On Geometry of 2-nondegenerate, Hypersurface-type Cauchy–Riemann Structures Encoded by Dynamical Legendrian Contact Structures

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A motivating question

- What local invariants distinguish real hypersurfaces in $\mathbb{C}^n$?
- (1933) Cartan answered this question for $\mathbb{C}^2$ by constructing canonical absolute parallelisms
- To what extent can Cartan's solution to the local equivalence problem be extended to higher dimensions?

Extensions to Higher Dimensions

Tanaka and Chern–Moser generalized Cartan's solution to the local equivalence problem for hypersurfaces of abitrary dimension that are *Levi-nondegenerate*.[1]

- Tanaka applies an algebraic method of *Tanaka prolongation*.
- Chern and Moser applied *Cartan's method of equivalence* and a *method of normal forms* respectively.
- Today Levi-nondegerate CR geometry is well understood within the general framework of parabolic geometry.

* These methods require the Levi-nondegeneracy assumption.

[1] reference: Tanaka [8], Chern and Moser [1]

Maximally Symmetric Levi-nondegenerate CR Structures

Maximally symmetric Levi-nondegenerate CR manifolds with a given Levi form include

- spheres in $\mathbb{C}^{n+1}$
  – algebra of infinitesimal symmetries is $\mathfrak{su}(n+1,1)$
- and more generally, Hermitian quadrics,

$$\left\{ (z_0,\ldots,z_{n+1}) \in \mathbb{P}\left(\mathbb{C}^{n+2}\right) \,\middle|\, \sum_{j=1}^{n} \epsilon_j |z_j|^2 = i z_0 \overline{z_{n+1}} - i\overline{z_0} z_{n+1} \right\}$$

for some $\epsilon_1,\ldots,\epsilon_n = \pm 1$.
– algebra of infinitesimal symmetries is $\mathfrak{su}(p+1,q+1)$

Beyond Nondegeneracy

Comparatively little is known about Levi-degenerate hypersurfaces in $\mathbb{C}^n$.

- Following Levi-nondegenerate structures, 2-*nondegenerate* structures form the next natural class to address.
- Absolute parallelisms were constructed only recently for all 5-dimensional CR manifolds.[2]
- Absolute parallelisms were constructed recently for a special class of CR manifolds whose maximally symmetric models have semisimple symmetry groups.[3]
- Absolute parallelisms were constructed recently for the class of 2-nondegenerate CR structures having a *regular CR symbol*.[4]

New methods are required to treat the remaining Levi-degenerate structures!

[2]reference: Isaev and Zaitsev [3], Medori and Spiro [4], Merker and Pocchiola [5]
[3]reference: Gregorovič [2]
[4]reference: Porter and Zelenko [6]

Hypersurface-type CR Structures

Let M be a real hypersurface of $\mathbb{C}^{n+1}$ with odd dimension $2n+1$

$D = TM \cap iTM$ denotes the maximal complex subbundled of TM

$J : D \to D$ denotes multiplication by i

$H \subset \mathbb{C} \otimes_{\mathbb{R}} TM$ denotes the i-eigenspace of J

Definition

(M, H) is a Cauchy–Riemann (CR) manifold of hypersurface-type

Intrinsic CR Structure Definition

Alternatively, a CR structure H can be defined as a complex vector bundle in $\mathbb{C} \otimes_{\mathbb{R}} TM$ satisfying

- $\text{rank}_{\mathbb{C}}(H) = \frac{\dim(M)-1}{2} = n$ . . . . (hypersurface-type)
- $[H, H] \subset H$ . . . . (integrable)
- $H \cap \overline{H} = 0$ . . . . (totally real)

The Levi Form

The *Levi form* $\mathcal{L}$ is a field of Hermitian forms given by

$$\mathcal{L}(X_p, Y_p) := \frac{i}{2}\left[X, \overline{Y}\right]_p \pmod{H_p \oplus \overline{H}_p} \quad \forall X, Y \in \Gamma(H)$$

taking values in $\mathbb{C} \otimes_{\mathbb{R}} T_pM / \left(H_p \oplus \overline{H}_p\right) \cong \mathbb{C}$.

Let K denote the *Levi kernel*, i.e., the kernel of $\mathcal{L}$.

