Normal forms and symmetries for (2,3,5) and (3,5) distributions

111 years after E.Cartan's 5 variables paper

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Do we need Cartan tensor?

Yes. By many reasons, probably the most important is as follows:

without Cartan tensor there is no way to answer the following important question: if a given (2,3,5) distribution is flat or not?

I will give one of equivalent definitions of a flat (2,3,5) distributins using one of my tools: characteristic matrix of an endowed 5-dim Lie algebra.

Endowed 5-dim algebras and their characteristic matrices

By definition, an endowed 5-dim algebra is a 5-dim Lie algebra $\mathcal{A}$ endowed with a generating 2-plane $\mathcal{P} = span(a_1, a_2)$ meaning that the vectors

$$a_1, a_2,\ a_3 = [a_1, a_2],\ a_4 = [a_1, a_3],\ a_5 = [a_2, a_3]$$

are linearly independent.

Any endowed 5-dim algebra can be described by the following 3×5 matrix $(r_{i,j})$ that I call a characteristic matrix:

$$\begin{array}{ccc} [a_1, a_4] & = & r_{1,1}a_1 + \cdots + r_{1,5}a_5 \\ [a_1, a_5] = [a_2, a_4] & = & r_{2,1}a_1 + \cdots + r_{2,5}a_5 \\ [a_2, a_5] = & = & r_{3,1}a_1 + \cdots + r_{3,5}a_5 \end{array}$$

All other structure equations follow from the Jacobi identity.

Homogeneous (2,3,5) distribution induced by an andowed 5-dim algebra

Let $(\mathcal{A}, \mathcal{P})$ be an endowed 5-dim algebra.

Let (G, id) be a nbhd of id of the Lie group G of $\mathcal{A}$,
so that $\mathcal{P}$ is a 2-plane in $T_{\mathrm{id}}G$.

Push $\mathcal{P}$ to (G, id) by the flows of left invariant vector fields on G.

We obtain a local (germ at id) (2,3,5) distribution on (G, id).

It is homogeneous becuse its symmetry algebra either is $\mathcal{A}$ or contains $\mathcal{A}$.

The most known classical theorem

Theorem 1. (Cartan, Tanaka). Let D be the germ at $0 \in \mathbb{R}^5$ of a (2,3,5) distribution. The following are equivalent:

1. D is equivalent to a distribution induced by the endowed 5-dim algebra with the zero characteristic matrix.

2. The symmetry algebra of D is $\mathfrak{g}_2$

3. The Cartan tensor of D vanishes at any point near $0 \in \mathbb{R}^5$

Definition. If the conditions of Theorem 1. hold, the (2,3,5) distribution is called flat.

Question. What can be said about the (2,3,5) distribution induced by the endowed 5-dim algebra with non-zero characteristic matrix?
a. Can it be flat? **b**. Its symmetry algebra?

I will give complete answers. Quick answer to **a**: yes

Less known very important theorem

Theorem 2. (Cartan) If D is not flat, its symmetry algebra has dim ≤ 7.

My proof of Theorem 2 is as follows.

Theorem 3. (M. Zh) Let D be the germ at $0 \in \mathbb{R}^5$ of a non-flat (2,3,5) distribution. The isotropy subalgebra at 0 of the symmetry algebra of D is, as an abstract Lie algebra, one of the following:

a. 1-dim; **b**. 2-dim non-Abelian; **c.** $\mathfrak{sl}(2)$.

Each of these cases is realizable.
In case **c**, the Cartan tensor of D at 0 is equal to 0.

Theorem 2. is a logical corollary of Theorems 3. and 1.

How I prove Theorem 3? My base theorem for non-flat distributions

My proof of Theorem 3, and of many other theorems, uses the following statement that for me plays the key role.

Theorem 4 (the base theorem for non-flat distributions (M. Zh.)
The isotropy subalgebra of the symmetry algebra any non-flat (2,3,5) distribution germ at $0 \in \mathbb{R}^5$ does not contain vector fields with the zero linear approximation at 0.

Theorem 5 (logical corollary of Theorem 4)
The isotropy subalgebra $\mathcal{I}$ of the symmetry algebra any non-flat (2,3,5) distribution germ D at $0 \in \mathbb{R}^5$ is isomorphic (as an abstract Lie alegbra) to the Lie algebra $j_0^1\mathcal{I}$ (the linear approximation of $\mathcal{I}$).

