Normal forms and symmetries for (2,3,5) and (3,5) distributions

111 years after E.Cartan's 5 variables paper

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Do we need Cartan tensor?

Yes. By many reasons, probably the most important is as follows:

without Cartan tensor there is no way to answer the following important question: if a given (2,3,5) distribution is flat or not?

I will give one of equivalent definitions of a flat (2,3,5) distributins using one of my tools: characteristic matrix of an endowed 5-dim Lie algebra.

Endowed 5-dim algebras and their characteristic matrices

By definition, an endowed 5-dim algebra is a 5-dim Lie algebra $\mathcal{A}$ endowed with a generating 2-plane $\mathcal{P} = span(a_1, a_2)$ meaning that the vectors

$$a_1, a_2, \; a_3 = [a_1, a_2], \; a_4 = [a_1, a_3], \; a_5 = [a_2, a_3]$$

are linearly independent.

Any endowed 5-dim algebra can be described by the following 3×5 matrix $(r_{i,j})$ that I call a characteristic matrix:

$$\begin{array}{ccc} [a_1, a_4] & = & r_{1,1}a_1 + \cdots + r_{1,5}a_5 \\ [a_1, a_5] = [a_2, a_4] & = & r_{2,1}a_1 + \cdots + r_{2,5}a_5 \\ [a_2, a_5] = & = & r_{3,1}a_1 + \cdots + r_{3,5}a_5 \end{array}$$

All other structure equations follow from the Jacobi identity.

Homogeneous (2,3,5) distribution induced by an andowed 5-dim algebra

Let $(\mathcal{A}, \mathcal{P})$ be an endowed 5-dim algebra.

Let (G, id) be a nbhd of id of the Lie group G of $\mathcal{A}$,
so that $\mathcal{P}$ is a 2-plane in $T_{\mathrm{id}}G$.

Push $\mathcal{P}$ to (G, id) by the flows of left invariant vector fields on G.

We obtain a local (germ at id) (2,3,5) distribution on (G, id).

It is homogeneous becuse its symmetry algebra either is $\mathcal{A}$ or contains $\mathcal{A}$.

The most known classical theorem

Theorem 1. (Cartan, Tanaka). Let D be the germ at $0 \in \mathbb{R}^5$ of a (2,3,5) distribution. The following are equivalent:

1. D is equivalent to a distribution induced by the endowed 5-dim algebra with the zero characteristic matrix.

2. The symmetry algebra of D is $\mathfrak{g}_2$

3. The Cartan tensor of D vanishes at any point near $0 \in \mathbb{R}^5$

Definition. If the conditions of Theorem 1. hold, the (2,3,5) distribution is called flat.

Question. What can be said about the (2,3,5) distribution induced by the endowed 5-dim algebra with non-zero characteristic matrix? **a**. Can it be flat? **b**. Its symmetry algebra?

I will give complete answers. Quick answer to **a**: yes

Less known very important theorem

Theorem 2. (Cartan) If D is not flat, its symmetry algebra has dim ≤ 7.

My proof of Theorem 2 is as follows.

Theorem 3. (M. Zh) Let D be the germ at $0 \in \mathbb{R}^5$ of a non-flat (2,3,5) distribution. The isotropy subalgebra at 0 of the symmetry algebra of D is, as an abstract Lie algebra, one of the following:

a. 1-dim; **b**. 2-dim non-Abelian; **c.** $\mathfrak{sl}(2)$.

Each of these cases is realizable.
In case **c**, the Cartan tensor of D at 0 is equal to 0.

Theorem 2. is a logical corollary of Theorems 3. and 1.

How I prove Theorem 3?
My base theorem for non-flat distributions

My proof of Theorem 3, and of many other theorems,
uses the following statement that for me plays the key role.

Theorem 4 (the base theorem for non-flat distributions (M. Zh.)
The isotropy subalgebra of the symmetry algebra any non-flat (2,3,5) distribution germ at $0 \in \mathbb{R}^5$ does not contain vector fields with the zero linear approximation at 0.

Theorem 5 (logical corollary of Theorem 4)
The isotropy subalgebra $\mathcal{I}$ of the symmetry algebra any non-flat (2,3,5) distribution germ D at $0 \in \mathbb{R}^5$ is isomorphic (as an abstract Lie alegbra) to the Lie algebra $j_0^1\mathcal{I}$ (the linear approximation of $\mathcal{I}$).

