An introduction to supergravity in 11 dimensions

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Partly based on joint works with P. de Medeiros and J. Figueroa-O'Farrill

Plan of the series of talks:

First part:

- $d = 11$ Supergravity
- Detour on Lie superalgebras (including the Poincaré superalgebra)
- Killing spinor equations and Killing superalgebras
- Brane solutions

Second part:

- Homogeneity theorem
- Filtered deformations
- Spencer cohomology and Killing spinors
- Maximally supersymmetric backgrounds

Third part:

- Clifford algebras
- PDEs on spinor bilinears
- Highly supersymmetric backgrounds

Killing superalgebras

Let (M, g, F) be Lorentzian mnfd (M, g), $\dim M = 11$, with closed $F \in \Omega^4(M)$ and endowed with a spinor bundle $S(M) \longrightarrow M$ (the fiber $S(M)_x \cong S = \mathbb{R}^{32}$). The *bosonic equations of supergravity* are two coupled PDEs [Cremmer-Julia-Scherk '78]:

$$\left.\begin{aligned} \mathrm{Ric}(X,Y) &= \tfrac{1}{2} g(i_X F, i_Y F) - \tfrac{1}{6} g(X,Y)|F|^2 \\ d * F &= \tfrac{1}{2} F \wedge F \end{aligned}\right\} \quad (*)$$

A *symmetry* of a solution of $(*)$ is a pair (ξ, ϵ) given by

(i) a Killing vector field for g preserving also F;

(ii) a (generalized) Killing spinor, *i.e.*, a section ϵ of $S(M)$ parallel w.r.t. the *"superconnection"*

$$D_X \epsilon = \nabla_X \epsilon - \tfrac{1}{24}\big(X \cdot F - 3F \cdot X\big) \cdot \epsilon \,.$$

Killing superalgebras

Thm[Figueroa-O'Farrill, Meessen, Philip '05] The $\mathbb{Z}_2$-graded v.s. $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of symmetries of a supergravity background (M, g, F) has a natural structure of a Lie superalgebra, called the *Killing superalgebra*.

Ex. (M, g) Minkowski, $F = 0$, $D = \nabla$ then $\mathfrak{k}_{\bar{1}} \cong S$, $\mathfrak{k}_{\bar{0}} \cong \mathfrak{so}(V) \oplus V$ and $\mathfrak{k} = \mathfrak{p}$.

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The Killing superalgebra is useful algebraic invariant of a supergravity bkgd:

- late '90s: first general check of AdS/CFT correspondence;
- early 2000s: it contracts under Penrose limit;
- mid 2000s: *homogeneity conjecture* by Meessen, i.e., if

$$\dim(\mathfrak{k}_{\bar{1}}) > \tfrac{1}{2} \dim S = 16 \qquad (1)$$

then bkgd is locally homogeneous.

Homogeneity theorem

Thm[Figueroa-O'Farrill, Hustler '12] If $\dim(\mathfrak{k}_{\bar{1}}) > 16$ then bkgd is loc. hom.

Proof.

(M, g, F) is locally homogeneous $\Leftrightarrow$ the evaluation map $\mathrm{ev}_x : \mathfrak{k}_{\bar{0}} \to T_x M$ is surjective for all $x \in M$. *Claim:* It is already so when restricted to $[\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}] \subset \mathfrak{k}_{\bar{0}}$.

In other words, we claim that if $S' \subset S$ with $\dim S' > 16$, then $\kappa(S', S') = V$. If this is not the case, then there are $v \neq 0$ such that $v \perp \kappa(S', S')$, i.e.,

$$0 = \eta(v, \kappa(s_1, s_2)) = \langle s_1, v \cdot s_2 \rangle$$

for all $s_1, s_2 \in S'$. This says that Clifford multiplication by v sends S' to $(S')^{\perp}$.

Now $\dim(S')^{\perp} < 16$, so Clifford multiplication by v has non-trivial Kernel in S, hence v is lightlike (remember that $v \cdot (v \cdot s) = -\eta(v, v)s$ for all $s \in S$ by definition of Clifford multiplication).

Homogeneity theorem

We have seen that $\kappa(S', S')^{\perp} \subset V$ consists of lightlike vectors and it is non-trivial, hence $\dim \kappa(S', S')^{\perp} = 1$ and $\kappa(S', S')^{\perp} = \mathbb{R}v$. Now

$$\kappa(S', S') = (\mathbb{R}v)^{\perp} = \mathbb{R}v \oplus W$$

for some spacelike 9-dimensional vector space W. It is a general fact that the Dirac current $\kappa(s, s)$ of any $s \in S$ is causal: either lightlike or timelike. For $s' \in S'$, this implies that $\kappa(s', s') \in \mathbb{R}v$ and by polarization $\kappa(S', S') \subset \mathbb{R}v$.