Theorem 6 (M.Zh). Let D and $\mathcal{I}$ be as in Corollary 5. Then $j_0^1\mathcal{I}$ is isomorphic to one of **a, b, c** in Theorem 3.

Theorem 3. is a logical corollary of Theorems 5 and 6

My approach is based on:

In order to prove Cartan-Tanaka classical Theorem 1, my base Theorem 4, and Theorem 6 (Theorems 2,3,5 are, as I explained, their corollaries), and many other statements that will be formulated in this mini-course, I use the following, for (2,3,5) distributions D:

A. A theorem explaining the Cartan tensor of D at a fixed point

B. A theorem on characteristic polynomial of D :
a generalization of Cartan tensor at a fixed point

C. A theorem on almost exact normal form for all possible D

Theorems **A,B,C** for local Riemannian metrics

Given the germ at $0 \in \mathbb{R}^n$ of a Riemannian metric on $\mathbb{R}^n$,
take an orthonormal basis $V_1, ..., V_n$ of vector fields,
and a local coordinate system $x = (x_1, ..., x_n)$.

Describe the metric by an $n \times n$ matrix $M(x)$
with the functional entries $M_{i,j}(x) = V_j(x_i)$.

Theorem 7. (Gauss lemma; M. Zh).
1. For a suitable orthonormal basis and suitable local coordinates one has

$$M(x) = I + A(x), \ A^{\mathrm{tr}}(x) = A(x), \ A(x) \cdot x \equiv 0, \ x = (x_1, \dots, x_n)^{\mathrm{tr}}$$

2.This normal form is exact up to transformations
$A(x) \to R^{\mathrm{tr}} A(Rx) R$ where R is a constant orthogonal matrix.

3. The metric is flat if and only if $A(x) \equiv 0$.

Remark. The local coordinates in this normal form are normal (geodesic).

Proof of Theorem 7 by the Poincare way

What I call the Poincare way is the way used in the whole local analysis and physics (but unfortunately so far not by many people in local differential geometry), especially after the resonance normal form obtained by Poincare for vector fields, that is used and developed in thousands of works.

We have $M(x) = I + A^{(1)}(x) + A^{(2)}(x) + \cdots$ where (i) denotes the homogeneous part of degree i. Assume we have normalized $A^{(i)}(x)$ for $i < k$ and want to normalize $A^{(k)}(x)$. For that, let us change the local coordinates and the basis as follows:

$$x \to x + \Phi^{(k)}(x),$$
$$(V_1, ..., V_n)^{\mathrm{tr}} = \exp(S^{(k)}(x)) \cdot (V_1, ..., V_n)^{\mathrm{tr}}, \; S^{(k)}(x) \in \mathfrak{so}(n)$$

Proof of Theorem 7 by the Poincare way (continuation)

These transformation preserve already normalized $A^{(i)}(x), i > k$ and

$$A^{(k)}(x) \to A^{(k)}(x) + \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x)$$

The first statement of Theorem 7 now follows from the following claim.

Claim 8. One of the complement spaces to the image of the operator

$$\left(\Phi^{(k)}(x),\ S^{(k)}(x)\right) \to \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x), \quad S^{(k)}(x) \in \mathfrak{so}(n)$$

is the space of symmetric matrices
$A^{(k)}(x)$ such that $A^{(k)}(x) \cdot x \equiv 0$, $x = (x_1, ..., x_n)^{\mathrm{tr}}$.

Proof of Claim 8 by GR inner products

The simplest exercise. For the inner product $< ax^i, bx^i >= i!ab$ in the space of homogeneous degree i polynomials of one variable one has

$$p(x) = ax^k,\ q(x) = bx^{k-1} \implies < p'(x), q(x) >=< p(x), q(x)x >$$

and consequently $\Big(p(x) \to p'(x)\Big)^*$ is the operator $q(x) \to q(x)x$.

The inner product in this exercise can be easily and naturally extended to homogeneous vector functions and matrices with homogeneous entries. I call these inner products GR inner products because they were introduced and used for many nice theorems on normal forms for vector fields by my first teacher Genrich Ruvimovich Belitskii.