Theorem 6 (M.Zh). Let D and $\mathcal{I}$ be as in Corollary 5. Then $j_0^1\mathcal{I}$ is isomorphic to one of **a, b, c** in Theorem 3.

Theorem 3. is a logical corollary of Theorems 5 and 6

My approach is based on:

In order to prove Cartan-Tanaka classical Theorem 1, my base Theorem 4, and Theorem 6 (Theorems 2,3,5 are, as I explained, their corollaries), and many other statements that will be formulated in this mini-course, I use the following, for (2,3,5) distributions D:

A. A theorem explaining the Cartan tensor of D at a fixed point

B. A theorem on characteristic polynomial of D :
a generalization of Cartan tensor at a fixed point

C. A theorem on almost exact normal form for all possible D

Theorems **A,B,C** for local Riemannian metrics

Given the germ at $0 \in \mathbb{R}^n$ of a Riemannian metric on $\mathbb{R}^n$,
take an orthonormal basis $V_1, ..., V_n$ of vector fields,
and a local coordinate system $x = (x_1, ..., x_n)$.

Describe the metric by an $n \times n$ matrix $M(x)$
with the functional entries $M_{i,j}(x) = V_j(x_i)$.

Theorem 7. (Gauss lemma; M. Zh).
1. For a suitable orthonormal basis and suitable local coordinates one has

$$M(x) = I + A(x),\ A^{\mathrm{tr}}(x) = A(x),\ A(x) \cdot x \equiv 0,\ x = (x_1, \dots, x_n)^{\mathrm{tr}}$$

2.This normal form is exact up to transformations
$A(x) \to R^{\mathrm{tr}} A(Rx) R$ where R is a constant orthogonal matrix.

3. The metric is flat if and only if $A(x) \equiv 0$.

Remark. The local coordinates in this normal form are normal (geodesic).

Proof of Theorem 7 by the Poincare way

What I call the Poincare way is the way used in the whole local analysis and physics (but unfortunately so far not by many people in local differential geometry), especially after the resonance normal form obtained by Poincare for vector fields, that is used and developed in thousands of works.

We have $M(x) = I + A^{(1)}(x) + A^{(2)}(x) + \cdots$ where (i) denotes the homogeneous part of degree i. Assume we have normalized $A^{(i)}(x)$ for $i < k$ and want to normalize $A^{(k)}(x)$. For that, let us change the local coordinates and the basis as follows:

$$x \to x + \Phi^{(k)}(x),$$
$$(V_1, ..., V_n)^{\mathrm{tr}} = \exp(S^{(k)}(x)) \cdot (V_1, ..., V_n)^{\mathrm{tr}}, \ S^{(k)}(x) \in \mathfrak{so}(n)$$

Proof of Theorem 7 by the Poincare way (continuation)

These transformation preserve already normalized $A^{(i)}(x), i > k$ and

$$A^{(k)}(x) \to A^{(k)}(x) + \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x)$$

The first statement of Theorem 7 now follows from the following claim.

Claim 8. One of the complement spaces to the image of the operator

$$\left(\Phi^{(k)}(x),\ S^{(k)}(x)\right) \to \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x), \quad S^{(k)}(x) \in \mathfrak{so}(n)$$

is the space of symmetric matrices
$A^{(k)}(x)$ such that $A^{(k)}(x) \cdot x \equiv 0$, $x = (x_1, ..., x_n)^{\mathrm{tr}}$.

Proof of Claim 8 by GR inner products

The simplest exercise. For the inner product $< ax^i, bx^i >= i!ab$ in the space of homogeneous degree i polynomials of one variable one has

$$p(x) = ax^k, \; q(x) = bx^{k-1} \implies < p'(x), q(x) >=< p(x), q(x)x >$$

and consequently $\Big(p(x) \to p'(x)\Big)^*$ is the operator $q(x) \to q(x)x$.

The inner product in this exercise can be easily and naturally extended to homogeneous vector functions and matrices with homogeneous entries. I call these inner products GR inner products because they were introduced and used for many nice theorems on normal forms for vector fields by my first teacher Genrich Ruvimovich Belitskii.

