{R}^n$, take an orthonormal basis $V_1, ..., V_n$ of vector fields, and a local coordinate system $x = (x_1, ..., x_n)$.

Describe the metric by an $n \times n$ matrix $M(x)$ with the functional entries $M_{i,j}(x) = V_j(x_i)$.

Theorem 7. (Gauss lemma; M. Zh).
1. For a suitable orthonormal basis and suitable local coordinates one has

$$M(x) = I + A(x), \; A^{\mathrm{tr}}(x) = A(x), \; A(x) \cdot x \equiv 0, \; x = (x_1, \ldots, x_n)^{\mathrm{tr}}$$

2.This normal form is exact up to transformations $A(x) \to R^{\mathrm{tr}} A(Rx) R$ where R is a constant orthogonal matrix.

3. The metric is flat if and only if $A(x) \equiv 0$.

Remark. The local coordinates in this normal form are normal (geodesic).

Proof of Theorem 7 by the Poincare way

What I call the Poincare way is the way used in the whole local analysis and physics (but unfortunately so far not by many people in local differential geometry), especially after the resonance normal form obtained by Poincare for vector fields, that is used and developed in thousands of works.

We have $M(x) = I + A^{(1)}(x) + A^{(2)}(x) + \cdots$ where (i) denotes the homogeneous part of degree i. Assume we have normalized $A^{(i)}(x)$ for $i < k$ and want to normalize $A^{(k)}(x)$. For that, let us change the local coordinates and the basis as follows:

$$x \to x + \Phi^{(k)}(x),$$
$$(V_1, ..., V_n)^{\mathrm{tr}} = \exp(S^{(k)}(x)) \cdot (V_1, ..., V_n)^{\mathrm{tr}}, \; S^{(k)}(x) \in \mathfrak{so}(n)$$

Proof of Theorem 7 by the Poincare way (continuation)

These transformation preserve already normalized $A^{(i)}(x), i > k$ and

$$A^{(k)}(x) \to A^{(k)}(x) + \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x)$$

The first statement of Theorem 7 now follows from the following claim.

Claim 8. One of the complement spaces to the image of the operator

$$\left(\Phi^{(k)}(x),\ S^{(k)}(x)\right) \to \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x), \quad S^{(k)}(x) \in \mathfrak{so}(n)$$

is the space of symmetric matrices
$A^{(k)}(x)$ such that $A^{(k)}(x) \cdot x \equiv 0$, $x = (x_1, ..., x_n)^{\mathrm{tr}}$.

Proof of Claim 8 by GR inner products

The simplest exercise. For the inner product $< ax^i, bx^i >= i!ab$ in the space of homogeneous degree i polynomials of one variable one has

$$p(x) = ax^k,\ q(x) = bx^{k-1} \implies < p'(x), q(x) >=< p(x), q(x)x >$$

and consequently $\Big(p(x) \to p'(x)\Big)^*$ is the operator $q(x) \to q(x)x$.

The inner product in this exercise can be easily and naturally extended to homogeneous vector functions and matrices with homogeneous entries. I call these inner products GR inner products because they were introduced and used for many nice theorems on normal forms for vector fields by my first teacher Genrich Ruvimovich Belitskii.

Those who did the exercise above will easily give an explicit formula for GR inner products and will prove that with respect to them $\Big(\big(\Phi^{(k)}(x)\big)'\Big)^*$ is the operator $A^{(k)}(x) \to A^{(k)}(x) \cdot x, x = (x_1, ..., x_n)^{\mathrm{tr}}$ which implies Claim 8.

Proof of the remaining part of Theorem 7

In the normalization way used the transformations that differ from the identity by $(i), i \geq 1$ and did not use transformations of degree (0) that preserve I, i.e. we did not use the action of $O(n)$.

The constructed normal form is "good" meaning that it is preserved by the action of $O(n)$: it is easy to see that $O(n)$ acts as follows $A(x) \to R^{\mathrm{tr}} A(Rx)R, \ R \in O(n)$ which preserves the equations $A(x) = A^{\mathrm{tr}}(x)$ and $A(x) \cdot x \equiv 0, x = (x_1, ..., x_n)^{\mathrm{tr}}$.

The fact that the constructed normal form is exact follows (modulo simple general claims related to the Poincare way) form the following statement that can be easily checked:

Claim. For $k \geq 1$ the kernel of the operator

$$\left(\Phi^{(k)}(x),\ S^{(k)}(x)\right) \to \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x), \quad S^{(k)}(x) \in \mathfrak{so}(n)$$

is trivial.

The characteristic matrix and the Riemannian curvature tensor at a fixed point

For non-flat Riemannian metric germ at $0 \in \mathbb{R}^5$ we have

$$M = I + A^{(k)}(x) + A^{(k+1)}(x) + \cdots$$

and the equations $A^{(k)}(x) = \left(A^{(k)}(x)\right)^{\mathrm{tr}}$ and $A^{(k)}(x) \cdot x \equiv 0$, $x = (x_1, ..., x_n)^{\mathrm{tr}}$ imply $k \geq 2$ (for $k = 1$ the matrix $A^{(1)}(x)$ is zero).

The matrix $A^{(k)}(x)$ in the normal form might be called the characteristic matrix.

If $k = 2$ the characteristic matrix can be identified with the curvature tensor at $0 \in \mathbb{R}^5$.

If $k > 2$ then the curvature tensor vanishes at $0 \in \mathbb{R}^5$ which does not mean, of course, that it vanishes at other points near 0.

Straightforward generalizations

For germs on $\mathbb{R}^n$ of conformal structures (metrics up to multiplication by a non-vanishing function) we use a change of coordinates and a change of orthonormal basis as above, and we multiply $M(x)$ by a nin-vanishing function H, so that the transformation of $A^{(k)}(x)$ is as follows:

$$A^{(k)}(x) \to A^{(k)}(x) + \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x) + H^{(k)}(x)I$$

and exactly in the same way we obtain the normal form $M = I + A(x)$,

$$A^{\mathrm{tr}}(x) = A(x),\ A(x) \cdot x \equiv 0,\ \mathrm{trace}\, A(x) \equiv 0$$

If $n \geq 4$ the homogeneous decomposition of $A(x)$ starts, as well as for metrics, with $A^{(2)}$ and $A^{(2)}$ can be identified with the Weyl tensor; if $n = 3$ it starts with $A^{(3)}$ and $A^{(3)}$ can be identified with the Cotton tensor. If $n = 2$ the equations above imply $A(x) \equiv 0$.

Generalization to Einstein metric and Einstein conformal structure are also straightforward. Maybe they can be used for some problems posed in Pawel Nurowski's lectures?

Is it possible to do a similar work for (2,3,5) distributions?

The first answer is no because unlike Riemannian metrics or conformal structures we cannot describe the class of all (2,3,5) distributions in the form $(0) + (1) + (2) + \cdots$ with a fixed (0) and arbitrary $(1), (2), \ldots$.

The second answer is yes due to a very good quasi.

Quasi-homogeneity

Let $E = x_1\partial_{x_1} + \cdots + x_n\partial_{x_n}$

Definition. A function $f(x)$ is called homogeneous of degree d if $E(f) = df$. A vector field V is called homogeneous of degree d if $[E, V] = dV$.

Example. The vector field $x_1^{r_1} \cdot \cdots \cdot x_n^{r_n}\partial_{x_k}$ is homogeneous of degree $r_1 + \cdots + r_n - 1$.

Let now $E_\lambda = \lambda x_1\partial_{x_1} + \cdots + \lambda_n x_n\partial_{x_n}$

Definition. Replace E by E_λ in the definition above. A function $f(x)$ or a vector field V in that definition is called quasi-homogeneous of degree d with respect to the weights $\lambda_1, ..., \lambda_n$.

Example. With respect to the weights $1, 1, 2, 3, 3$ there are non-zero vector field on $\mathbb{R}^5$ of any degree $d \geq -3$.

Andre Bellaiche theorem

Theorem (A. Bellaiche). Let $V_1, ..., V_k$ be any bracket generating tuple of vector field germs at $0 \in \mathbb{R}^n$. In suitable local coordinates

$$V_i = N_i^{[-1]} + N_i^{[0]} + N_i^{[1]} + \cdots$$

where $[i]$ denotes the quasi-homogeneous part with respect to the natural weights defined by the growth vector of the tuple.