Those who did the exercise above will easily give an explicit formula for GR inner products and will prove that with respect to them $\Big(\big(\Phi^{(k)}(x)\big)'\Big)^*$ is the operator $A^{(k)}(x) \to A^{(k)}(x) \cdot x, x = (x_1, ..., x_n)^{\rm tr}$ which implies Claim 8.

Proof of the remaining part of Theorem 7

In the normalization way used the transformations that differ from the identity by $(i), i \geq 1$ and did not use transformations of degree (0) that preserve I, i.e. we did not use the action of $O(n)$.

The constructed normal form is "good" meaning that it is preserved by the action of $O(n)$: it is easy to see that $O(n)$ acts as follows $A(x) \to R^{\mathrm{tr}} A(Rx)R,\ R \in O(n)$ which preserves the equations $A(x) = A^{\mathrm{tr}}(x)$ and $A(x) \cdot x \equiv 0, x = (x_1, ..., x_n)^{\mathrm{tr}}$.

The fact that the constructed normal form is exact follows (modulo simple general claims related to the Poincare way) form the following statement that can be easily checked:

Claim. For $k \geq 1$ the kernel of the operator

$$\left(\Phi^{(k)}(x),\ S^{(k)}(x)\right) \to \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x),\ \ S^{(k)}(x) \in \mathfrak{so}(n)$$

is trivial.

The characteristic matrix and the Riemannian curvature tensor at a fixed point

For non-flat Riemannian metric germ at $0 \in \mathbb{R}^5$ we have

$$M = I + A^{(k)}(x) + A^{(k+1)}(x) + \cdots$$

and the equations $A^{(k)}(x) = \left(A^{(k)}(x)\right)^{\mathrm{tr}}$ and $A^{(k)}(x) \cdot x \equiv 0$, $x = (x_1, ..., x_n)^{\mathrm{tr}}$ imply $k \geq 2$ (for $k = 1$ the matrix $A^{(1)}(x)$ is zero).

The matrix $A^{(k)}(x)$ in the normal form might be called the characteristic matrix.

If $k = 2$ the characteristic matrix can be identified with the curvature tensor at $0 \in \mathbb{R}^5$.

If $k > 2$ then the curvature tensor vanishes at $0 \in \mathbb{R}^5$ which does not mean, of course, that it vanishes at other points near 0.

Straightforward generalizations

For germs on $\mathbb{R}^n$ of conformal structures (metrics up to multiplication by a non-vanishing function) we use a change of coordinates and a change of orthonormal basis as above, and we multiply $M(x)$ by a nin-vanishing function H, so that the transformation of $A^{(k)}(x)$ is as follows:

$$A^{(k)}(x) \to A^{(k)}(x) + \left(\Phi^{(k)}(x)\right)' + S^{(k)}(x) + H^{(k)}(x)I$$

and exactly in the same way we obtain the normal form $M = I + A(x)$,

$$A^{\mathrm{tr}}(x) = A(x),\ A(x) \cdot x \equiv 0,\ \mathrm{trace}\, A(x) \equiv 0$$

If $n \geq 4$ the homogeneous decomposition of $A(x)$ starts, as well as for metrics, with $A^{(2)}$ and $A^{(2)}$ can be identified with the Weyl tensor; if $n = 3$ it starts with $A^{(3)}$ and $A^{(3)}$ can be identified with the Cotton tensor. If $n = 2$ the equations above imply $A(x) \equiv 0$.

Generalization to Einstein metric and Einstein conformal structure are also straightforward. Maybe they can be used for some problems posed in Pawel Nurowski's lectures?

Is it possible to do a similar work for (2,3,5) distributions?

The first answer is no because unlike Riemannian metrics or conformal structures we cannot describe the class of all (2,3,5) distributions in the form $(0) + (1) + (2) + \cdots$ with a fixed (0) and arbitrary $(1), (2), \ldots$.

The second answer is yes due to a very good quasi.

Quasi-homogeneity

Let $E = x_1\partial_{x_1} + \cdots + x_n\partial_{x_n}$

Definition. A function $f(x)$ is called homogeneous of degree d if $E(f) = df$. A vector field V is called homogeneous of degree d if $[E, V] = dV$.

Example. The vector field $x_1^{r_1} \cdot \cdots \cdot x_n^{r_n}\partial_{x_k}$ is homogeneous of degree $r_1 + \cdots + r_n - 1$.