2-nondegeneracy

For $v \in K_p$ we define the antilinear operator $\mathrm{ad}_v : H_p/K_p \to H_p/K_p$ by taking $V \in \Gamma(K)$ such that $V_p = v$ and setting

$$\mathrm{ad}_v(X_p + K_p) := [V, \overline{X}]_p \mod K_p \oplus \overline{H}_p \qquad \forall X \in \Gamma(H).$$

- H is *1-nondegenerate* (i.e., Levi-nondegenerate) if $K = 0$.
- H is *2-nondegenerate* if $K \neq 0$ and $\mathrm{ad}_v \neq 0$ for all $v \in K \setminus \{0\}$.

Heisenberg Algebras in 2-nondegenerate Structures

For $p \in M$, set

$$\mathfrak{g}_{-2}(p) := \mathfrak{g}_{-2,0}(p) := \mathbb{C}T_pM/(H_p \oplus \overline{H}_p), \quad \mathfrak{g}_{-1,1}(p) := H_p/K_p,$$

and

$$\mathfrak{g}_{-1,-1}(p) := \overline{H}_p/\overline{K}_p.$$

If H is 2-nondegenerate then, with $\mathfrak{g}_{-1}(p) := \mathfrak{g}_{-1,1}(p) \oplus \mathfrak{g}_{-1,-1}(p)$,

$$\mathfrak{g}_{-}(p) := \mathfrak{g}_{-2}(p) \oplus \mathfrak{g}_{-1}(p)$$

has the structure of a Heisenberg algebra with nontrivial Lie brackets given by

$$[v, \overline{w}] := i\mathcal{L}(v, w) \qquad \forall\, v, w \in \mathfrak{g}_{-1,1}(p).$$

Preliminaries: Derivations of the Heisenberg Algebra

For $v \in K_p$, the map ad_v determines an element

$$\widetilde{\mathrm{ad}_v} \in \mathfrak{csp}(\mathfrak{g}_{-1}(p)) \cong \mathfrak{der}(\mathfrak{g}_{-}(p))$$

defined by

$$\widetilde{\mathrm{ad}_v}(x) := \begin{cases} 0 & \text{if } x \in H_p/K_p \\ \mathrm{ad}_v(\overline{x}) & \text{if } x \in \overline{H}_p/\overline{K}_p. \end{cases}$$

We define

$$\mathfrak{g}_{0,2}(p) := \left\{ \widetilde{\mathrm{ad}_v} \in \mathfrak{der}(\mathfrak{g}_{-}(p)) \;\middle|\; v \in K_p \right\}$$

and define

$$\mathfrak{g}_{0,-2}(p) := \overline{\mathfrak{g}_{0,2}(p)}.$$

CR Symbols of 2-nondegenerate Structures

The *CR symbol* of a 2-nondegenerate, hypersurface-type structure H at $p \in M$, introduced in [6, Porter and Zelenko], is the space

$$\mathfrak{g}^0(p) := \mathfrak{g}_-(p) \oplus \mathfrak{g}_{0,0}(p) \oplus \mathfrak{g}_{0,-2}(p) \oplus \mathfrak{g}_{0,2}(p)$$

together with the involution induced on $\mathfrak{g}_-(p)$ induced by conjugation on $\mathbb{C}T_pM$, where

$$\mathfrak{g}_{0,0}(p) := \left\{ v \in \mathfrak{der}\big(\mathfrak{g}_-(p)\big) \;\middle|\; [v, \mathfrak{g}_{i,j}(p)] \subset \mathfrak{g}_{i,j}(p)\, \forall (i,j) \right\}.$$

In general $[\mathfrak{g}_{0,2}, \mathfrak{g}_{0,-2}]$ is not a subspace of $\mathfrak{g}^0$.
The CR symbol $\mathfrak{g}^0(p)$ is *regular* if it is a Lie subalgebra of $\mathfrak{g}_-(p) \rtimes \mathfrak{der}\big(\mathfrak{g}_-(p)\big)$.

Parallelisms for Regular CR Symbols

Theorem (Porter–Zelenko [6])

For a 2-nondegenerate CR structure of hypersurface type on a manifold of arbitrary odd dimension with a constant regular CR symbol, one can assign a canonical absolute parallelism on a fiber bundle over M.