Those who did the exercise above will easily give an explicit formula for GR inner products and will prove that with respect to them $\Big(\big(\Phi^{(k)}(x)\big)'\Big)^*$ is the operator $A^{(k)}(x) \to A^{(k)}(x) \cdot x, x = (x_1, ..., x_n)^{\mathrm{tr}}$ which implies Claim 8.

Proof of the remaining part of Theorem 7

In the normalization way used the transformations that differ from the identity by $(i), i \geq 1$ and did not use transformations of degree (0) that preserve I, i.e. we did not use the action of $O(n)$.

The constructed normal form is "good" meaning that it is preserved by the action of $O(n)$: it is easy to see that $O(n)$ acts as follows $A(x) \to R^{\mathrm{tr}} A(Rx)R,\ R \in O(n)$ which preserves the equations $A(x) = A^{\mathrm{tr}}(x)$ and $A(x) \cdot x \equiv 0, x = (x_1, ..., x_n)^{\mathrm{tr}}$.

The fact that the constructed normal form is exact follows (modulo simple general claims related to the Poincare way) form the following statement that can be easily checked:

Claim. For $k \geq 1$ the kernel of the operator

$$\left(\Phi^{(k)}(x),\ S^{(k)}(x)\right) \to \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x),\quad S^{(k)}(x) \in \mathfrak{so}(n)$$

is trivial.

The characteristic matrix and the Riemannian curvature tensor at a fixed point

For non-flat Riemannian metric germ at $0 \in \mathbb{R}^5$ we have

$$M = I + A^{(k)}(x) + A^{(k+1)}(x) + \cdots$$

and the equations $A^{(k)}(x) = \left(A^{(k)}(x)\right)^{\mathrm{tr}}$ and $A^{(k)}(x) \cdot x \equiv 0$, $x = (x_1, ..., x_n)^{\mathrm{tr}}$ imply $k \geq 2$ (for $k = 1$ the matrix $A^{(1)}(x)$ is zero).

The matrix $A^{(k)}(x)$ in the normal form might be called the characteristic matrix.

If $k = 2$ the characteristic matrix can be identified with the curvature tensor at $0 \in \mathbb{R}^5$.

If $k > 2$ then the curvature tensor vanishes at $0 \in \mathbb{R}^5$ which does not mean, of course, that it vanishes at other points near 0.

Straightforward generalizations

For germs on $\mathbb{R}^n$ of conformal structures (metrics up to multiplication by a non-vanishing function) we use a change of coordinates and a change of orthonormal basis as above, and we multiply $M(x)$ by a nin-vanishing function H, so that the transformation of $A^{(k)}(x)$ is as follows:

$$A^{(k)}(x) \to A^{(k)}(x) + \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x) + H^{(k)}(x)I$$

and exactly in the same way we obtain the normal form $M = I + A(x)$,

$$A^{\mathrm{tr}}(x) = A(x),\ A(x) \cdot x \equiv 0,\ \mathrm{trace}\, A(x) \equiv 0$$

If $n \geq 4$ the homogeneous decomposition of $A(x)$ starts, as well as for metrics, with $A^{(2)}$ and $A^{(2)}$ can be identified with the Weyl tensor; if $n = 3$ it starts with $A^{(3)}$ and $A^{(3)}$ can be identified with the Cotton tensor. If $n = 2$ the equations above imply $A(x) \equiv 0$.

Generalization to Einstein metric and Einstein conformal structure are also straightforward. Maybe they can be used for some problems posed in Pawel Nurowski's lectures?

Is it possible to do a similar work for (2,3,5) distributions?

The first answer is no because unlike Riemannian metrics or conformal structures we cannot describe the class of all (2,3,5) distributions in the form $(0) + (1) + (2) + \cdots$ with a fixed (0) and arbitrary $(1), (2), \ldots$.

The second answer is yes due to a very good quasi.

Quasi-homogeneity

Let $E = x_1\partial_{x_1} + \cdots + x_n\partial_{x_n}$

Definition. A function $f(x)$ is called homogeneous of degree d if $E(f) = df$. A vector field V is called homogeneous of degree d if $[E, V] = dV$.

Example. The vector field $x_1^{r_1} \cdot \cdots \cdot x_n^{r_n}\partial_{x_k}$ is homogeneous of degree $r_1 + \cdots + r_n - 1$.