Example of the natural weights. Let V_1, V_2 be vector field germs at $0 \in \mathbb{R}^2$ such that $V_1(0) \neq 0$ and V_2 and all repeated brackets of V_1, V_2 of lenghth ≤ 99 are at 0 proportional to $V_1(0)$, and one of the brackets of lenghth 100 is not. Then the growth vector is $(1, 1, .., 1, 2)$ with 1 repeated 100 times, and the natital weights are $(1, 101)$.

Starting point

In what $n = 5$ and follows $[i]$ denotes quasi-homogeneity of degree i with respect to the weights 1,1,2,3,3 that are natural for (2,3,5) distributions.

The Bellaiche theorem implies that any (2,3,5) distribution germ D can be described by vector fields

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + [0] + [1] + \cdots, \quad \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \in [-1]$$

and the distribution described by the vector fields N_1, N_2 is
a flat (2,3,5) distribution.
It can be identified with the nilpotent approximation=symbol of D.

The linear operator
(linearization of the pseudo-group action)

We have a pseudo-group that consists of local diffeos Φ (change of coordinates) and non-singilar 2×2 matrices $\widehat{H}(x)$ corresponding to the change of basis V_1, V_2 of a (2,3,5) distribution. It acts as follows:

$$(\Phi, \widehat{H}).\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \widehat{H}\begin{pmatrix} \Phi_* V_1 \\ \Phi_* V_2 \end{pmatrix}$$

The Lie algebra of this pseudo-group is $(Z, H(x))$ where Z is a vector field germ and $H(x)$ is any 2×2 matrix. We need the linearization at id of the map

$$(\Phi, \widehat{H}).\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \widehat{H}\begin{pmatrix} \Phi_* N_1 \\ \Phi_* N_2 \end{pmatrix}$$

It is the linear operator

$$L_N: \ (Z, H) \to \begin{pmatrix} [Z, N_1] \\ [Z, N_2] \end{pmatrix} + H\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

Working with the operator L_N

In the same way as for Riemannian metrics we obtain a normal form

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[0]} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[1]} + \cdots$$

where $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[i]}$ belongs to any fixed beforehand complementary space to the image of the operator L_N restricted to $[i+1]$.

End of the first lecture

Some of the comments of the listeners are in the next page

Some of the comments of the listeners to the first lecture

Bronek Jakubczyk: The Poincare way approach to classification of Riemannian metrics is contained in the book:
Jerry Kijowski, Geometria rozniczkowa jako narzedzie nauk przyrodniczych
Bronek presented me this book, sorry that I forgot to mention it

Borya Kruglikov: Theorem 4, that I call the base theorem for non-flat (2,3,5) distributions, has a huge generalization: Theorem 1.3 in the paper
B.Krugkikov, D.The, Jet-determination of symmetries of parabolic geometries, Math. Ann., 2018. I should discuss with Borya or Dennis: one of the assumption of their theorem is that the geometry is torsion free. How to prove that the geometry of (2,3,5) distributions is torsion free? It is not excluded that proving Th. 1.4 I am proving, implicitly, namely that, in another language.

Igor Zelenko: I mentioned Igor's fundamental form, constructed by abnormal curves, but did not say - sorry! that Igor has it now not only for (2,3,5) distributions (the case that it can be identified with the Cartan tensor, according to Igor's Phd Thesis) but for all $(2, n)$ bracket-generating distributions (maybe with a constant growth vector only? or with the max growth vector only? Jointly with Borya Doubrov? To disciss with Igor)

A way to obtain exact normal form for (2,3,5) distribution germs at $0 \in \mathbb{R}^5$ and Cartan tensor at 0

Take any pair $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ of vector fields on $\mathbb{R}^5$, in local coordinates $x_1, \dots x_5$ such that N_1, N_2 are quasi-homogeneous with respect to the weights $w(x_1) = w(x_2) = 1, w(x_3) = 2, w(x_4) = w(x_5) = 3$ and the vector fields $N_1, N_2, N_3 = [N_1, N_2], N_4 = [N_1, N_3], N_5 = [N_2, N_3]$ are linearly independent at 0.

Conceptually, the choice of N (=symbol) is irrelevant (all N are diffeomorphic) but the choice can be good or not good.

Examples: (a) $N_1 = \partial_{x_1}, \; N_2 = \partial_{x_2} + x_1\partial_{x_3} + x_3\partial_{x_4} + x_1^2\partial_{x_5}$
(in Monge form $z' = (y'')^2$)

(b) $N_1 = \partial_{x_1} + x_2 U, \; N_2 = \partial_{x_2} - x_1 U, \; U = \partial_z + x_1\partial_{y_1} + x_2\partial_{y_2}$

For me (b) is good and (a) is not.

Consider the linear operators

$$L_N : \left(Z^{[i]}, H^{[i]}\right) \to [Z^{[i]}, N] + H^{[i]}N \in [i-1]$$

where Z is a vector field, H is a 2×2 matrix whose entries are functions, and $[i]$ denotes: quasi-homogeneous of degree i.

Take any complementary subspace $W^{[i-1]}$ to the image of $L_N^{[i]}$:

$$[i-1] = \mathrm{Image}\left(L_N^{[i]}\right) \oplus W^{[i-1]}$$

Definition. $W^{[i]}$, $i \geq 0$ is good if it respects the group of quasi-homogeneous degree 0 symmetries of N:

$$\mathrm{Sym}^{[0]}(N) = \exp\left(Z^{[0]}, H^{[0]}\right), \ \left(Z^{[0]}, H^{[0]}\right) \in \ker L_N^{[0]}$$

i.e. $\mathrm{Sym}^{[0]}(N).W^{[i]} \subseteq W^{[i]}$.

Do good complementary subspaces exist?

In most (but not all) problems I know (good complementary subspaces can be defined in the same way for all geometrix stuctures) they exist.
It does not depend on the choice of N, it depends on the problem only.
I do not know general theorems about that.
I do not know how deep/interesting is this question.

Whether or not the complementary subspaces $W^{[i]}$ are goog we have:

• Any (2,3,5) distribution germ is equivalen to

$$N + \sum_{i \geq 0} V^{[i]}, \ V^{[i]} \in W^{[i]}, \ i \geq 0 \tag{1}$$

• A (2,3,5) distrribution is flat (equivalent to N)
if and only if in this normal form $V^{[i]} = 0$ for any $i \geq 0$.

• If $\ker L_N^{[i]} = \{0\}$ for all $i \geq 1$ and if $W^{[i]}$ are good for all $i \geq 0$ then the normal form is exact modulo the action of $\mathrm{Sym}^{[0]}(N)$, i.e. if two equivalent distributions have normal form (1) then they can be brough one to the other by some $(\Phi, H) \in \mathrm{Sym}^{[0]}(N)$.

But the assumption $\ker L_N^{[i]} = \{0\}$ for all $i \geq 1$, which makes the classification problem rather simple, holds true for Riemannian metrics (see my first lecture) but not for most of other geometric structures (not for conformal structures and not for (2,3,5) distributions).

• Let $k \geq 0$ be the minimal integer such that $W^{[k]} \neq \{0\}$, i.e. $L_N^{[k+1]}$ is not onto. If $W^{[k]}$ is good (see the definition above) then modolo the action of $\mathrm{Sym}^{[0]}(N)$ it is an invariant of the distribution, i.e. if two equivalent distributions have normal form (1), with $V, \widetilde{V} \in W^{[k]}$ then $(\Phi, H).V = \widetilde{V}$ for some $(\Phi, H) \in \mathrm{Sym}^{[0]}(N)$.

I will say that it is "the first invariant".

Assume we know nothing about (2,3,5) distributions. What to expect?

Since all (2,3,5) distributions have the same (up to equivalence) symbol N, let us believe that the linear operators $L_N^{[i]}$ have max rank, for all i.

If so (and it is so) we can compute the dimensions of the kernel of $L_N^{[i]}$ and the dimensions of the complementary space to its image by computing

$$\Delta_i = \dim\left(\text{the target space of } L_N^{[i]}\right) - \dim\left(\text{the source space of } L_N^{[i]}\right)$$

which is a simple exercise in combinatorics; in 10-15 minutes one can calculate that for $-3 \leq i \leq 3$ the Δ_i is negative:

$$\Delta_{-3} = \Delta_3 = \Delta_{-1} = \Delta_1 = -2;\ \Delta_{-2} = \Delta_2 = -1; \Delta_0 = -4$$

and for $i \geq 4$ the Δ_i is positive: $\Delta_4 = 5$ and fastly growth for $i \geq 5$.