- a bigraded analog of Tanaka prolongation is used for construction of the absolute paralellism
- the resulting bundle's dimension can be calculated from the CR symbol
- there exists a unique maximally symmetric CR structure among all 2-nondegenerate CR structures with constant *regular* symbol $\mathfrak{g}^0$
- if $\dim M = 2n + 1$, the CR symbol is *regular*, and $\operatorname{rank} K = 1$ then the algebra of infinitesimal symmetries has dimension less than or equal to $n^2 + 7$.

An Outline of Fundamental Results

D.S., I. Zelenko (2020) - we establish the following results

- *construction of canonical absolute paralellisms* for *recoverable* 2-nondegenerate CR structures of hypersurface type, through a reduction of CR structures to special flag structures on Levi leaf spaces and the application of a microlocal version of Tanaka prolongation
- through reduction of the absolute parallelism, we obtain strong algebraic constraints on the local invariants of homogeneous models, expressed in terms of *reduced modified CR symbols*
- if $\dim M >> \mathrm{rank} K > 1$ then generic CR symbols cannot be exhibited on homogeneous models
- there exists a unique maximally symmetric CR structure among all 2-nondegenerate CR structures with constant *reduced modified CR symbol*

The Levi Leaf Space

$\mathbb{C}M$ denotes the complexification defined (locally) in coordinates by replacing real coordinates with complex coordinates.

$K \oplus \overline{K}$ is integrable and induces *the Levi foliation* of $\mathbb{C}M$.

Define the *Levi leaf space* $\mathcal{N}$ to be the leaf space of this foliation, with natural projection denoted by

$$\pi : \mathbb{C}M \to \mathcal{N}.$$

The distribution

$$\mathcal{D} := \pi_*(H \oplus \overline{H})$$

is a contact distribution on $\mathcal{N}$. Set

$$\mathrm{LG}(\mathcal{D}_\gamma) := \text{Lagrangian Grassmannian modeled on } \mathcal{D}_\gamma.$$

Dynamical Legendrian Contact Structures

For $p \in \mathbb{C}M$, regard $J^+(p) := \pi_* H_p$ and $J^-(p) := \pi_* \overline{H}_p$ as points in $\mathrm{LG}\big(\mathcal{D}_{\pi(p)}\big)$. Note

$$\mathcal{D}_{\pi(p)} = J^+(p) \oplus J^-(p) \qquad \forall\, p \in \mathbb{C}M$$

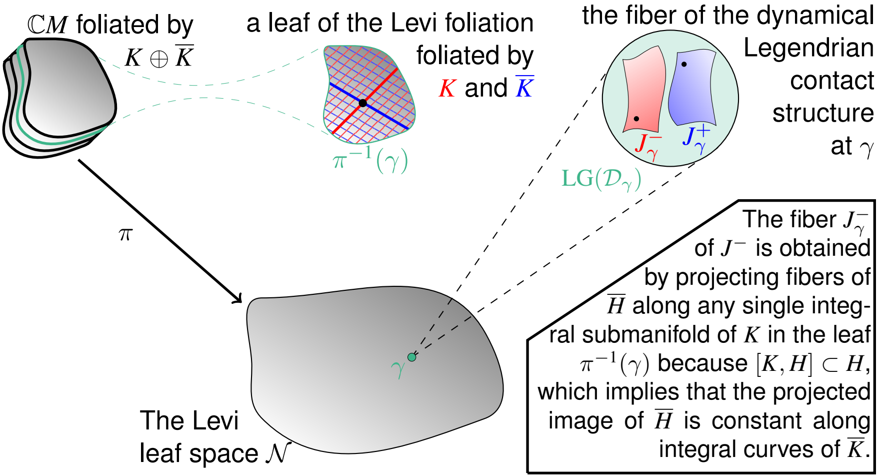
Setting $J^\pm := \mathrm{Im}(J^\pm)$, after possibly shrinking $\mathbb{C}M$,

$$(J^+, J^-)$$

defines a *dynamical Legendrian contact structure* on $\mathcal{N}$ with

$$\dim_{\mathbb{C}}(J^\pm_\gamma) = \mathrm{rank}_{\mathbb{C}} K \qquad \forall\, \gamma \in \mathcal{N}.$$

DLCS Diagram



Recovering CR Structures From DLC Structures

A CR structure H is recoverable if and only if H is the unique involutive subdistribution of $H + \overline{K}$ of rank $n = \operatorname{rank} H$, transversal to $\overline{K}$ and containing K.