Let now $E_\lambda = \lambda x_1\partial_{x_1} + \cdots + \lambda_n x_n\partial_{x_n}$

Definition. Replace E by E_λ in the definition above. A function $f(x)$ or a vector field V in that definition is called quasi-homogeneous of degree d with respect to the weights $\lambda_1, ..., \lambda_n$.

Example. With respect to the weights $1, 1, 2, 3, 3$ there are non-zero vector field on $\mathbb{R}^5$ of any degree $d \geq -3$.

Andre Bellaiche theorem

Theorem (A. Bellaiche). Let $V_1, ..., V_k$ be any bracket generating tuple of vector field germs at $0 \in \mathbb{R}^n$. In suitable local coordinates

$$V_i = N_i^{[-1]} + N_i^{[0]} + N_i^{[1]} + \cdots$$

where $[i]$ denotes the quasi-homogeneous part with respect to the natural weights defined by the growth vector of the tuple.

Example of the natural weights. Let V_1, V_2 be vector field germs at $0 \in \mathbb{R}^2$ such that $V_1(0) \neq 0$ and V_2 and all repeated brackets of V_1, V_2 of lenghth ≤ 99 are at 0 proportional to $V_1(0)$, and one of the brackets of lenghth 100 is not. Then the growth vector is $(1, 1, .., 1, 2)$ with 1 repeated 100 times, and the natital weights are $(1, 101)$.

Starting point

In what $n = 5$ and follows $[i]$ denotes quasi-homogeneity of degree i with respect to the weights 1,1,2,3,3 that are natural for (2,3,5) distributions.

The Bellaiche theorem implies that any (2,3,5) distribution germ D can be described by vector fields

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + [0] + [1] + \cdots , \quad \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \in [-1]$$

and the distribution described by the vector fields N_1, N_2 is
a flat (2,3,5) distribution.
It can be identified with the nilpotent approximation=symbol of D.

The linear operator (linearization of the pseudo-group action)

We have a pseudo-group that consists of local diffeos Φ (change of coordinates) and non-singilar 2×2 matrices $\widehat{H}(x)$ corresponding to the change of basis V_1, V_2 of a (2,3,5) distribution. It acts as follows:

$$(\Phi, \widehat{H}).\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \widehat{H}\begin{pmatrix} \Phi_* V_1 \\ \Phi_* V_2 \end{pmatrix}$$

The Lie algebra of this pseudo-group is $(Z, H(x))$ where Z is a vector field germ and $H(x)$ is any 2×2 matrix. We need the linearization at id of the map

$$(\Phi, \widehat{H}).\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \widehat{H}\begin{pmatrix} \Phi_* N_1 \\ \Phi_* N_2 \end{pmatrix}$$

It is the linear operator

$$L_N : \ (Z, H) \to \begin{pmatrix} [Z, N_1] \\ [Z, N_2] \end{pmatrix} + H\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

Working with the operator L_N

In the same way as for Riemannian metrics we obtain a normal form

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[0]} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[1]} + \cdots$$

where $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[i]}$ belongs to any fixed beforehand complementary space to the image of the operator L_N restricted to $[i+1]$.

End of the first lecture

Some of the comments of the listeners are in the next page

Some of the comments of the listeners to the first lecture

Bronek Jakubczyk: The Poincare way approach to classification of Riemannian metrics is contained in the book:
Jerry Kijowski, Geometria rozniczkowa jako narzedzie nauk przyrodniczych
Bronek presented me this book, sorry that I forgot to mention it

Borya Kruglikov: Theorem 4, that I call the base theorem for non-flat (2,3,5) distributions, has a huge generalization: Theorem 1.3 in the paper
B.Krugkikov, D.The, Jet-determination of symmetries of parabolic geometries, Math. Ann., 2018. I should discuss with Borya or Dennis: one of the assumption of their theorem is that the geometry is torsion free. How to prove that the geometry of (2,3,5) distributions is torsion free? It is not excluded that proving Th. 1.4 I am proving, implicitly, namely that, in another language.