Therefore the symmetry algebra of N is the sum of the vector spaces

$$\mathrm{sym}^{[-3]} + \mathrm{sym}^{[-2]} + \mathrm{sym}^{[-1]} + \mathrm{sym}^{[0]} + \mathrm{sym}^{[1]} + \mathrm{sym}^{[2]} + \mathrm{sym}^{[3]}$$

of dimensions $2, 1, 2, 4, 2, 1, 2$.

Here, as above, $[i]$ denotes homogeneous of degree i, therefore $[\mathrm{sym}^{[i]}, \mathrm{sym}^{[j]}] \subseteq \mathrm{sym}^{[i+j]}$.

This is how a simple combinatorics leads to $\mathfrak{g}_2$.

Now, we have a normal form for the quasi-3-jet of a (2,3,5) distribution:

$$N + W^{[3]},\ W^{[3]} \text{ is a complimentary space to the image of } L_N^{[4]}$$

and since by combinatorics $\Delta_4 = 5$ we have (privided that it is true that $L_N^{[4]}$ has max rank) $\dim W^{[3]} = 5$.

Can we find a good $W^{[3]}$? Yes, it is simple, a good $W^{[3]}$ comes out almost "automatically" after few hours of work provided the choice of N is good (if not - we also can obtain a good $W^{[3]}$, but not "automatically, and it will be very involved, and few weeks instead of few hours).

The nilpotent approximation (symbol)

$$N_1 = \partial_{x_1} + x_2 U, \;\; N_2 = \partial_{x_2} - x_1 U, \;\; U = \partial_z + x_1 \partial_{y_1} + x_2 \partial_{y_2}$$

is good because for it it is quick to find a good $W^{[3]}$, and, what is more important, it has a very simple form

$$W^{[3]} = F\Big(x_2 U, \; -x_1 U\Big)$$
$$F = c_{4,0}x_1^4 + c_{3,1}x_1^3 x_2 + c_{2,2}x_1^2 x_2^2 + c_{1,3}x_1 x_2^4 + c_{0,4}x_2^4$$

Now, for my N we have

$$\mathrm{Sym}_N^{[0]} = (\Phi_T, H_T), \ T \in \mathfrak{gl}(2), \ H_T = T^{\mathrm{tr}},$$
$$\Phi_T : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ z \to \det T \cdot z, \ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \det T \cdot T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

which brings $W^{[3]}$ with a fixed $F(x)$, $x = (x_1, x_2)$ to $F(Tx)$.

Therefore the first invariant in the classification of (2,3,5) distribution germs is a homogeneous degree 4 polynomial $F(x_1, x_2)$ defined up to a linear non-singular change of x_1, x_2.

This is how the Poincare way plus few hours simple calculations give the Cartan tensor at $0 \in \mathbb{R}^5$.

The Cartan tensor at 0 is what I called above the first invariant.

It is a complete invariant in the classification of quasi-3-jets of (2,3,5) distributions.

Obtaining good $W^{[i]}$ for all $i \geq 3$ requires more work,
but not much more if one uses GR inner products, see the first lecture.

The following claim generelizes the previous claim from $i = 3$ to any $i \geq 3$:

- For the symbol

$$N_1 = \partial_{x_1} + x_2 U, \ N_2 = \partial_{x_2} - x_1 U, \ U = \partial_z + x_1 \partial_{y_1} + x_2 \partial_{y_2}$$

the subspaces

$$W^{[i]} = F^{[i+1]}\Big(x_2 U, \ -x_1 U\Big)$$

with $F^{[i+1]}$ in the ideal $\mathbf{I}$ given below are good for any $i \geq 3$.

$\mathbf{I}$ is the ideal in the space of functions generated by:
$x_1^\alpha x_2^\beta, \ \alpha + \beta = 4; \ x_1^\alpha x_2^\beta z, \ \alpha + \beta = 3; \ x_1^\alpha x_2^\beta z^2, \ \alpha + \beta = 2$ and
$x_1 z(x_1 y_2 - x_2 y_1), \ x_2 z(x_1 y_2 - x_2 y_1), \ (x_1 y_2 - x_2 y_1)^2$.

NORMAL FORM

In this way we obtain the following normal form that serves for all (2,3,5) distribution germs:

$$V_1 = N_1 + x_2 F(x_1\partial_{y_1} + x_2\partial_{y_2}),\ \ V_2 = N_2 - x_1 F(x_1\partial_{y_1} + x_2\partial_{y_2})$$
$$F = F(x_1, x_2, z, y_1, y_2) \in \text{the ideal } \mathbf{I} \text{ generated by}$$
$$x_1^\alpha x_2^\beta,\ \alpha+\beta = 4;\ x_1^\alpha x_2^\beta z,\ \alpha+\beta = 3,\ \ x_1^\alpha x_2^\beta z^2,\ \alpha+\beta = 2,$$
$$x_1 z(x_1 y_2 - x_2 y_1),\ x_2 z(x_1 y_2 - x_2 y_1),\ (x_1 y_2 - x_2 y_1)^2.$$

The distribution is flat if and only if $F \equiv 0$.

• If it is not flat then $F = F^{[m]} + F^{[m+1]} + \cdots,\ m \geq 4,\ F^{[m]} \neq 0$
In this case $F^{[m]}$ up to the linear transformation
$F^{[m]} \to F^{[m]}(Tx, \det Tz,\ \det T \cdot Ty),\ x = (x_1, x_2), y = (y_1, y_2), T \in GL(2)$
that I call <u>characteristic polynomial</u> is an invariant.

• if $m = 4$ <u>the characteristic polynomial is the Cartan tensor at 0.</u>

The main Cartan theorem from the normal form

Let us prove that if the Cartan tensor vanishes not only at 0 but at any point near 0 then in the bormal form $F \equiv 0$ and consequently the distribution is flat.

Assume by contradiction that $F \not\equiv 0$. Then $F = F^{[m]} + F^{[m+1]} + \cdots$ where $m \geq 5$, $F^{[m]} \neq 0$ and $F^{[m]} \in$**I**. Here **I** is the ideal given in the previous page.

Let $m = 5$. In this case $F^{[5]} = P^{(5)}(x_1, x_2) + zP^{(3)}(x_1, x_2)$ where $P^{(5)}(x_1, x_2)$,$P^{(3)}(x_1, x_2)$ are homogeneus polynomials of degrees 5, 3.

Let us compute the Cartan tensor at the point $P_\epsilon = (\epsilon, 0, 0, 0, 0)$ up to $o(\epsilon)$. It is very easy to see that the quasi-3-jet the distribution at P_ϵ is equivalent to my normal form with $\widehat{F}^{[4]} = \frac{\partial F^{[5]}}{\partial x_1} + o(\epsilon)$.

If $\widehat{F}^{[4]} \in \mathbf{I}$ then $\widehat{F}^{[4]}$ is the Cartan tensor at P_ϵ up to $o(\epsilon)$ and then $\widehat{F}^{[4]} = 0$, i.e. $\frac{\partial F^{[5]}}{\partial x_1} = 0$. In the same way, taking the point $P_\epsilon = (0, \epsilon, 0, 0, 0)$, we obtain $\frac{\partial F^{[5]}}{\partial x_2} = 0$. Then $F^{[5]} = 0$, contradiction.

The life is a bit more complicated: it is not true that $\widehat{F}^{[4]} \in \mathbf{I}$, because $z\frac{\partial F^{(3)}}{\partial x_1} \notin \mathbf{I}$.

Within quasi-homogeneous degree 4 functions we have a certain equivalence modulo $\mathbf{I}$ and I have to find a function in $\mathbf{I}$ that is equivalent to $z\frac{\partial F^{(3)}}{\partial x_1} \notin \mathbf{I}$.

Certainly I know everything about the equivalence modulo $\mathbf{I}$ (it is very simple), in particular:

- for $\alpha + \beta = 2$ one has $x_1^\alpha x_2^\beta z \sim 0 \mod \mathbf{I}$

which is both good and bad.

