An introduction to supergravity in 11 dimensions

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Partly based on joint works with P. de Medeiros and J. Figueroa-O'Farrill

Plan of the series of talks:

First part:

- $d = 11$ Supergravity
- Detour on Lie superalgebras (including the Poincaré superalgebra)
- Killing spinor equations and Killing superalgebras
- Brane solutions

Second part:

- Homogeneity theorem
- Filtered deformations
- Spencer cohomology and Killing spinors
- Maximally supersymmetric backgrounds

Third part:

- Rudiments of spinorial algebra & spin geometry
- PDEs on spinor bilinears
- Highly supersymmetric backgrounds

Rudiments of spinorial algebra & spin geometry

Let (V, η) be a vector space with a positive-definite inner product. The *Clifford algebra* $\mathcal{C}\ell(V)$ associated to (V, η) is associative algebra with a unity generated by V with the relation $v^2 = -\eta(v, v)1$ for all $v \in V$.

Rem I. By polarization, the Clifford relations are equivalent to

$$v \cdot w + w \cdot v = -2\eta(v, w)$$

for all $v, w \in V$. It follows that $\mathcal{C}\ell(V) = \mathcal{C}\ell(V)_{\bar{0}} \oplus \mathcal{C}\ell(V)_{\bar{1}}$ is a $\mathbb{Z}_2$-graded algebra.

Rem II. If $\{e_i\}$ is an orthonormal basis of V, then

$$e_i \cdot e_j = \begin{cases} -e_j \cdot e_i & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$$

Setting the alternating product of generators

$$e_{i_1 \cdots i_p} := \tfrac{1}{p!} \sum_{\sigma \in S_p} (-1)^{|\sigma|} e_{i_{\sigma(1)}} \cdots e_{i_{\sigma(p)}}$$

gives a vector space isomorphism $\mathcal{C}\ell(V) \cong \Lambda^\bullet V = \mathbb{R} \oplus V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^m V$.

Exercises on Clifford algebras

Show that

$$v \cdot \alpha = v \wedge \alpha - \imath_v \alpha \tag{1}$$

$$\alpha \cdot v = (-1)^{|\alpha|} \left(v \wedge \alpha + \imath_v \alpha \right) \tag{2}$$

$$(v \wedge w) \cdot \alpha = v \wedge w \wedge \alpha + \imath_v \imath_w \alpha - v \wedge \imath_w \alpha + w \wedge \imath_v \alpha \tag{3}$$

$$\alpha \cdot (v \wedge w) = v \wedge w \wedge \alpha + \imath_v \imath_w \alpha + v \wedge \imath_w \alpha - w \wedge \imath_v \alpha \tag{4}$$

for all $v, w \in V$ and $\alpha \in \Lambda^\bullet V$.

Table II. $C\ell_{r,s}$ in the box (r,s)

8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128) \oplus \mathbb{R}(128)$	$\mathbb{R}(256)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64) \oplus \mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
5	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32) \oplus \mathbb{H}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$	$\mathbb{H}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**

Rudiments of spinorial algebra & spin geometry

$$\mathrm{Spin}(V) = \{g = v_1 \cdots v_{2k} \mid v_i \in V \text{ s.t } \eta(v_i, v_i) = +1\} \subset \mathcal{C}\ell(V)$$

$$\mathfrak{so}(V) \overset{\cong}{\to} \mathcal{C}\ell(V) \text{ via } v \wedge w \mapsto \frac{1}{4}[v, w] = \frac{1}{4}(v \cdot w - w \cdot v)$$

Important fact: $g \in \mathrm{Spin}(V)$, $v \in V \Rightarrow g \cdot v \cdot g^{-1} \in V$. This is the 2-fold cover $\mathrm{Ad} : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ (archetipical example to have in mind: $\mathrm{Spin}(3) \cong \mathrm{Sp}(1) \to \mathrm{SO}(3) = \mathrm{SO}(\mathrm{Im}\,\mathbb{H})$)

Rudiments of spinorial algebra & spin geometry

Let (M,g) be an orientable Riemannian manifold and

$$SO(M) = \{u : \mathbb{R}^m \to T_xM \text{ orientation-preserving linear isomorphism s.t. } u^*g = \eta \mid x \in M\}$$

the bundle of oriented orthonormal frames.