Igor Zelenko: I mentioned Igor's fundamental form, constructed by abnormal curves, but did not say - sorry! that Igor has it now not only for (2,3,5) distributions (the case that it can be identified with the Cartan tensor, according to Igor's Phd Thesis) but for all $(2, n)$ bracket-generating distributions (maybe with a constant growth vector only? or with the max growth vector only? Jointly with Borya Doubrov? To disciss with Igor)

A way to obtain exact normal form for (2,3,5) distribution germs at $0 \in \mathbb{R}^5$ and Cartan tensor at 0

Take any pair $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ of vector fields on $\mathbb{R}^5$, in local coordinates $x_1, \dots x_5$ such that N_1, N_2 are quasi-homogeneous with respect to the weights $w(x_1) = w(x_2) = 1, w(x_3) = 2, w(x_4) = w(x_5) = 3$ and the vector fields $N_1, N_2, N_3 = [N_1, N_2], N_4 = [N_1, N_3], N_5 = [N_2, N_3]$ are linearly independent at 0.

Conceptually, the choice of N (=symbol) is irrelevant (all N are diffeomorphic) but the choice can be good or not good.

Examples: (a) $N_1 = \partial_{x_1},\ N_2 = \partial_{x_2} + x_1\partial_{x_3} + x_3\partial_{x_4} + x_1^2\partial_{x_5}$
(in Monge form $z' = (y'')^2$)

(b) $N_1 = \partial_{x_1} + x_2 U,\ N_2 = \partial_{x_2} - x_1 U,\ U = \partial_z + x_1\partial_{y_1} + x_2\partial_{y_2}$

For me (b) is good and (a) is not.

Consider the linear operators

$$L_N : \left(Z^{[i]}, H^{[i]}\right) \to [Z^{[i]}, N] + H^{[i]} N \in [i-1]$$

where Z is a vector field, H is a 2×2 matrix whose entries are functions, and $[i]$ denotes: quasi-homogeneous of degree i.

Take any complementary subspace $W^{[i-1]}$ to the image of $L_N^{[i]}$:

$$[i-1] = \mathrm{Image}\left(L_N^{[i]}\right) \oplus W^{[i-1]}$$

Definition. $W^{[i]}$, $i \geq 0$ is good if it respects the group of quasi-homogeneous degree 0 symmetries of N:

$$\mathrm{Sym}^{[0]}(N) = \exp\left(Z^{[0]}, H^{[0]}\right), \ \left(Z^{[0]}, H^{[0]}\right) \in \ker L_N^{[0]}$$

i.e. $\mathrm{Sym}^{[0]}(N).W^{[i]} \subseteq W^{[i]}$.

Do good complementary subspaces exist?

In most (but not all) problems I know (good complementary subspaces can be defined in the same way for all geometrix stuctures) they exist.
It does not depend on the choice of N, it depends on the problem only.
I do not know general theorems about that.
I do not know how deep/interesting is this question.

Whether or not the complementary subspaces $W^{[i]}$ are goog we have:

- Any (2,3,5) distribution germ is equivalen to

$$N + \sum_{i \geq 0} V^{[i]}, \ V^{[i]} \in W^{[i]}, \ i \geq 0 \tag{1}$$

- A (2,3,5) distrribution is flat (equivalent to N)

if and only if in this normal form $V^{[i]} = 0$ for any $i \geq 0$.

• If $\ker L_N^{[i]} = \{0\}$ for all $i \geq 1$ and if $W^{[i]}$ are good for all $i \geq 0$ then the normal form is exact modulo the action of $\mathrm{Sym}^{[0]}(N)$, i.e. if two equivalent distributions have normal form (1) then they can be brough one to the other by some $(\Phi, H) \in \mathrm{Sym}^{[0]}(N)$.

But the assumption $\ker L_N^{[i]} = \{0\}$ for all $i \geq 1$, which makes the classification problem rather simple, holds true for Riemannian metrics (see my first lecture) but not for most of other geometric structures (not for conformal structures and not for (2,3,5) distributions).

• Let $k \geq 0$ be the minimal integer such that $W^{[k]} \neq \{0\}$, i.e. $L_N^{[k+1]}$ is not onto. If $W^{[k]}$ is good (see the definition above) then modolo the action of $\mathrm{Sym}^{[0]}(N)$ it is an invariant of the distribution, i.e. if two equivalent distributions have normal form (1), with $V, \widetilde{V} \in W^{[k]}$ then $(\Phi, H).V = \widetilde{V}$ for some $(\Phi, H) \in \mathrm{Sym}^{[0]}(N)$.