It is good because the argumet above gives us $P^{(5)}(x_1, x_2) = 0$.

It is bad because the argument above does not give us any information about $P^{(3)}(x_1, x_2) = 0$.

But the argument above is too straightforward. Why we computed Cartan tensor modulo $o(\epsilon)$ at the points $(\epsilon, 0, 0, 0, 0), (0, \epsilon, 0, 0, 0)$? Because the calculation is immediate. But there are other points. And we have two ways: either to work with some other points or to work modulo $o(\epsilon^2)$ rather than $o(\epsilon)$. The latter is doable but more involved.

What is worth to do is to use the 2-dimensional $\mathfrak{g}_2^{[-1]} = \ker L_N^{[-1]}$ and to move 0 to the points $A_{1,\epsilon}$ and $A_{2,\epsilon}$ by the ϵ-time flows of vector fields $\xi_1^{[-1]}, \xi_2^{[-1]}$ that span $\mathfrak{g}_2^{[-1]}$.

I calculated that at A_ϵ the quasi-3-jet of the distribution that has my normal form with $F^{[5]} = zP^{(3)}(x_1, x_2)$ is my normal form with $\widehat{F}^{[4]} = x_2 P^{(3)} \in \mathbf{I}$ and we obtain $P^{(3)} = 0$ as required.

I have proved, up to some calculations, that in my normal form $F^{[5]} = 0$.

The proofs that $F^{[6]} = F^{[7]} = 0$ are similar, even a bit easier.

The proof that $F^{[\geq 8]} = 0$ is also based on the same ideas (and we have the same argument for any $m \geq 8$).

The total proof with <u>all</u> details takes 5-6 pages.

Proof of the base theorem for non-flat distributions

Recall from the first lecture

Theorem 4 The isotropy subalgebra of the symmetry algebra any non-flat (2,3,5) distribution germ at $0 \in \mathbb{R}^5$ does not contain vector fields with the zero linear approximation at 0.

The constructed normal form reduces this statement to the claims that certain three linear operators have the trivial kernel. These operators have, as parameters, $F^{[m]} \neq 0$ in my normal form, which makes the proof rather long (but simple), around 5 pages.

But for the cae $m = 4$ - the case that Cartan tensor does not vanish at 0, the proof is $1/2$ of a page.

Classification of isotropy subalgebras

Given a 2×2 matrix T define the following linear vector field:

$$V_T = \left\langle Tx, \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} \right\rangle + \mathrm{trace}\, T \cdot z\partial_z + \left\langle (T + \mathrm{trace}\, T) y, \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \end{pmatrix} \right\rangle$$

Here $x = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix}, y = \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \end{pmatrix}$ and $< ., >$ is the standard inner product.

Using the base theorem (see the previous page) which I proved from the constructed normal form, and the normal form itself, it takes 1-2 pages to prove the following theorem.

Theorem

(M. Zh). The isotropy subalgebra of the symmetry algebra of any non-flat (2,3,5) distribution germ is diffeomorphic to one of the following:

1. A 1-dim algebra spanned by V_T with

a. any fixed traceless T

b. $T = \mathrm{diag}(-\frac{p}{q}, 1)$ where p and q are integers, $1 \leq p < q$

2. A 2-dim non-Abelian algebra spanned by V_{T_1}, V_{T_2} where

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

3. The $\mathfrak{sl}(2)$ spanned by $V_{T_1}, V_{T_2}, V_{T_3}$ where T_1, T_2, T_3 is a basis of $\mathfrak{sl}(2)$.

If Cartan tensor does not vanish at 0 then the realizable cases are as follows: case 2; case 1.b with $p = 1, q = 3$, and

1.a. with $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or or $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Homogeneous (2,3,5) distributions and endowed symmetry algebras

Given a homogeneous (2,3,5) distribution D, germ at $0 \in \mathbb{R}^5$, take its abstract symmetry algebra $\mathcal{A}$, abstract isotropy subalgebra $\mathcal{I} \subset \mathcal{A}$, and a 2-plane $\mathcal{P} \subset \mathcal{A}$ that corresponds to $D(0)$. $\mathcal{P}$ is defined modulo $\mathcal{I}$.

Definition. $(\mathcal{A}, \mathcal{I}, \mathcal{P})$ = endowed symmetry algebra of $\mathcal{D}$.

Properties: 1. $\dim \mathcal{A} = 5 + \dim \mathcal{I}$; 2. $[\mathcal{I}, \mathcal{P}] \subseteq \mathcal{P} \mod \mathcal{I}$

3. Let $\mathcal{P} = span(a_1, a_2)$. The vectors

$a_1, a_2, a_3 = [a_1, a_2], a_4 = [a_1, a_3], a_5 = [a_2, a_3]$ span $\mathcal{A}/\mathcal{I}$.

4. $\mathcal{I}$ does not contain non-zero ideals of $\mathcal{A}$.

Definition. A triple $(\mathcal{A}, \mathcal{I}, \mathcal{P})$ with 1.-4. = abstract endowed algebra

Homogeneous (2,3,5) distributions = abstract endowed algebras

The pair $(\mathcal{A}, \mathcal{I})$ (without $\mathcal{P}$) = abstract transitive algebra

Any abstract transitive algebra induces a Lie algebra of vector field germs on $(\mathbb{R}^5, 0) \sim (\exp \mathcal{A}/\exp \mathcal{I}, \mathrm{id})$ with the isotropy $\mathcal{I}$.

Theorem. (Nagano-Sussmann). Two transitive Lie algebras of vector field germs are diffeomorphic if and only if the corresponding abstract transitive Lie algebras are isomorphic.

Corollary 1. Any endowed algebra $(\mathcal{A}, \mathcal{I}, \mathcal{P})$ induces a (2,3,5) distribution germ on $(\mathbb{R}^5, 0) \sim (\exp \mathcal{A}/\exp \mathcal{I}, \mathrm{id})$ which will be denoted $(\mathcal{A}, \mathcal{I}, \mathcal{P})\uparrow$.

Corollary 2. Two homogeneous (2,3,5) distribution germs are diffeomorphic if and only if their endowed symmetry algebras are isomorphic as abstract endowed algebras.

Important:

1.The symmetry algebra of $(\mathcal{A},\mathcal{I},\mathcal{P})\uparrow$ is either $\mathcal{A}$ or contains $\mathcal{A}$ and is bigger than $\mathcal{A}$.

2. The symmetry algebra of $(\mathcal{A},\mathcal{I},\mathcal{P})\uparrow$ is $\mathcal{A}$ if and only if $(\mathcal{A},\mathcal{I},\mathcal{P})\not\subset(\widetilde{\mathcal{A}},\widetilde{\mathcal{I}},\mathcal{P})$, for any abstract endowed algebra $(\widetilde{\mathcal{A}},\widetilde{\mathcal{I}},\mathcal{P})$ with $\dim\widetilde{\mathcal{I}}>\dim\mathcal{I}$.
If $(\mathcal{A},\mathcal{I},\mathcal{P})\subset(\widetilde{\mathcal{A}},\widetilde{\mathcal{I}},\mathcal{P})$ then the symmetry algebra of $(\mathcal{A},\mathcal{I},\mathcal{P})\uparrow$ is $\mathcal{A}$ or contains $\mathcal{A}$.