For $Z \subset \mathrm{Hom}(V, W)$, define $\partial : \mathrm{Hom}(V, Z) \to \mathrm{Hom}(V \wedge V, W)$ by

$$\partial(f)(v_1, v_2) = f(v_1)v_2 - f(v_2)v_1, \quad v_1, v_2 \in V, \; f \in \mathrm{Hom}(V, Z). \qquad (1)$$

Define the first prolongation $Z_{(1)}$ of Z to be the kernel of ∂.

Proposition

A 2 non-degenerate hypersurface type CR structure H is recoverable in a neighborhood of a point p if and only if the first prolongation $\left(\mathrm{ad}K_p\right)_{(1)}$ of the space $\mathrm{ad}K_p$ vanishes.

If $\operatorname{rank} K = 1$ then the original CR structure given by H is recoverable near p if and only if $\operatorname{rank}(\mathrm{ad}_v) > 1$ for any v generating K_p.

Moduli Spaces of CR Symbols

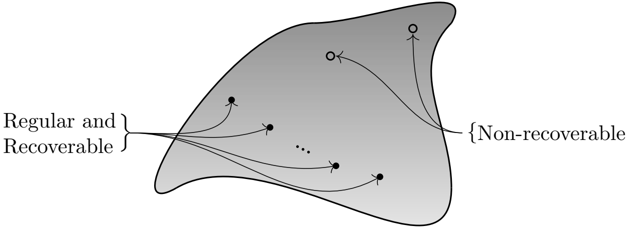
CR symbols are uniquely determined by

1. $\mathbb{R}\ell$, a real line of nondegenerate Hermition forms
2. $\{\mathrm{ad}_v \mid v \in K_p\}$, a $\mathrm{rank}\,K$-dimensional vector space of ℓ-selfadjoint antilinear operators

where ℓ is the Hermitian form on $\mathfrak{g}_{-1,1}(p)$ induced by $\mathcal{L}$.

Fix $n > 2$ and $\mathrm{rank}\,K = 1$

The space of CR symbols for a given signature of $\mathcal{L}$ on manifolds of dimension $2n+1$



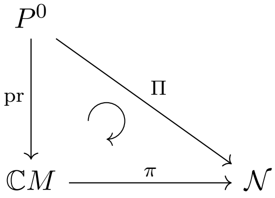
Frames Adapted to the DLC Structures

Fix a model abstract CR symbol $\mathfrak{g}^0 = \mathfrak{g}_- \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2}$.

$\mathrm{pr} : P^0 \to \mathbb{C}M$ is the bundle whose fiber $\mathrm{pr}^{-1}(p)$ over $p \in M$ is comprised of all *adapted frames at p*, which are the Lie algebra isomorphisms ψ satisfying

- $\psi : \mathfrak{g}_- \to \mathfrak{g}_-(p)$
- $\psi([y_1, y_2]) = [\psi(y_1), \psi(y_2)] \quad \forall\, y_1, y_2 \in \mathfrak{g}_-$
- $\psi(\mathfrak{g}_{i,j}) = \mathfrak{g}_{i,j}(p) \quad \forall\, (i,j) \in \{(-1, \pm 1), (-2, 0)\}$
- $\psi^{-1} \circ \mathfrak{g}_{0,\pm 2}(p) \circ \psi = \mathfrak{g}_{0,\pm 2}$

There is also a projection $\Pi : P^0 \to \mathcal{N}$ that factors through π:



Modified CR Symbols

Fix $\psi_0 \in P^0$. Let $\psi : (-\varepsilon, \varepsilon) \to P^0_{\Pi(\psi_0)}$ be a curve in $P^0_{\Pi(\psi_0)} = \Pi^{-1}\big(\Pi(\psi_0)\big)$ with

$$\psi(0) = \psi_0,$$

and define $\theta_0 : T_{\psi_0} P^0_{\Pi(\psi_0)} \to \mathfrak{csp}\big(\mathcal{D}_{\Pi(\psi_0)}\big)$ by

$$\theta_0\big(\psi'(0)\big) := \psi_0^{-1} \circ \psi'(0).$$

The *modified CR symbol* of the structure H at $\psi \in P^0$ is

$$\mathfrak{g}^{0,\mathrm{mod}}(\psi) := \mathfrak{g}_-\big(\mathrm{pr}(\psi)\big) \oplus \mathfrak{g}_0^{\mathrm{mod}}(\psi)$$

where

$$\mathfrak{g}_0^{\mathrm{mod}}(\psi) := \theta_0\left(T_\psi P^0_{\Pi(\psi)}\right).$$

Regular frames

Definition

A point $\psi_0 \in P^0$ is *regular* (with respect to the Tanaka prolongation) if the maps $\psi \mapsto \dim \mathfrak{g}_k^{\mathrm{mod}}(\psi)$ are constant in a neighborhood of ψ_0 for all $k \geq 0$.

Canonical absolute parallelisms

Theorem (S. and Zelenko)

Fix a CR symbol $\mathfrak{g}^0$ and a corresponding modified CR symbol $\mathfrak{g}^{0,\mathrm{mod}}$ so that its universal Tanaka prolongation $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{mod}})$ is finite dimensional.

1. *Given a 2-nondegenerate, hypersurface-type CR structure with symbol $\mathfrak{g}^0$ such that there exists a regular point ψ_0 (w.r.t. the Tanaka prolongation) in the bundle P^0 of this structure with $\mathfrak{g}^{0,\mathrm{mod}}(\psi_0) = \mathfrak{g}^{0,\mathrm{mod}}$, there exists a bundle over a neighborhood $\mathcal{O}$ of ψ_0 in P^0 of dimension equal to $\dim \mathfrak{u}(\mathfrak{g}^{0,\mathrm{mod}})$ that admits a canonical absolute parallelism.*
2. *The dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate, hypersurface-type CR structure of the previous item is not greater than $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{mod}})$.*

Level Sets of Modified Symbols

Fix $\mathfrak{g}^{0,\mathrm{mod}}$. The *level set of $\mathfrak{g}^{0,\mathrm{mod}}$* in P^0 is

$$P^0(\mathfrak{g}^{0,\mathrm{mod}}) := \left\{\psi \in P^0 \mid \mathfrak{g}^{0,\mathrm{mod}}(\psi) = \mathfrak{g}^{0,\mathrm{mod}}\right\}.$$

Suppose

$$\mathrm{pr}\left(P^0(\mathfrak{g}^{0,\mathrm{mod}})\right) = \mathbb{C}M.$$

Applying θ_0 to a tangent space in $P^0(\mathfrak{g}^{0,\mathrm{mod}})$ at ψ gives a subspace $\mathfrak{g}^{0,\mathrm{red}}(\psi) \subset \mathfrak{g}^{0,\mathrm{mod}}(\psi)$.

For homogeneous models this can be iterated to obtain a maximally reduced symbol $\mathfrak{g}^{0,\mathrm{red}}$ with level set $P^{0,\mathrm{red}}$.

Homogeneity and Lie Algebras

Proposition

If $\psi \mapsto \mathfrak{g}^{0,\mathrm{red}}(\psi)$ *is constant on* $P^{0,\mathrm{red}}$ *then* $\mathfrak{g}^{0,\mathrm{red}}(\psi)$ *is a Lie subalgebra of* $\mathfrak{g}_-(\mathrm{pr}(\psi)) \rtimes \mathfrak{csp}(\mathcal{D}_{\Pi(\psi)})$.

This provides a strong algebraic constraint on the local invariants of homogeneous models.

Generic Nonexistence of Homogeneous Models

Theorem (S. and Zelenko)

For any fixed rank $r > 1$, in the set of all CR symbols associated with 2-nondegenerate, hypersurface-type CR manifolds of odd dimension greater than $4r+1$ with rank r Levi kernel and with reduced Levi form of arbitrary signature, the CR symbols not associated with any homogeneous model are generic. For $r = 1$, the same statement holds if the reduced Levi form is sign-definite, that is, when the CR structure is pseudoconvex.