I will say that it is "the first invariant".

Assume we know nothing about (2,3,5) distributions. What to expect?

Since all (2,3,5) distributions have the same (up to equivalence) symbol N, let us believe that the linear operators $L_N^{[i]}$ have max rank, for all i.

If so (and it is so) we can compute the dimensions of the kernel of $L_N^{[i]}$ and the dimensions of the complementary space to its image by computing

$$\Delta_i = \dim\left(\text{the target space of } L_N^{[i]}\right) - \dim\left(\text{the source space of } L_N^{[i]}\right)$$

which is a simple exercise in combinatorics; in 10-15 minutes one can calculate that for $-3 \leq i \leq 3$ the Δ_i is negative:

$$\Delta_{-3} = \Delta_3 = \Delta_{-1} = \Delta_1 = 2;\ \Delta_{-2} = \Delta_2 = -1; \Delta_0 = -4$$

and for $i \geq 4$ the Δ_i is positive: $\Delta_4 = 5$ and fastly growth for $i \geq 5$.

Therefore the symmetry algebra of N is the sum of the vector spaces

$$\mathrm{sym}^{[-3]} + \mathrm{sym}^{[-2]} + \mathrm{sym}^{[-1]} + \mathrm{sym}^{[0]} + \mathrm{sym}^{[1]} + \mathrm{sym}^{[2]} + \mathrm{sym}^{[3]}$$

of dimensions $2, 1, 2, 4, 2, 1, 2$.

Here, as above, $[i]$ denotes homogeneous of degree i, therefore $[\mathrm{sym}^{[i]}, \mathrm{sym}^{[j]}] \subseteq \mathrm{sym}^{[i+j]}$.

This is how a simple combinatorics leads to $\mathfrak{g}_2$.

Now, we have a normal form for the quasi-3-jet of a (2,3,5) distribution:

$$N + W^{[3]},\ W^{[3]} \text{ is a complimentary space to the image of } L_N^{[4]}$$

and since by combinatorics $\Delta_4 = 5$ we have (privided that it is true that $L_N^{[4]}$ has max rank) $\dim W^{[3]} = 5$.

Can we find a good $W^{[3]}$? Yes, it is simple, a good $W^{[3]}$ comes out almost "automatically" after few hours of work provided the choice of N is good (if not - we also can obtain a good $W^{[3]}$, but not "automatically, and it will be very involved, and few weeks instead of few hours).

The nilpotent approximation (symbol)

$$N_1 = \partial_{x_1} + x_2 U, \;\; N_2 = \partial_{x_2} - x_1 U, \;\; U = \partial_z + x_1 \partial_{y_1} + x_2 \partial_{y_2}$$

is good because for it it is quick to find a good $W^{[3]}$, and, what is more important, it has a very simple form

$$W^{[3]} = F\Big(x_2 U, \; -x_1 U\Big)$$
$$F = c_{4,0} x_1^4 + c_{3,1} x_1^3 x_2 + c_{2,2} x_1^2 x_2^2 + c_{1,3} x_1 x_2^4 + c_{0,4} x_2^4$$

Now, for my N we have

$$\mathrm{Sym}_N^{[0]} = (\Phi_T, H_T),\ \ T \in \mathfrak{gl}(2),\ \ H_T = T^{\mathrm{tr}},$$
$$\Phi_T:\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \ z \to \det T \cdot z,\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \det T \cdot T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

which brings $W^{[3]}$ with a fixed $F(x)$, $x = (x_1, x_2)$ to $F(Tx)$.

Therefore the first invariant in the classification of (2,3,5) distribution germs is a homogeneous degree 4 polynomial $F(x_1, x_2)$ defined up to a linear non-singular change of x_1, x_2.

This is how the Poincare way plus few hours simple calculations give the Cartan tensor at $0 \in \mathbb{R}^5$.

The Cartan tensor at 0 is what I called above the first invariant.

It is a complete invariant in the classification of quasi-3-jets of (2,3,5) distributions.

Obtaining good $W^{[i]}$ for all $i \geq 3$ requires more work,
but not much more if one uses GR inner products, see the first lecture.