3. Two non-isomorphic abstract endowed algebras might induce diffeomorphic distributions.

Base theorem for classification of homogeneous distributions: endowed symmetry algebras of dim 6,7. In all cases we have $[a_1, a_2] = a_3, [a_1, a_3] = a_4, [a_2, a_3] = a_5$

$[(7,5,1,0)_{\mu_1,\mu_2}]: \quad \mathcal{P} = span(a_1, a_2),\ \mathcal{I} = span(b_1, b_2); \quad [b_1, b_2] = b_2,$ $[a_1, b_1] = -a_1,\ [a_2, b_2] = a_1,\ [a_1, b_2] = [a_2, b_1] = 0$ $[a_1, a_4] = [a_1, a_5] = 0,\ [a_2, a_5] = \mu_1 a_3 + \mu_2 b_2$ $(\mu_1, \mu_2) \sim (k\mu_1, k^2\mu_2)\ k \neq 0;\ \mu_2 \neq \frac{9\mu_1^2}{100};\ \mathrm{CT} = \pm x_1^4$
$[S^{\pm}_{\mu_1,\mu_2}]: \quad \mathcal{P} = span(a_1, a_2),\ \mathcal{I} = span(b)$ $[a_1, b] = -a_2, [a_2, b] = \pm a_1$ $[a_1, a_4] = \mu_1 a_3 + \mu_2 b,\ [a_1, a_5] = 0,\ [a_2, a_5] = \pm(\mu_1 a_3 + \mu_2 b)$ $(\mu_1, \mu_2) \sim (k\mu_1, k^2\mu_2)\ + : k > 0,\ - : k \neq 0;\ \mu_2 \neq \frac{9\mu_1^2}{100}$ $+ : \mathrm{CT} = \pm (x_1^2 + x_2^2)^2;\ - : \mathrm{CT} = \pm x_1^2 x_2^2$
$[\mathfrak{sl}(2) \ltimes \mathfrak{h}_1]^{\pm}: \quad \mathcal{P} = span(a_1, a_2),\ \mathcal{I} = span(b)$ $[a_1, b] = -2b, [a_2, b] = a_1$ $[a_1, a_4] = 2a_3 + a_4,\ [a_1, a_5] = a_5,\ [a_2, a_5] = \pm(2a_2 + a_3 - a_4)$ $\mathrm{CT} = -x_1^4$, for both $\pm$

Important

• The symmetry algebra $\mathfrak{sl}(2) \ltimes \mathfrak{h}_1$ is Borya Doubrov's discovery.

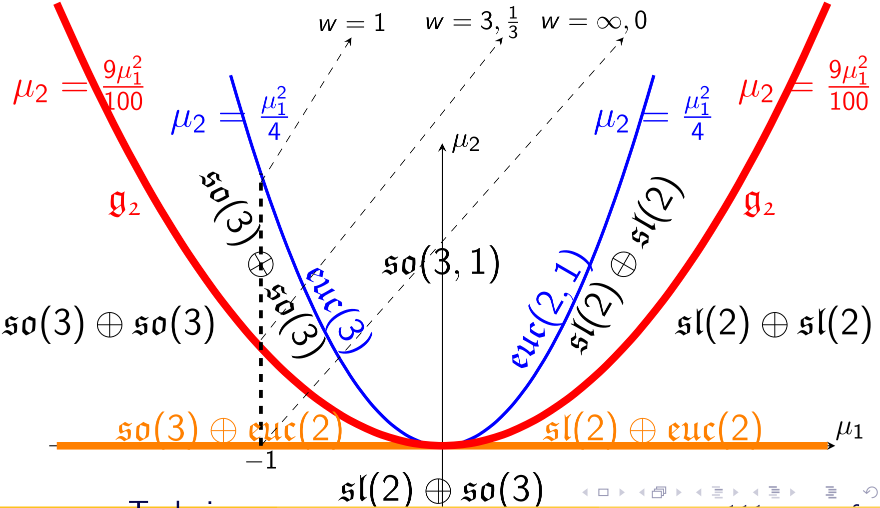
• The 7-dim Lie algebra $\mathcal{A}$ in $[(7,5,1,0)_{\mu_1,\mu_2}]$ is solvable:
$\dim \mathcal{A}^2 = 5, \dim (A^2)^2 = 1$ whence the notation (7,5,1,0)

• The pair $(\mu_1, \mu_2) \sim (k\mu_1, k^2\mu_2), k \neq 0$ is an invariant not only of the endowed algebra but of the Lie algebra too.

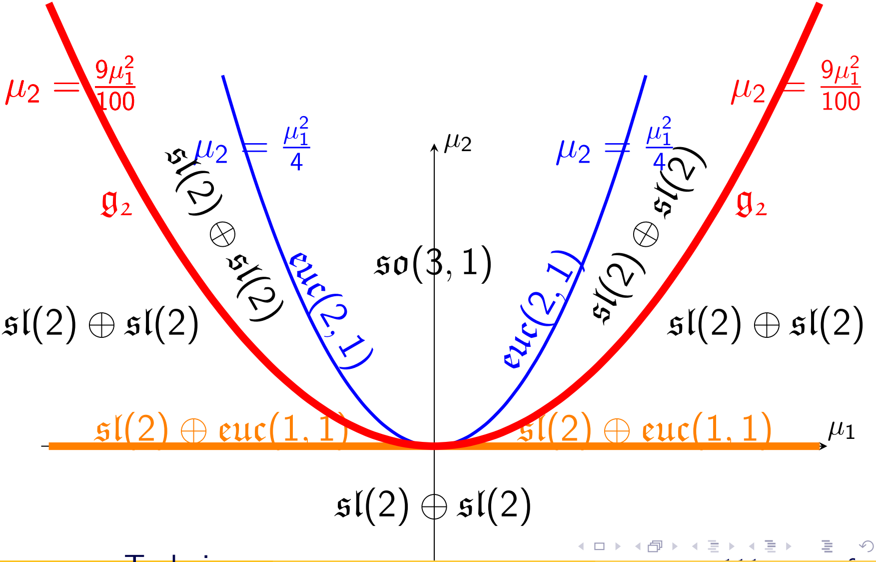
• In the endowed algebras $[S^{\pm}_{\mu_1,\mu_2}]$ the pair $(\mu_1, \mu_2) \sim (k\mu_1, k^2\mu_2)$, with $k > 0$ for $+$ and $k \neq 0$ for $-$ is an invariant, but NOT for the Lie algebra. The Lie algebra is isomorphic to one of 9 fixed Lie algebras, see the figures in the next pages.

• Over $\mathbb{C}$ in all normal forms with $\pm$ we have $+ \sim -$;
$k > 0$ shuld be replace by $k \neq 0, k \in \mathbb{C}$. No other changes.

The symmetry algebras of homogeneous (2,3,5) distributions $S^{+}_{\mu_1,\mu_2} \uparrow$, $(\mu_1, \mu_2) \sim (k\mu_1, k^2\mu_2)$, $\underline{k > 0}$. Dashed line: two rolloing balls: the endowed symmetry algebra $S^{+}_{-1,\mu}$, $\mu = \frac{1}{(w+\frac{1}{w})^2} \leq \frac{1}{4}$ where w is the ratio of the radii



The symmetries algebras of homogeneous (2,3,5) distributions $S^-_{\mu_1,\mu_2}\uparrow$, $(\mu_1,\mu_2)\sim(k\mu_1,k^2\mu_2)$, $k\neq 0$.



Proof of the base theorem on homogeneous (2,3,5) distributions

We considet the case that the distribution is not flat, then the Cartan tensor CT does not vanish at 0. The normal form just for quasi-3-jet (see the second lecture) at 0 in its terms, and the theorem that the isotropy does not contain non-zero vector fields with the zero linear approximation (for non-vanishing at 0 CT it is simple) imply a part of structure equations for the endowed symmetry algebra of any homogeneous (2,3,5) distribution, namely:

Case 1. CT $= \pm x_1^4$ and the isotropy $\mathcal{I}$ is 2-dim, $\mathcal{I} = span(b_1, b_2)$. In this case $b_1 = x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4} + 2x_5\partial_{x_5} + h.o.t.$
$b_2 = x_1\partial_{x_2} + x_4\partial_{x_5} + h.o.t.$ and $\mathcal{P} = span(a_1, a_2)$ where
$a_1 = \partial_{x_1} + \cdots$, $a_2 = \partial_{x_2} + \cdots$, so that we have $[b_1, b_2] = -b_2$ and
$[a_1, b_1] = *b_1 + *b_2$, $[a_2, b_1] = a_2 + *b_1 + *b_2$
$[a_1, b_2] = a_2 + *b_1 + *b_2$, $[a_2, b_2] = *b_1 + *b_2$, each $*$ is some number.

Case 2. CT $= \pm x_1^4$ and isotropy $\mathcal{I}$ is 1-dim, $\mathcal{I} = span(b_1)$ where b_1 is from Case 1.

Case 3. CT $= \pm x_1^4$ and isotropy $\mathcal{I}$ is 1-dim, $\mathcal{I} = span(b_2)$ where b_2 is from Case 1.