Constructing Homogeneous Models (slide 1 of 3)

The bigrading $\mathfrak{g}_{-1} = \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1}$ induces a decomposition $\mathfrak{csp}(\mathfrak{g}_{-1}) = (\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,-2} \oplus (\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,0} \oplus (\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,2}$.

For a given reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$, we define

$$\mathfrak{g}_{0,0}^{\mathrm{red}} := \mathfrak{g}^{0,\mathrm{red}} \cap (\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,0} \subset \mathfrak{g}_{0,0}.$$

Supposing that $\mathfrak{g}^{0,\mathrm{red}}$ is a subalgebra of $\mathfrak{g}_{-1} \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$, let $G^{0,\mathrm{red}}$ and $G_{0,0}^{\mathrm{red}} \subset G^{0,\mathrm{red}}$ be connected Lie groups with Lie algebras $\mathfrak{g}^{0,\mathrm{red}}$ and $\mathfrak{g}_{0,0}^{\mathrm{red}}$.

Constructing Homogeneous Models (slide 2 of 3)

Note that $\mathfrak{g}_-$ (and in particular $\mathfrak{g}_{-1,1}$) is a subspace of $\mathfrak{g}^{0,\text{red}}$

Let $\widehat{D}^{\text{flat}}_{-1,1}$ be the left-invariant distribution on $G^{0,\text{red}}$ equal to $\mathfrak{g}_{-1,1}$ at the identity.

Let $\widehat{D}^{\text{flat}}_{0,2}$ be the left-invariant distribution on $G^{0,\text{red}}$ equal to

$$\mathfrak{g}_0^{\text{red}} \cap \Big((\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,0} \oplus (\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,2} \Big)$$

at the identity.

Since $\mathfrak{g}_{-1,1}$ and $\mathfrak{g}_0^{\text{red}} \cap \Big((\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,0} \oplus (\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,2} \Big)$ are invariant under the adjoint action of $G_{0,0}^{\text{red}}$, the push-forward of each $\widehat{D}_{i,j}^{\text{flat}}$ to

$$M_0^{\mathbb{C}} := G^{0,\text{red}} / G_{0,0}^{\text{red}}$$

is a well defined distribution, which we denote by $D_{i,j}^{\text{flat}}$.

$H^{\text{flat}} := D_{-1,1}^{\text{flat}} \oplus D_{0,2}^{\text{flat}}$ defines a 2-nondegenerate, hypersurface-type CR structure on $M_0^{\mathbb{C}}$, a “complexified CR manifold”.

Taking M_0 to be the fixed point set of an appropriately chosen involution on $M_0^{\mathbb{C}}$, (M_0, H^{flat}) is a homogeneous CR manifold whose bundle P^0 admits a reduction with constant reduced modified symbol $\mathfrak{g}^{0,\text{red}}$.

Maximally Symmetric Models

Assume that given a CR symbol $\mathfrak{g}^0$ there exist a reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ with finite dimensional Tanaka prolongation.

Theorem (S. and Zelenko)

1. *Given a 2-nondegenerate, hypersurface-type CR structure such that the corresponding bundle P^0 admits a subbundle $P^{0,\mathrm{red}}$ with the reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$, there exists a bundle over $P^{0,\mathrm{red}}$ of dimension equal to $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ that admits a canonical absolute parallelism.*

Continuing the theorem from the previous slide:

Theorem (S. and Zelenko)

2. *The dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate, hypersurface-type CR structure of item (1) is not greater than $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$. Moreover, if we assume that the CR symbol $\mathfrak{g}^0$ is recoverable then the algebra of infinitesimal symmetries of the flat CR structure with constant reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ is isomorphic to the real part $\Re\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ of $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$.*

Continuing the theorem from the previous slide:

Theorem (S. and Zelenko)

3. *If the CR symbol* $\mathfrak{g}^0$ *is recoverable then any CR structure with the constant reduced modified symbol* $\mathfrak{g}^{0,\mathrm{red}}$ *whose algebra of infinitesimal symmetries has dimension equal to* $\dim_{\mathbb{C}} \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ *is locally to equivalent to the flat CR structure with reduced modified symbol* $\mathfrak{g}^{0,\mathrm{red}}$.