The following claim generelizes the previous claim from $i = 3$ to any $i \geq 3$:

- For the symbol

$$N_1 = \partial_{x_1} + x_2 U, \;\; N_2 = \partial_{x_2} - x_1 U, \;\; U = \partial_z + x_1 \partial_{y_1} + x_2 \partial_{y_2}$$

the subspaces

$$W^{[i]} = F^{[i+1]}\Big(x_2 U, \; -x_1 U\Big)$$

with $F^{[i+1]}$ in the ideal $\mathbf{I}$ given below are good for any $i \geq 3$.

$\mathbf{I}$ is the ideal in the space of functions generated by:
$x_1^{\alpha} x_2^{\beta}, \; \alpha + \beta = 4; \; x_1^{\alpha} x_2^{\beta} z, \; \alpha + \beta = 3; \; x_1^{\alpha} x_2^{\beta} z^2, \; \alpha + \beta = 2$ and
$x_1(x_1 y_2 - x_2 y_1), \; x_2(x_1 y_2 - x_2 y_1), \; (x_1 y_2 - x_2 y_1)^2.$

NORMAL FORM

In this way we obtain the following normal form that serves for all (2,3,5) distribution germs:

$$V_1 = \partial_{x_1} + x_2(1+F)U, \;\; V_2 = \partial_{x_2} - x_1(1+F)U, \;\; U = \partial_z + x_1\partial_{y_1} + x_2\partial_{y_2}$$
$$F = F(x_1, x_2, z, y_1, y_2) \in \text{the ideal } \mathbf{I} \text{ generated by}$$
$$x_1^\alpha x_2^\beta, \;\; \alpha+\beta = 4; \;\; x_1^\alpha x_2^\beta z, \;\; \alpha+\beta = 3, \quad x_1^\alpha x_2^\beta z^2, \;\; \alpha+\beta = 2,$$
$$x_1(x_1y_2 - x_2y_1), \;\; x_2(x_1y_2 - x_2y_1), \;\; (x_1y_2 - x_2y_1)^2.$$

The distribution is flat if and only if $F \equiv 0$.

• If it is not flat then $F = F^{[m]} + F^{[m+1]} + \cdots, \;\; m \geq 4, \;\; F^{[m]} \neq 0$
In this case $F^{[m]}$ up to the linear transformation
$F^{[m]} \to F^{[m]}(Tx, \det Tz, \; \det T \cdot Ty), \; x = (x_1, x_2), y = (y_1, y_2), T \in GL(2)$
that I call characteristic polynomial is an invariant.

• if $m = 4$ the characteristic polynomial is the Cartan tensor at 0.

The main Cartan theorem from the normal form

Let us prove that if the Cartan tensor vanishes not only at 0 but at any point near 0 then in the bormal form $F \equiv 0$ and consequently the distribution is flat.

Assume by contradiction that $F \not\equiv 0$. Then $F = F^{[m]} + F^{[m+1]} + \cdots$ where $m \geq 5$, $F^{[m]} \neq 0$ and $F^{[m]} \in$**I**. Here **I** is the ideal given in the previous page.

Let $m = 5$. In this case $F^{[5]} = P^{(5)}(x_1, x_2) + zP^{(3)}(x_1, x_2)$ where $P^{(5)}(x_1, x_2)$,$P^{(3)}(x_1, x_2)$ are homogeneus polynomials of degrees 5, 3.

Let us compute the Cartan tensor at the point $P_\epsilon = (\epsilon, 0, 0, 0, 0)$ up to $o(\epsilon)$. It is very easy to see that the quasi-3-jet the distribution at P_ϵ is equivalent to my normal form with $\widehat{F}^{[4]} = \frac{\partial F^{[5]}}{\partial x_1} + o(\epsilon)$.

If $\widehat{F}^{[4]} \in \mathbf{I}$ then $\widehat{F}^{[4]}$ is the Cartan tensor at P_ϵ up to $o(\epsilon)$ and then $\widehat{F}^{[4]} = 0$, i.e. $\frac{\partial F^{[5]}}{\partial x_1} = 0$. In the same way, taking the point $P_\epsilon = (0, \epsilon, 0, 0, 0)$, we obtain $\frac{\partial F^{[5]}}{\partial x_2} = 0$. Then $F^{[5]} = 0$, contradiction.

The life is a bit more complicated: it is not true that $\widehat{F}^{[4]} \in \mathbf{I}$, because $z\frac{\partial F^{(3)}}{\partial x_1} \notin \mathbf{I}$.