Case 4. CT $= \pm x_1^2 x_2^2$ or CT$=\pm(x_1^2 + x_2^2)^2$. These cases are similar. The isotropy $\mathcal{I}$ must be 1-dim, $\mathcal{I} = span(b)$. In the first case
$b = x_1\partial_{x_1} - x_2\partial_{x_2} + x_4\partial_{x_4} - x_5\partial_{x_5} + h.o.t.$, and $\mathcal{P} = span(a_1, a_2)$ where
$a_1 = \partial_{x_1} + \cdots, a_2 = \partial_{x_2} + \cdots$ so that
$[a_1, b] = a_1 + w_1 b, [a_2, b] = -a_2 + w_2 b.$

Case 5. CT $= x_1^3 x_2$. The isotropy $\mathcal{I}$ must be 1-dim, $\mathcal{I} = span(b)$, and
$b = x_1\partial_{x_1} - 3x_2\partial_{x_2} - 2x_3\partial_{x_3} - -x_4\partial_{x_4} - 5x_5\partial_{x_5} + h.o.t.$, and
$\mathcal{P} = span(a_1, a_2)$ where $a_1 = \partial_{x_1} + \cdots, a_2 = \partial_{x_2} + \cdots$ so that
$[a_1, b] = a_1 + w_1 b, [a_2, b] = -3a_2 + w_2 b.$

Consider Case 4 with CT$= x_1^2 x_2^2$. We may assume $w_1 = w_2 = 0$ $(a_1 \to a_1 - w_1 b, a_2 - w_2 b)$. Recall that $a_3 = [a_1, a_2], a_4 = [a_1, a_3], a_5 = [a_2, a_3]$ so that $[a_3, b] = 0, [a_4, b] = a_4, [a_5, b] = -a_5$. Let

$$
\begin{aligned}
[a_1, a_4] &= t_{1,1} a_1 + \cdots + t_{1,5} a_5 + t_{1,6} b \\
[a_1, a_5] &= t_{2,1} a_1 + \cdots + t_{2,5} a_5 + t_{2,6} b \\
[a_2, a_5] &= t_{3,1} a_1 + \cdots + t_{3,5} a_5 + t_{3,6} b
\end{aligned}
$$

The Jacobi identity for the triples $(a_1, a_4, b), (a_1, a_4, b), (a_2, a_5, b)$ immediately gives

$t_{1,1} = \cdots = t_{1,6} = 0,\ t_{3,1} = \cdots = t_{3,6} = 0,\ t_{2,1} = t_{2,2} = t_{2,4} = t_{2,5} = 0$ and we have $[a_1, a_4] = 0, [a_2, a_5] = 0, [a_1, a_5] = \mu_1 a_3 + \mu_2 b$. Now we can compute all structure equations and to see for which μ_1, μ_2 they hold. They hold for any μ_1, μ_2. The obtained normal form is isomorphic to the normal form $S^-_{\mu_1,\mu_2}$ in the table in the base theorem.

Consider now Case 5 with CT= $x_1^3x_2$. Exactly like in Case 4 we obtain $[a_1, a_4] = \mu b, [a_1, a_5] = 0, [a_2, a_5] = 0$. Again, compute all the structure equations and check the remaining Jacobi identities. Unlike the previous case they imply $\mu = 0$. Therefore the endowed algebra contains 5-dim endowed subalgebra with the zero characteristic matrix. But then CT=0, contradiction.
We have proved that for homogeneous distributions with symmetry algebras of dim ≥ 6: CT $\neq x_1^3x_2$.

Case 1 above, with 7-dim endowed symmetry algebras, can be treated by the same powerful method: working with Jacobi identities, it requires a bit more time. We obtain the normal form $[(7,5,1,0)_{\mu_1,\mu_2}]$ in the table in the base theorem.

For Case 2, with CT=$\pm x_1^4$ and $\mathcal{I} = span(x_1\partial_{x_1} - x_2\partial_{x_2} + x_4\partial_{x_4} - x_5\partial_{x_5})$, the way is the same. We obtain certain family of endowed algebras but it is easy to see that for any values of parameters this endowed algebra is an endowed subalgebra of $[(7,5,1,0)_{\mu_1,\mu_2}]$. Therefore Case 2 is not realizable.

Remains Case 3, with CT$=\pm x_1^4$ and $\mathcal{I} = span(x_1\partial_{x_2} + x_2\partial_{x_5})$. A simple, as above, work with Jacobi identities splits Case 3 onto 3.a and 3.b. In case 3.a we obtain an endowed subalgebra of $[(7,5,1,0)_{\mu_1,\mu_2}]$, therefore this case is not realizable.

The first level Jacobi identities in case 3.b show that we do not have an endowed subalgebra of $[(7,5,1,0)_{\mu_1,\mu_2}]$. But, since everything above matches Cartan so well, it is easy to believe that the second level Jacobi identities give us that the 6-dim endowed algebra contains a 5-dim endowed subalgebra with the zero characteristic matrix, then it induces a flat adistribution, like when we dealt with potential CT$=x_1^3x_2$. I wanted it, and I proved it, making a calculational mistake.

Nevertheless, the fact that an endowed algebra $\mathcal{E}$ does not contain an endowed 5-dim subalgebra with the zero characteristic matrix does <u>not</u> mean that $\mathcal{E}\uparrow$ is not flat, it might be flat!

The last hope that Cartan is right was as follows. Look at the structure equations for the endowed algebra $[\mathfrak{sl}(2) \ltimes \mathfrak{h}_1]$ in the base theorem. We can take away the isotropy b, what remains is an endowed 5-dim algebra. It is a solvable (5,4,2,0) algebra, meaning that the square is 4-dim and the square of the square is 2-dim Abelian. In the classification of (5,4,2,0) endowed algebras (as well as non-endowed) there is an invariant $\lambda \in \mathbb{R}$, a normal form $[(5,4,2,0)]^{\pm}_{\lambda}$. For the (5,4,2,0) endowed subalgebra of $[\mathfrak{sl}(2) \ltimes \mathfrak{h}_1]$ this invariant takes just one value $\lambda = \lambda^*$. The last hope was that CT for $[(5,4,2,0)]^{\pm}_{\lambda^*}$ is 0.

This hope was promising because there exist, and exactly one, λ for which CT=0, so that the induced distribution is flat (and not "through" a 6-dim or a 7-dim subalgebra of $\mathfrak{g}_2$, directly!), it is so for certain $\lambda = \lambda^{**}$. But $\lambda^{**} \neq \lambda^* \implies$ Borya is absolutely right and Cartan not.

The last, and non-trivial part of the base theorem, is to prove that the distributions $[(7,5,1,0)_{\mu_1,\mu_2}]\uparrow$ and $[S^{\pm}_{\mu_1,\mu_2}]\uparrow$ are flat if and only if $\mu_2 = \frac{9\mu_1^2}{100}$.

The problem is that a straightforward realization of the endowed algebras $[(7,5,1,0)_{\mu_1,\mu_2}]$ and $[S^{\pm}_{\mu_1,\mu_2}]$ is doable but very involved task. And after that one also need to realize the induced distributions by vector fields (it is easier), before applying an algorithm for computing the Cartan tensor at $0 \in \mathbb{R}^5$, to see for which μ_1, μ_2 it is 0.

Fortunately, there is another, much better way. The endowed algebras $[(7,5,1,0)_{\mu_1,\mu_2}]$ and $[S^{\pm}_{\mu_1,\mu_2}]$ contain endowed 5-dim subalgebra $\mathcal{E}$ that can be easily calculated, and the pair (μ_1, μ_2) is "flat" if and only if the Cartan tensor of $\mathcal{E}\uparrow$ is 0. For the latter I have an explicit formula in terms of the characteristic matrix (few hours by hand, $1/10$ sec by Matematica).

In fact, over $\mathbb{R}$ the $[S^{\pm}_{\mu_1,\mu_2}]$ contains endowed 5-dim subalgebras not for all μ_1, μ_2 (for example, when the non-endowed Lie algebra is $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ or $\mathfrak{so}(3,1)$), but in order to decide if a distribution is flat or not it suffices to work over $\mathbb{C}$, and over $\mathbb{C}$ no problems.

Another advantage of finding 5-dim endowed subalgebras $\mathcal{E}$ of endowed algebras $\mathcal{U}$ of dim 6 and 7 is as follows:

• It is very easy to realize $\mathcal{E}$ by vector fields.

• Let A_1, A_2 be vector fields that realize the generating plane $\mathcal{P} = span(a_1, a_2)$ in $\mathcal{E}$.