Let now $E_\lambda = \lambda x_1\partial_{x_1} + \cdots + \lambda_n x_n\partial_{x_n}$

Definition. Replace E by E_λ in the definition above. A function $f(x)$ or a vector field V in that definition is called quasi-homogeneous of degree d with respect to the weights $\lambda_1, ..., \lambda_n$.

Example. With respect to the weights $1, 1, 2, 3, 3$ there are non-zero vector field on $\mathbb{R}^5$ of any degree $d \geq -3$.

Andre Bellaiche theorem

Theorem (A. Bellaiche). Let $V_1, ..., V_k$ be any bracket generating tuple of vector field germs at $0 \in \mathbb{R}^n$. In suitable local coordinates

$$V_i = N_i^{[-1]} + N_i^{[0]} + N_i^{[1]} + \cdots$$

where $[i]$ denotes the quasi-homogeneous part with respect to the natural weights defined by the growth vector of the tuple.

Example of the natural weights. Let V_1, V_2 be vector field germs at $0 \in \mathbb{R}^2$ such that $V_1(0) \neq 0$ and V_2 and all repeated brackets of V_1, V_2 of lenghth ≤ 99 are at 0 proportional to $V_1(0)$, and one of the brackets of lenghth 100 is not. Then the growth vector is $(1, 1, .., 1, 2)$ with 1 repeated 100 times, and the natital weights are $(1, 101)$.

Starting point

In what $n = 5$ and follows $[i]$ denotes quasi-homogeneity of degree i with respect to the weights 1,1,2,3,3 that are natural for (2,3,5) distributions.

The Bellaiche theorem implies that any (2,3,5) distribution germ D can be described by vector fields

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + [0] + [1] + \cdots, \quad \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \in [-1]$$

and the distribution described by the vector fields N_1, N_2 is
a flat (2,3,5) distribution.
It can be identified with the nilpotent approximation=symbol of D.

The linear operator (linearization of the pseudo-group action)

We have a pseudo-group that consists of local diffeos Φ (change of coordinates) and non-singilar 2×2 matrices $\widehat{H}(x)$ corresponding to the change of basis V_1, V_2 of a (2,3,5) distribution. It acts as follows:

$$(\Phi, \widehat{H}). \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \widehat{H} \begin{pmatrix} \Phi_* V_1 \\ \Phi_* V_2 \end{pmatrix}$$

The Lie algebra of this pseudo-group is $(Z, H(x))$ where Z is a vector field germ and $H(x)$ is any 2×2 matrix. We need the linearization at id of the map

$$(\Phi, \widehat{H}). \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \widehat{H} \begin{pmatrix} \Phi_* N_1 \\ \Phi_* N_2 \end{pmatrix}$$

It is the linear operator

$$L_N : \ (Z, H) \to \begin{pmatrix} [Z, N_1] \\ [Z, N_2] \end{pmatrix} + H \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

Working with the operator L_N

In the same way as for Riemannian metrics we obtain a normal form

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[0]} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[1]} + \cdots$$

where $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{[i]}$ belongs to any fixed beforehand complementary space to the image of the operator L_N restricted to $[i+1]$.

End of the first lecture

Some of the comments of the listeners are in the next page

Some of the comments of the listeners to the first lecture

Bronek Jakubczyk: The Poincare way approach to classification of Riemannian metrics is contained in the book:
Jerry Kijowski, Geometria rozniczkowa jako narzedzie nauk przyrodniczych
Bronek presented me this book, sorry that I forgot to mention it

Borya Kruglikov: Theorem 4, that I call the base theorem for non-flat (2,3,5) distributions, has a huge generalization: Theorem 1.3 in the paper
B.Krugkikov, D.The, Jet-determination of symmetries of parabolic geometries, Math. Ann., 2018. I should discuss with Borya or Dennis: one of the assumption of their theorem is that the geometry is torsion free. How to prove that the geometry of (2,3,5) distributions is torsion free? It is not excluded that proving Th. 1.4 I am proving, implicitly, namely that, in another language.

Igor Zelenko: I mentioned Igor's fundamental form, constructed by abnormal curves, but did not say - sorry! that Igor has it now not only for (2,3,5) distributions (the case that it can be identified with the Cartan tensor, according to Igor's Phd Thesis) but for all $(2, n)$ bracket-generating distributions (maybe with a constant growth vector only? or with the max growth vector only? Jointly with Borya Doubrov? To disciss with Igor)