Within quasi-homogeneous degree 4 functions we have a certain equivalence modulo $\mathbf{I}$ and I have to find a function in $\mathbf{I}$ that is equivalent to $z\frac{\partial F^{(3)}}{\partial x_1} \notin \mathbf{I}$.

Certainly I know everything about the equivalence modulo $\mathbf{I}$ (it is very simple), in particular:

- for $\alpha + \beta = 2$ one has $x_1^\alpha x_2^\beta z \sim 0 \mod \mathbf{I}$

which is both good and bad.

It is good because the argumet above gives us $P^{(5)}(x_1, x_2) = 0$.

It is bad because the argument above does not give us
any information about $P^{(3)}(x_1, x_2) = 0$.

But the argument above is too straightforward. Why we computed Cartan tensor modulo $o(\epsilon)$ at the points $(\epsilon, 0, 0, 0, 0), (0, \epsilon, 0, 0, 0)$? Because the calculation is immediate. But there are other points. And we have two ways: either to work with some other points or to work modulo $o(\epsilon^2)$ rather than $o(\epsilon)$. The latter is doable but more involved.

What is worth to do is to use the 2-dimensional $\mathfrak{g}_2^{[-1]} = \ker L_N^{[-1]}$
and to move 0 to the points $A_{1,\epsilon}$ and $A_{2,\epsilon}$ by the ϵ-time flows
of vector fields $\xi_1^{[-1]}, \xi_2^{[-1]}$ that span $\mathfrak{g}_2^{[-1]}$.

I calculated that at A_ϵ the quasi-3-jet of the distribution that has my normal form with $F^{[5]} = zP^{(3)}(x_1, x_2)$ is my normal form with $\widehat{F}^{[4]} = x_2 P^{(3)} \in \mathbf{I}$ and we obtain $P^{(3)} = 0$ as required.

I have proved, up to some calculations, that in my normal form $F^{[5]} = 0$.

The proofs that $F^{[6]} = F^{[7]} = 0$ are similar, even a bit easier.

The proof that $F^{[\geq 8]} = 0$ is also based on the same ideas (and we have the same argument for any $m \geq 8$).

The total proof with <u>all</u> details takes 5-6 pages.

Proof of the base theorem for non-flat distributions

Recall from the first lecture

Theorem 4 The isotropy subalgebra of the symmetry algebra any non-flat (2,3,5) distribution germ at $0 \in \mathbb{R}^5$ does not contain vector fields with the zero linear approximation at 0.

The constructed normal form reduces this statement to the claims that certain three linear operators have the trivial kernel. These operators have, as parameters, $F^{[m]} \neq 0$ in my normal form, which makes the proof rather long (but simple), around 5 pages.

But for the cae $m = 4$ - the case that Cartan tensor does not vanish at 0, the proof is 1/2 of a page.

Classification of isotropy subalgebras

Given a 2×2 matrix T define the following linear vector field:

$$V_T = \left\langle Tx, \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} \right\rangle + \mathrm{trace}\, T \cdot z\partial_z + \left\langle (T + \mathrm{trace}\, T)y, \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \end{pmatrix} \right\rangle$$

Here $x = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix}, y = \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \end{pmatrix}$ and $< ., >$ is the standard inner product.

Using the base theorem (see the previous page) which I proved from the constructed normal form, and the normal form itself, it takes 1-2 pages to prove the following theorem.

Theorem

(M. Zh). The isotropy subalgebra of the symmetry algebra of any non-flat (2,3,5) distribution germ is diffeomorphic to one of the following:

1. A 1-dim algebra spanned by V_T with

a. any fixed traceless T

b. $T = \mathrm{diag}(-\frac{p}{q}, 1)$ where p and q are integers, $1 \leq p < q$

2. A 2-dim non-Abelian algebra spanned by V_{T_1}, V_{T_2} where

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

3. The $\mathfrak{sl}(2)$ spanned by $V_{T_1}, V_{T_2}, V_{T_3}$ where T_1, T_2, T_3 is a basis of $\mathfrak{sl}(2)$.

If Cartan tensor does not vanish at 0 then the realizable cases are as follows: case 2; case 1.b with $p = 1, q = 3$, and

1.a. with $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or or $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$