• Then the same vector fields A_1, A_2 describe the distribution which is diffeomorphic to $\mathcal{U}\uparrow$.

What to do?

1. Decompose all endowed 5-dim algebras onto the following groups:

1. Endowed subalgebras of $[(7,5,1,0)_{\mu_1,\mu_2}]$ for some (non-fixed) μ_1, μ_2

2. Endowed subalgebras of $[S^{\pm}_{\mu_1,\mu_2}]$ for some (non-fixed) μ_1, μ_2

3. Endowed subalgebras of $[\mathfrak{sl}(2) \ltimes \mathfrak{h}_1]^{\pm}$

4. Endowed subalgebras that are not in **1., 2., 3.**
but which induce a flat distribution

5. Everything else

6. Complete classification of everything else with respect to isomorphisms of endowed 5-dim algebras. It gives classification of all homogeneous distributions with 5-dim symmetry algebras, in most forms one wishes (endowed algebras, polynomial vector fields, Monge forms, ...)

7. Does $[(7,5,1,0)_{\mu_1,\mu_2}]$, $[S^{\pm}_{\mu_1,\mu_2}]$ with <u>fixed</u> μ_1, μ_2 have an endowed 5-dim subalgebra? If yes - we have everything we wish for it, too.
And what we have if not?

Answer to **1.**

Proposition. A 5-dim endowed algebra $\mathcal{E}$ is an endowed subalgebra of the endowed algebra $[(7, 5, 1, 0)_{\mu_1,\mu_2}]$ if and only if one of the following:

1. $\mathcal{E}$ is a (5,3,1,0) or a (5,4,1,0) solvable algebra
2. $\mathcal{E}$ is a special (5,3,0) algebra, where

Definition. An endowed (5,3,0) algebra is special if its reduced characteristic 3×2 matrix (the last two columns of the characteristic matrix) is equivalent to a matrix of the form

$$\begin{pmatrix} * & * \\ a & * \\ 0 & 3a \end{pmatrix}, \quad a \in \mathbb{R}, \ * \text{ is any number.}$$

Answer to 2

Proposition. A 5-dim endowed algebra $\mathcal{E}$ is an endowed subalgebra of the endowed algebra $[S^{\pm}_{\mu_1,\mu_2}]$ if and only if one of the following:

1. The Lie algebra in $\mathcal{E}$ is any of the real forms of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C} \times \mathbb{C}$ or $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}^2$

2. $\mathcal{E}$ is an endowed (5,3,0) algebra whose reduced characteristic 3×2 matrix is equivalent to

$$\begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3. $\mathcal{E}$ is an endowed algebra $[(5,4,2,0)]_\lambda$ with a special value of the invariant $\lambda = \lambda_1^*$

Answers to 3-5

• The endowed algebra $[\mathfrak{sl}(2) \ltimes \mathfrak{h}_1]^{\pm}$ has one and only one endowed subalgebra; it is "visible" (just remove the isotropy *b* from the structure equations) and it is an endowed algebra $[(5,4,2,0)]_\lambda$ with a special value of the invariant $\lambda = \lambda_2^*$

• There are only two (over $\mathbb{C}$-one) endowed 5-dim subalgebra of $\mathfrak{g}_2$ which is not an endowed subalgebra of one of the endowed symmetry algebras of dim 6,7. It is, again, $[(5,4,2,0)]_\lambda$ with a new special value of the invariant $\lambda = \lambda_3^*$

• Everything else is not a big set. It is:

1. non-special (5,3,0) endowed algebras
2. endowed algebras $[(5,4,2,0)]_\lambda$ with the invariant $\lambda \neq \lambda_1^*, \lambda_2^*, \lambda_3^*$
3. two endowed algebras $[\mathfrak{sl}(2) \ltimes \mathbb{C}^2]^{\pm}$

Answers to 6,7

Complete classification of everything else with respect to isomorphisms of endowed 5-dim algebras, equivalently classification of homogeneous (2,3,5) distributions with 5-dim symmetry algebras, is not a big deal, but one should find "good" normal forms, especially a 2-parameter normal form for (5,3,0) endowed algebras.

We have already agreed that in many cases, if not all, what is good and not good normal form depends on a person.

In fact, not only on a person: it is a function of two variables:
person and time.

• Over $\mathbb{C}$ any endowed symmetry algebra of dim 6,7 has (as well as $\mathfrak{g}_2$) and endowed 5-dim subalgebra, i.e. any homogeneous (2,3,5) distribution is left-invariant.

• Over $\mathbb{R}$ it is not so only for $[S^{\pm}]_{\mu_1,\mu_2}$ with (μ_1, μ_2) such that the Lie algebra is one of $\mathfrak{so}(3) \oplus \mathfrak{so}(3), \mathfrak{so}(3,1), \mathfrak{euc}(3)$.

Certain endowed 5-dim algebras, over $\mathbb{C}$

The (5,3,0) endowed algebras are given in terms of the reduced characteristic matrix (the last two columns of the characteristic matrix; the other endowed algebras are given in terms of characteristic matrix.

$\begin{pmatrix} 2\lambda_1 - 1 & 0 \\ \lambda_2 & \lambda_1 \\ 0 & 2\lambda_2 - 1 \end{pmatrix}$ $[(5,3,0)^A_{\lambda_1,\lambda_2}]$	$(\lambda_1,\lambda_2) \sim S_3 \circ (\lambda_1,\lambda_2)$ S_3 is generated by involutions $(\lambda_1,\lambda_2) \to (\lambda_2,\lambda_1),\ (\lambda_1,\lambda_2) \to (\lambda_1,\ 1-\lambda_1-\lambda_2)$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}$ $[(5,3,0)^B]$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ $[(5,3,0)^C]$
$\begin{pmatrix} 0 & (\lambda-1)(2\lambda-1) & 1-2\lambda^2 & 3\lambda-1 & 0 \\ 0 & 0 & 0 & 0 & \lambda \\ 0 & 1 & -\frac{\lambda}{\lambda-1} & \frac{1}{\lambda-1} & 0 \end{pmatrix}$ $[(5,4,2,0)_\lambda],\ \lambda \neq 1$	
$\begin{pmatrix} 0 & \lambda & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{\lambda^2-1} & -1 & -\frac{\lambda}{\lambda^2-1} & 0 \end{pmatrix}$ $\lambda \neq 0 : \mathfrak{sl}(2) \oplus (\mathbb{C} \ltimes \mathbb{C})]^A_\lambda;\ \lambda \sim -\lambda$	$\begin{pmatrix} 4 & 0 & 4 & 0 & 1 \\ 0 & -4 & 0 & 1 & 0 \\ -4 & 0 & 4 & 0 & -1 \end{pmatrix}$ $[\mathfrak{sl}(2) \ltimes \mathbb{C}^2]$

Classification of non-flat homogeneous (2,3,5) distributions over $\mathbb{C}$, in terms of 5-dim endowed algebras