Def. A *spin structure* on (M,g) is principal $\mathrm{Spin}(V)$-bundle $\mathrm{Spin}(M) \to M$ together with commutative diagram of bundle morphisms

$$\begin{array}{ccc} \mathrm{Spin}(M) & & \\ \downarrow & \searrow & \\ & & M \\ & \nearrow & \\ \mathrm{SO}(M) & & \end{array}$$

which restricts fiberwise to $\mathrm{Ad} : \mathrm{Spin}(V) \to SO(V)$. The vector bundle $S(M) = \mathrm{Spin}(M) \times_{\mathrm{Spin}(V)} S$ is called *spinor bundle* (S = irrep. of $\mathcal{C}\ell(V)$).

Example

$M = S^m \cong \mathrm{SO}(m+1)/\mathrm{SO}(m) \cong \mathrm{Spin}(m+1)/\mathrm{Spin}(m)$

$$
\begin{array}{ccc}
\mathrm{SO}(M) \cong \mathrm{SO}(m+1) \ni & g = \begin{pmatrix} g_1 & g_2 & \cdots & g_{m+1} \end{pmatrix} \\
\downarrow & \downarrow \\
M & g_1
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Spin}(M) \cong \mathrm{Spin}(m+1) & \searrow & \\
\downarrow & & M \\
\mathrm{SO}(m+1) & \nearrow &
\end{array}
$$

Spinor fields satisfying special PDEs

Def. A spinor field $\epsilon \in \Gamma(S(M))$ is called

- *parallel* if $\nabla_X \epsilon = 0 \ \forall X \in \mathfrak{X}(M)$;
- *Killing* if there is constant λ such that $\nabla_X \epsilon = \lambda X \cdot \epsilon \ \forall X \in \mathfrak{X}(M)$.

Thm[Friedrich '80s] If a Riemannian manifold (M, g) has a non-trivial parallel spinor, then $\mathrm{Ric} = 0$.

Proof. Clearly $\nabla_X \epsilon = 0 \Rightarrow R(X, Y)\epsilon = 0$, where $R \in \Lambda^2 T^* M \otimes \mathrm{End}(S(M))$, so the Clifford trace $0 = 4 \sum_j e_j \cdot R(e_i, e_j)\epsilon$. Now

$$R(e_i, e_j) = \frac{1}{2} \sum_{k,l} R_{ijkl} e_k \wedge e_l \Rightarrow R(e_i, e_j)\epsilon = \frac{1}{4} \sum_{k,l} R_{ijkl} e_k \cdot e_l \cdot \epsilon$$

and substituting into the Clifford trace leads to $0 = \sum_{j,k,l} R_{ijkl} e_j \cdot e_k \cdot e_l \cdot \epsilon$.

Now

$$\begin{aligned} 0 &= \sum_{j,k,l} R_{ijkl} e_j \cdot e_k \cdot e_l \cdot \epsilon \\ &= \sum_{j,k,l} R_{ijkl} \big(e_{jkl} - \eta_{jk} e_l + \eta_{jl} e_k \big) \cdot \epsilon \\ &= \sum_{j,k,l} R_{ijkl} \big(e_{jkl} + 2\eta_{jl} e_k \big) \cdot \epsilon \end{aligned}$$

The first term vanishes due to Bianchi Identity $R_{ijkl} + R_{iljk} + R_{iklj} = 0$ so we are left with $0 = -2\sum_{j,k,l} R_{jikl}\eta_{jl} e_k \cdot \epsilon = -2\sum_k \mathrm{Ric}_{ik}\, e_k \cdot \epsilon$. Equivalently, if we look at the Ricci tensor as an endomorphism, we have $\mathrm{Ric}(X) \cdot \epsilon = 0$ for all $X \in \mathfrak{X}(M)$, so that $\mathrm{Ric}(X) = 0$ for all $X \in \mathfrak{X}(M)$, which is our claim ■