A 7-dimensional Example of a Homogeneous Model with Non-regular CR Symbol

Fix a basis $(e_0, \ldots, e_4)$ of $\mathfrak{g}_-$ with

$$[e_1, e_4] = [e_2, e_3] = e_0 \iff H_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathfrak{g}_{-1,-1} := \mathrm{span}_{\mathbb{C}}\{e_1, e_2\} \quad \text{and} \quad \mathfrak{g}_{-1,1} := \mathrm{span}_{\mathbb{C}}\{e_3, e_4\}$$

Let $\mathfrak{g}_0^{\mathrm{red}} \subset \mathfrak{csp}(\mathfrak{g}_{-1})$ to be spanned by

Ω C

$$\begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & 0 & i \\ \frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & -i & 0 & \frac{-i}{\sqrt{2}} \\ 1 & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

and the 4×4 identity matrix. Set $\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}_- \rtimes \mathfrak{g}_0^{\mathrm{red}}$.

Maximally Symmetric 7-dimensional Models

H_ℓ	C	Ω	symmetry group dimension
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	8
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$	8
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ \sqrt{\frac{3}{4}} & 0 \end{pmatrix}$	9
$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}$	0	10 if $\eta = 0$ 15 if $\eta = 1$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0	15
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	0	16

Here $\epsilon \in \{-1, 1\}$ and $\eta \in \{0, 1\}$

Maximum Dimension Of Symmetry Groups For Rank 1 Levi Kernel

For $\operatorname{rank} K = 1$, a CR symbol is encoded in the pair $(\mathbb{R}\ell, \mathbb{C}A)$. Canonical forms for these pairs are obtained [7, S. and Zelenko].

Theorem

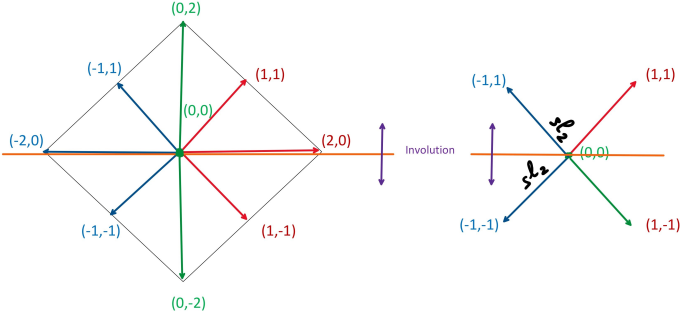
If the reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$ is associated with a rank 1 nonregular structure then $\mathfrak{u}\left(\mathfrak{g}^{0,\mathrm{red}}\right) = \mathfrak{g}^{0,\mathrm{red}}$.

Combining this with [6, Porter and Zelenko] yields

Theorem

A sharp upper bound for the dimension of the group of symmetries of a homogeneous 2-nondegenerate, hypersurface-type CR manifold with a rank 1 Levi kernel is $\frac{1}{4}(\dim M - 1)^2 + 7$.

Description Of The Symmetry Algebra Of The Maximally Symmetric 7-dimensional Model



Compexification of the symmetry algebra is

$$\Big(\mathbb{C} \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})\Big) \ltimes (\mathbb{C}^3 \otimes \mathbb{C}^3) \qquad .$$

The symmetry algebra is the real part w.r.t. to the involution by the reflection (w.r.t. the orange line on the figure) and isomorphic to

$$\Big(\mathbb{R} \oplus \mathfrak{so}(3,1)\Big) \ltimes (\mathrm{Symm}^2_0(\mathbb{R}^4)).$$

Hypersurface realizations of maximally symmetric models

The hypersurfaces

$$\left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \;\middle|\; \Im(z_0 + z_1^2 \overline{z_n}) = z_1 \overline{z_2} + \overline{z_1} z_2 + \sum_{i=3}^{n-1} \varepsilon_i z_i \overline{z_i} \right\}$$

with $\varepsilon_i = \pm 1$ are 2-nondegenerate. Their Levi form's signature is determined by $\{\varepsilon_i\}$ and their algebras of infinitesimal symmetries attain the upper bound $\frac{1}{4}(\dim M - 1)^2 + 7 = n^2 + 7$.[5]

[5] reference: [6, Porter and Zelenko]

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