Parameters	Endowed symm. algebra	Symm. alg.
$1.[(5,3,0)^A_{\lambda_1,\lambda_2}]\uparrow$		
1.1. $\lambda_1 \neq -1, \lambda_2 \neq -1,\ \lambda_1+\lambda_2 \neq 2$ $(\lambda_1,\lambda_2) \neq (0,\frac{1}{2}),(\frac{1}{2},0)$ $(\lambda_1,\lambda_2) \sim S_3 \circ (\lambda_1,\lambda_2)$	$[(5,3,0)^A_{\lambda_1,\lambda_2}]$	$(5,3,0)^A_{\lambda_1,\lambda_2}$
1.2. $\lambda_1 = 1+\lambda,\ \lambda_2 = 1-\lambda$ $\lambda \neq \pm 1, \pm 2, \pm\frac{1}{2};\ \lambda \sim \pm\lambda, \pm\frac{1}{\lambda}$	$[(7,5,1,0)_{\mu,1}]$ $\mu = \frac{2(\lambda^2+1)}{\lambda^2-1}$	$(7,5,1,0)_{\mu,1}$
1.3. $\lambda_1 = 0,\ \lambda_2 = 2$	$[(7,5,1,0)_{1,0}]$	$(7,5,1,0)_{1,0}$
1.4. $\lambda_1 = 0,\ \lambda_2 = \frac{1}{2}$	$S_{1,0}$	$\mathfrak{sl}(2) \oplus \mathfrak{euc}^{\mathbb{C}}(2)$
2. $[(5,3,0)^B]\uparrow$		
	$[(5,3,0)^B]$	$(5,3,0)^B$
3. $[(5,3,0)^C]\uparrow$		
	$[(5,3,0)^C]$	$(5,3,0)^C$
4. $[(5,4,2,0)_\lambda],\ \lambda \neq 1$		
4.1. $\lambda \neq 0, \frac{1}{2}, \frac{4}{3}$	$[(5,4,2,0)_\lambda]$	$(5,4,2,0)_\lambda$
4.2. $\lambda = 0$	$S_{2,1}$	$\mathfrak{euc}^{\mathbb{C}}(3)$
4.3. $\lambda = \frac{1}{2}$	$[\mathfrak{sl}(2) \ltimes \mathfrak{h}_1]^{\pm}$	$\mathfrak{sl}(2) \ltimes \mathfrak{h}_1$
5. $[\mathfrak{sl}(2) \oplus (\mathbb{C} \ltimes \mathbb{C})]^A_\lambda \uparrow,\ \lambda \neq 0, \pm 1$		
$\lambda \neq \pm 3, \pm\frac{1}{3};\ \lambda \sim \pm\lambda, \pm\frac{1}{\lambda}$	$S_{\mu,1},\ \mu = \lambda + \frac{1}{\lambda}$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$
6. $[\mathfrak{sl}(2) \ltimes \mathbb{C}^2]\uparrow$		
	$[\mathfrak{sl}(2) \ltimes \mathbb{C}^2]$	$\mathfrak{sl}(2) \ltimes \mathbb{C}^2$

Realization of the normal forms for non-flat homogeneous (2,3,5) distributions over $\mathbb{C}$ by polynomial vector fields

1.a. $[(5,3,0)^A_{\lambda_1,\lambda_2}]\uparrow$ $\lambda_2 \neq 0,1$	$V_1 = \begin{pmatrix} 1 & 0 & \lambda_1 x_3 & \lambda_1 x_4 & (\lambda_1 - 1)x_5 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 & 1 & -1+\lambda_2 x_3 & -1+(\lambda_2-1)x_4 & -1+\lambda_2 x_5 \end{pmatrix}$
1.b. $[(5,3,0)^A_{\lambda_1,\lambda_2}]\uparrow$ $\lambda_1 = \lambda_2 = 0$	$V_1 = \begin{pmatrix} 1 & 0 & x_2 & 1 & x_5 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 & 1 & 0 & x_4 & 1 \end{pmatrix}$
2. $[(5,3,0)^B]\uparrow$	$V_1 = \begin{pmatrix} 1 & 0 & 1 & x_4 & x_5 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 & 1 & x_3 & 0 & x_4 \end{pmatrix}$
3. $[(5,3,0)^C]\uparrow$	$V_1 = \begin{pmatrix} 1 & 0 & x_3 & x_4 & x_3+x_5 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 & 1 & 1 & x_3 & x_4 \end{pmatrix}$
4.a. $[(5,4,2,0)_\lambda]\uparrow$ $\lambda \neq 1,2$	$V_1 = \begin{pmatrix} 1 & (\lambda-1)x_2 & x_3 & 2-\lambda+\lambda x_4 & (2\lambda-1)x_5 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 & 1 & 1 & x_3 & x_4 \end{pmatrix}$
4.b. $[(5,4,2,0)_\lambda]\uparrow$ $\lambda = 2$	$V_1 = \begin{pmatrix} 1 & x_2 & x_2+x_3 & x_2^2+2x_4 & x_2^3+3x_5 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 & 1 & 0 & 2x_3 & 1+3x_4 \end{pmatrix}$
5. $[\mathfrak{sl}(2) \oplus (\mathbb{C} \ltimes \mathbb{C})]_\lambda \uparrow$	$V_1 = \begin{pmatrix} x_1 & -x_2 & 1 & \lambda & -\lambda x_5 \end{pmatrix}$ $V_2 = \begin{pmatrix} 1+\frac{x_1^2}{2} & 1+\frac{x_2^2}{2} & \frac{x_1-x_2}{2} & 0 & 1 \end{pmatrix}$
6. $[\mathfrak{sl}(2) \ltimes \mathbb{C}^2]\uparrow$	$V_1 = \begin{pmatrix} 1 & \frac{x_2^2}{2} & -\frac{x_2}{2} & \frac{x_5}{\sqrt{2}} & 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} \frac{x_1^2}{2} & 1 & \frac{x_1}{2} & 1 & -\frac{x_4}{\sqrt{2}} \end{pmatrix}$

Realization of the normal forms for non-flat homogeneous (2,3,5) distributions over $\mathbb{C}$ by Monge form

$$\text{Monge form: } V_1 = \partial_{x_1},\ V_2 = \partial_{x_2} + x_1\partial_{x_3} + x_3\partial_{x_4} + F(x_1,x_2,x_3,x_4,x_5)\partial_{x_5}$$
$$\iff z' = F(y'', x, y', y, z);\quad \text{(2,3,5) at a point } p : \iff \frac{\partial^2 F}{\partial x_1^2}(p) \neq 0$$

Normal form	Equivalent to Monge form with $F =$:
1.a. $[(5,3,0)^A_{\lambda_1,\lambda_2}]\uparrow,\ \lambda_2 \neq 0,1$	$F = x_1^{1-\lambda_2} x_2^{\lambda_1-\lambda_2}$
1.b. $[(5,3,0)^A_{0,0}]\uparrow$	$F = \frac{e^{-x_1x_2}}{x_2}$
2. $[(5,3,0)^B]\uparrow$	$F = e^{-x_2}\ln(x_1)$
3. $[(5,3,0)^C]\uparrow$	$F = x_1(x_2^2 + \ln(x_1))$
4.a. $[(5,4,2,0)_\lambda]\uparrow,\ \lambda \neq 0,1,2$	$F = x_1^{\frac{\lambda}{2-\lambda}} + x_4$
4.b. $[(5,4,2,0)_\lambda]\uparrow,\ \lambda = 0$	$F = \ln(x_1) + x_4$
4.c. $[(5,4,2,0)_\lambda]\uparrow,\ \lambda = 2$	$F = e^{x_1} + x_4$
5. $[\mathfrak{sl}(2) \oplus (\mathbb{C} \ltimes \mathbb{C})]\uparrow$	$F = e^{x_2x_3}(2\lambda x_1 + x_3^2)^{\frac{\lambda+1}{2}} + x_1x_2x_3$
6. $[\mathfrak{sl}(2) \ltimes \mathbb{C}^2]\uparrow$	$F = x_1^{-\frac{1}{3}}(1 + x_3x_5)^2$

Remark. Borya Doubrov informed me that Travis Willse found a simpler Monge form for $[\mathfrak{sl}(2) \ltimes \mathbb{C}^2]\uparrow$: $F = x_1^{\frac{4}{3}} - x_5^2$

red lines $\lambda_1 = -1, \lambda_2 = -1, \lambda_1 + \lambda_2 = 1$
except cyan and blue points : CT$= \pm x_1^4$
(sign changes at blue points)
symm. alg. $(7,5,1,0)_{\mu,1}$

9 blue points
the S_3- orbit of
$\{(-1,-1),(-1,-\frac{3}{2})\}$
CT=0, symm. alg. $\mathfrak{g}$

6 cyan points
the S_3-orbit of $(2,0)$
CT$=x_1^4$; symm. alg.$(7,5,1,0)_{1,0}$

yellow curves
except blue points:
CT$= x_1^2 x_2(x_1 \pm x_2)$
(sign changes at blue point
symm. alg. 5-dim

3 green points
$(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0),(0,\frac{1}{2})$
CT$=x_1^2x_2^2$
symm. alg. $\mathfrak{sl}(2) \oplus \mathfrak{euc}(2,1)$

3 black points
$(\frac{3}{2},\frac{3}{2}),(\frac{3}{2},-2),(-2,\frac{3}{2})$
CT$= -x_1^2x_2^2$; symm. alg. 5-dim

the point $(\frac{1}{3},\frac{1}{3})$:
CT$= (x_1^2+x_2^2)^2$; symm. alg. 5-dim