Spinor fields satisfying special PDEs

Wang's classification of complete, simply connected, irreducible Riemannian manifolds admitting parallel spinors (1989):

Holonomy Representation	Geometry	Parallel spinors
$\mathrm{SU}(2n+1)$	Calabi-Yau	$(1,1)$
$\mathrm{SU}(2n)$	Calabi-Yau	$(2,0)$
$\mathrm{Sp}(n)$	Hyper-Kähler	$(n+1,0)$
G_2 $(\subset \mathrm{SO}(7))$	exceptional	1
$\mathrm{Spin}(7)$ $(\subset \mathrm{SO}(8))$	exceptional	1

Thm[Bär, Baum '90s] If a Riemannian manifold (M,g) has a non-trivial Killing spinor with Killing constant $\lambda \in \mathbb{C}$, then $\mathrm{Ric} = 4\lambda^2(m-1)g$, i.e., M is Einstein and $\lambda \in \mathbb{R}$ or $\lambda \in i\mathbb{R}$.

Supergravity

Let (M, g, F) be Lorentzian mnfd (M, g), $\dim M = 11$, with closed $F \in \Omega^4(M)$ and endowed with a spinor bundle $S(M) \longrightarrow M$ (the fiber $S(M)_x \cong S = \mathbb{R}^{32}$). The *bosonic equations of supergravity* are two coupled PDEs [Cremmer-Julia-Scherk '78]:

$$\mathrm{Ric}(X, Y) = \tfrac{1}{2} g(i_X F, i_Y F) - \tfrac{1}{6} g(X, Y)|F|^2$$

$$d * F = \tfrac{1}{2} F \wedge F$$

Killing superalgebra $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ where

$$\mathfrak{k}_{\bar{0}} = \left\{\xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0\right\}$$
$$\mathfrak{k}_{\bar{1}} = \left\{\epsilon \in \Gamma(S(M)) \mid \nabla_X \epsilon = \tfrac{1}{24}\left(X \cdot F - 3F \cdot X\right) \cdot \epsilon\right\}$$

High supersymmetry

It has long been suspected that there is some critical fraction of supersymmetry which forces the equations of motion of supergravity. In 2017, we gave following positive answer:

Thm[Figueroa-O'Farrill, A.S. '17] Let (M, g) be 11-dimensional Lorentzian mnfd with *closed* $F \in \Omega^4(M)$. If $\dim \mathfrak{k}_{\bar{1}} > 16$, then (i) (M, g, F) satisfies Einstein and Maxwell eqs (the bound is sharp) and (ii) $F = 0$ iff (M, g, F) is the flat model.

Main ingredients of the proof:

1. Filtered subdeformations,
2. PDEs satisfied by differential forms constructed out of Killing spinors,
3. the local homogeneity theorem.

Killing superalgebras as filtered deformations

Thm[Figueroa-O'Farrill, A.S. '17] Any Killing superalgebra $\mathfrak{k}$ is a filtered deformation of a graded subalgebra $\mathfrak{a} = V' \oplus S' \oplus \mathfrak{h}$ of Poincaré superalgebra $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$.

Explicitly:

$$
\begin{aligned}
[A,B] &= AB - BA & [A,s] &= As \\
[A,v] &= Av + \delta(A,v) & [v,s] &= \beta^\varphi(v,s) + X_v s \\
[v,w] &= \alpha(v,w) + \rho(v,w), & [s,s] &= \kappa(s,s) + \gamma^\varphi(s,s) - X_{\kappa(s,s)}
\end{aligned}
\tag{5}
$$

for $A, B \in \mathfrak{h}$, $v, w \in V'$, $s \in S'$, where

$$
\begin{aligned}
\alpha(v,w) &= X_v w - X_w v \\
\delta(A,v) &= [A, X_v] - X_{Av} \\
\rho(v,w) &= [X_v, X_w] - X_{\alpha(v,w)} + R(v,w)
\end{aligned}
\qquad
\begin{aligned}
\beta^\varphi(v,s) &= \tfrac{1}{24}(v \cdot \varphi - 3\varphi \cdot v) \cdot s, \\
\gamma^\varphi(s,s)(v) &= -2\kappa(\beta^\varphi(v,s), s).
\end{aligned}
$$

for some $\mathfrak{h}$-invariant $\varphi \in \Lambda^4 V$ and a map $X : V' \to \mathfrak{so}(V)$.

Jacobi Identities for Killing superalgebras

- $[\mathfrak{h}\mathfrak{h}\mathfrak{h}]$, $[\mathfrak{h}\mathfrak{h}S']$, $[\mathfrak{h}\mathfrak{h}V']$ are satisfied because $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{so}(V)$ that stabilizes S' and V';
- $[\mathfrak{h}S'S']$ and $[\mathfrak{h}S'V']$ are satisfied as $\mathfrak{h} < \mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$. E.g. for $A \in \mathfrak{h}$ and $s \in S'$, we have

$$\begin{aligned}[A,[s,s]] &= [A, \kappa(s,s) + \gamma^\varphi(s,s) - X_{\kappa(s,s)}] \\ &= A\kappa(s,s) + [A, \gamma^\varphi(s,s)] - X_{A\kappa(s,s)} \\ &= 2\kappa(As,s) + 2\gamma^\varphi(As,s) - 2X_{\kappa(As,s)} = 2[[A,s],s]\end{aligned}$$

  since κ and γ^φ are equivariant under $\mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$;
- $[\mathfrak{h}V'V']$ boilds down to $R : \Lambda^2 V' \to \mathfrak{so}(V)$ being $\mathfrak{h}$-equivariant;
- $[S'S'S']$ says that $[[s,s],s] = 0$ for all $s \in S'$ and it expands to

$$\gamma^\varphi(s,s)s = -\beta^\varphi(\kappa(s,s),s) \; .$$

  This is actually true for all $s \in S$ (it is one cocycle condition in $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$);

Jacobi Identities for Killing superalgebras

- $[S'S'V']$ Jacobi Identity. After a somewhat lengthy calculation and letting $\beta^\varphi_v(s) = \beta^\varphi(v, s)$ for all $v \in V$ and $s \in S$, this identity is equivalent to

$$\tfrac{1}{2}R(v, \kappa(s, s))w = \kappa\big((X_v\beta^\varphi)(w, s), s\big) - \kappa\big(\beta^\varphi_v(s), \beta^\varphi_w(s)\big) - \kappa\big(\beta^\varphi_w\beta^\varphi_v(s), s\big),$$

  for all $s \in S'$, $v \in V'$ and $w \in V$;

- $[S'V'V']$ expands to the following condition

$$R(v, w)s = (X_v\beta^\varphi)(w, s) - (X_w\beta^\varphi)(v, s) + [\beta^\varphi_v, \beta^\varphi_w](s),$$

  for all $s \in S'$ and $v, w \in V'$;

- $[V'V'V']$ expands to Bianchi Identities for R, algebraic and differential.

PDEs on Spinor Bilinears

For any section ε of $S(M)$ we may define *differential forms* on M as follows:

1. $\omega^{(1)} \in \Omega^1(M)$, where
$$\omega^{(1)}(X) = \langle \varepsilon, X \cdot \varepsilon \rangle$$
2. $\omega^{(2)} \in \Omega^2(M)$, where
$$\omega^{(2)}(X_1, X_2) = \langle \varepsilon, (X_1 \wedge X_2) \cdot \varepsilon \rangle$$
3. $\omega^{(5)} \in \Omega^5(M)$, where
$$\omega^{(5)}(X_1, \ldots, X_5) = \langle \varepsilon, (X_1 \wedge \ldots \wedge X_5) \cdot \varepsilon \rangle$$

The 1-form $\omega^{(1)}$ is the metric dual of Dirac current $\kappa = \kappa(\varepsilon, \varepsilon)$ of ε.

Prop. If $\varepsilon \in \mathfrak{k}_{\bar{1}}$ then:

$$d\omega^{(2)} = -\imath_\kappa F \tag{6}$$

$$d\omega^{(5)} = \imath_\kappa \star F - \omega^{(2)} \wedge F. \tag{7}$$

These imply that the supergravity *Maxwell eqs are satisfied* if $dF = 0$ and the space $\mathfrak{k}_{\bar{1}}$ of Killing spinors has $\dim \mathfrak{k}_{\bar{1}} > 16$.

Proof of PDEs on Spinor Bilinears

Proof.

We first rewrite

$$\begin{aligned}\nabla_Z \epsilon &= \tfrac{1}{24}\big(Z\cdot F - 3F\cdot Z\big)\cdot\epsilon \\ &= \tfrac{1}{24}\big(Z\wedge F - \imath_Z F\big)\cdot\epsilon - \tfrac{1}{8}\big(Z\wedge F + \imath_Z F\big)\cdot\epsilon \\ &= -\tfrac{1}{12}\big(Z\wedge F\big)\cdot\epsilon - \tfrac{1}{6}\big(\imath_Z F\big)\cdot\epsilon\end{aligned}$$

and then compute

$$\begin{aligned}(\nabla_Z\omega^{(2)})(X,Y) &= \langle\nabla_Z\epsilon, X\wedge Y\cdot\epsilon\rangle + \langle\epsilon, X\wedge Y\cdot\nabla_Z\epsilon\rangle \\ &= -\tfrac{1}{6}\big\langle\big(\imath_Z F\big)\cdot\epsilon, X\wedge Y\cdot\epsilon\big\rangle - \tfrac{1}{6}\big\langle\epsilon, X\wedge Y\cdot\big(\imath_Z F\big)\cdot\epsilon\big\rangle \\ &\quad - \tfrac{1}{12}\big\langle\big(Z\wedge F\big)\cdot\epsilon, X\wedge Y\cdot\epsilon\big\rangle - \tfrac{1}{12}\big\langle\epsilon, X\wedge Y\cdot\big(Z\wedge F\big)\cdot\epsilon\big\rangle \\ &= -\tfrac{1}{6}\big\langle\epsilon, \big(\imath_Z F\big)\cdot X\wedge Y\cdot\epsilon\big\rangle - \tfrac{1}{6}\big\langle\epsilon, X\wedge Y\cdot\big(\imath_Z F\big)\cdot\epsilon\big\rangle \\ &\quad + \tfrac{1}{12}\big\langle\epsilon, \big(Z\wedge F\big)\cdot X\wedge Y\cdot\epsilon\big\rangle - \tfrac{1}{12}\big\langle\epsilon, X\wedge Y\cdot\big(Z\wedge F\big)\cdot\epsilon\big\rangle\ .\end{aligned}$$

Using again the exercise on Clifford multiplication we get

Proof of PDEs on Spinor Bilinears – continued

$$
\begin{aligned}
(\nabla_Z \omega^{(2)})(X,Y) &= -\tfrac{1}{3}\left\langle \epsilon, X\wedge Y\wedge \left(\imath_Z F\right)\cdot\epsilon\right\rangle - \tfrac{1}{3}\left\langle \epsilon, \imath_X\imath_Y\imath_Z F\cdot\epsilon\right\rangle \\
&\quad + \tfrac{1}{6}\left\langle \epsilon, X\wedge \imath_Y\left(Z\wedge F\right)\cdot\epsilon\right\rangle - \tfrac{1}{6}\left\langle \epsilon, Y\wedge \imath_X\left(Z\wedge F\right)\cdot\epsilon\right\rangle \\
&= -\tfrac{1}{3}\left\langle \epsilon, X\wedge Y\wedge \left(\imath_Z F\right)\cdot\epsilon\right\rangle - \tfrac{1}{3}\left\langle \epsilon, \imath_X\imath_Y\imath_Z F\cdot\epsilon\right\rangle \\
&\quad + \tfrac{1}{6}g(Y,Z)\left\langle \epsilon, X\wedge F\cdot\epsilon\right\rangle - \tfrac{1}{6}g(X,Z)\left\langle \epsilon, Y\wedge F\cdot\epsilon\right\rangle \\
&\quad - \tfrac{1}{6}\left\langle \epsilon, X\wedge Z\wedge \left(\imath_Y F\right)\cdot\epsilon\right\rangle + \tfrac{1}{6}\left\langle \epsilon, Y\wedge Z\wedge \left(\imath_X F\right)\cdot\epsilon\right\rangle
\end{aligned}
$$

and skewsymmetrizing in X, Y and Z we finally arrive at

$$
\begin{aligned}
d\omega^{(2)}(X,Y,Z) &= (\nabla_X\omega^{(2)})(Y,Z) + (\nabla_Y\omega^{(2)})(Z,X) + (\nabla_Z\omega^{(2)})(X,Y) \\
&= -\left\langle \epsilon, \imath_X\imath_Y\imath_Z F\cdot\epsilon\right\rangle = -\omega^{(1)}(\imath_X\imath_Y\imath_Z F) = -\imath_\kappa\imath_X\imath_Y\imath_Z F \\
&= -(\imath_\kappa F)(X,Y,Z)
\end{aligned}
$$

that is $d\omega^{(2)} = -\imath_\kappa F$. Exercise: you are free to prove the identity for $d\omega^{(5)}$ in a similar fashion.

Proof of PDEs on Spinor Bilinears – the end

Let us then prove that the Maxwell eqs are satisfied if $dF = 0$ and $\dim \mathfrak{k}_{\bar{1}} > 16$. We first compute

$$
\begin{aligned}
0 &= \star \mathcal{L}_\kappa F = \mathcal{L}_\kappa \star F = d\imath_\kappa \star F + \imath_k d \star F \\
&= d(\omega^{(2)} \wedge F) + \imath_\kappa d \star F = d\omega^{(2)} \wedge F + \imath_\kappa d \star F \\
&= -\tfrac{1}{2}\imath_k (F \wedge F) + \imath_\kappa d \star F = \imath_\kappa \left(d \star F - \tfrac{1}{2} F \wedge F \right) .
\end{aligned}
$$

and then use the local homogeneity theorem. ■

High supersymmetry

Thm[Figueroa-O'Farrill, A.S. '17] Let (M, g) be 11-dimensional Lorentzian mnfd with *closed* $F \in \Omega^4(M)$. If $\dim \mathfrak{k}_{\bar{1}} > 16$, then (i) (M, g, F) satisfies Einstein and Maxwell eqs (the bound is sharp) and (ii) $F = 0$ iff (M, g, F) is the flat model.

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Sketch of proof of (i). The Jacobi identity $[S'S'V]$ in the filtered deformation gives

$$\tfrac{1}{2}R(v, \kappa(s,s))w = \kappa((X_v\beta^{\varphi})(w,s),s) - \kappa(\beta^{\varphi}_v(s), \beta^{\varphi}_w(s)) - \kappa(\beta^{\varphi}_w\beta^{\varphi}_v(s),s)$$

for all $s \in S'$ and $v, w \in V$. As $\kappa(S', S') = V$ by local homogeneity theorem, this fully determines the curvature R and, by a further contraction, the Ricci tensor

$$\mathrm{Ric}(v, \kappa(s,s)) = \tfrac{1}{2}g(\imath_v F, \imath_{e_i} F)\left\langle s, e^i \cdot s\right\rangle - \tfrac{1}{6}\|F\|^2\left\langle s, v \cdot s\right\rangle \\ - \tfrac{1}{6}\left\langle (v \wedge F \wedge F + 2\imath_v \delta F - v \wedge dF)\cdot s, s\right\rangle.$$

We then showed that the terms which depend on forms of different degree in $\odot^2 S' \subset \odot^2 S \cong \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V$ satisfy the eqs separately (not immediate: this embedding is diagonal) ■

Upshots

The theorem allows to establish a reconstruction result:

Def. A filtered subdeformation $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ of $\mathfrak{p}$ with $\dim \mathfrak{g}_{\bar{1}} > 16$ is *realizable* if it is constructed out of a closed 4-form $\varphi \in \Lambda^4 V$ as in (5).

Reconstruction thm[Figueroa-O'Farrill, A.S. '17] The highly supersymmetric bgkds, up to local equivalence, are in a *one-to-one correspondence* with maximal realizable filtered subdeformations $\mathfrak{g}$ of $\mathfrak{p}$ satisfying $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$, up to isomorphism of filtered subdeformations.

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Thanks!