In conclusion, we have $\kappa(S', S') = \mathbb{R}v \oplus W$ and $\kappa(S', S') \subset \mathbb{R}v$, which is a contradiction. ■

State of the art

- Local expressions for metric and 4-form of *low supersymmetric bkgds* have been derived solving the Killing spinor eqs: the G-structure method [Gauntlett, Gutowski, Pakis '03] and the spinorial geometry method [Gillard, Gran, Papadopoulos '05].
- There are (M, g) with parallel spinors that are not Ricci flat [Bryant '00].
- Other approaches like exceptional generalized geometry that apply for special compactifications [Coimbra, Strickland-Constable, Waldram '14].

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- There are (M, g) with parallel spinors that are not Ricci flat [Bryant '00].
- Other approaches like exceptional generalized geometry that apply for special compactifications [Coimbra, Strickland-Constable, Waldram '14].
- Classification of *highly supersymmetric bkgds* is largely open. Maximally supersymmetric bkgds are classified [Figueroa-O'Farrill, Papadopoulos '03] and there are non-existence results for 31 and 30 Killing spinors [Gran, Gutowski, Papadopoulos, Roest '07 & '10].
- There is one known bkgd with 26 Killing spinors [Michelson '02] and also bkgds with $24, 22, 20, 18$ [Gauntlett, Hull '02].

Killing superalgebras as filtered deformations

Thm[Figueroa-O'Farrill, A.S. '17] Any Killing superalgebra is a *filtered deformation* of a subalgebra of the Poincaré superalgebra $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}} = (\mathfrak{so}(V) \oplus V) \oplus S$.

What does this mean practically? The nonzero Lie brackets of $\mathfrak{p}$ are

$$[A, B] = AB - BA \,, \quad [A, s] = As \,, \quad [A, v] = Av \,, \quad [s, s] = \kappa(s, s) \,,$$

for all $A, B \in \mathfrak{so}(V)$, $s \in S$, $v \in V$. There exists a compatible *$\mathbb{Z}$-grading*

$$\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$$

where $\mathfrak{p}_{-2} = V$, $\mathfrak{p}_{-1} = S$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$. Compatibility means

(i) $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_{i+j}$ for all $i, j \in \mathbb{Z}$;

(ii) $\mathfrak{p}_{\bar{0}} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_0$ and $\mathfrak{p}_{\bar{1}} = \mathfrak{p}_{-1}$.

We are interested in graded subalgebras $\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 = V' \oplus S' \oplus \mathfrak{h}$ of $\mathfrak{p}$.

Killing superalgebras as filtered deformations

There is also a natural filtration $\mathfrak{a}^\bullet$ on $\mathfrak{a}$, i.e.

$$\mathfrak{a} = \mathfrak{a}^{-2} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^{-1} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^0 = \mathfrak{a}_0 \supset \mathfrak{a}^1 = 0 \; .$$

Def. A *filtered deformation* of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ with same underlying vector space as $\mathfrak{a}$ and a new Lie bracket $[-,-]$ which satisfies:

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Def. A *filtered deformation* of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ with same underlying vector space as $\mathfrak{a}$ and a new Lie bracket $[-,-]$ which satisfies:

(i) $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j} \oplus \mathfrak{a}_{i+j+1} \oplus \cdots$,

(ii) components of $[-,-]$ of zero degree coincide with original bracket.

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Def. A *filtered deformation* of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ with same underlying vector space as $\mathfrak{a}$ and a new Lie bracket $[-,-]$ which satisfies:

$$\begin{aligned}
[A, B] &= AB - BA \\
[A, v] &= Av + t\tau(A, v) \\
[A, s] &= As \\
[s, s] &= \kappa(s, s) + t\gamma(s, s) \\
[v, s] &= t\beta(v, s) \\
[v, w] &= t\alpha(v, w) + t^2\delta(v, w)
\end{aligned}$$

for some maps $\tau : \mathfrak{h} \otimes V' \to \mathfrak{h}$, $\gamma : \odot^2 S' \to \mathfrak{h}$, $\beta : V' \otimes S' \to S'$, $\alpha : \Lambda^2 V' \to V'$ and $\delta : \Lambda^2 V' \to \mathfrak{h}$ subject to the Jacobi Identities for all values of the parameter t.

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Killing superalgebras as filtered deformations

Thm[Figueroa-O'Farrill, A.S. '17] Any Killing superalgebra $\mathfrak{k}$ is a filtered deformation of a graded subalgebra $\mathfrak{a}$ of the Poincaré superalgebra $\mathfrak{p}$.

First let's start with a classical analogy [Kostant, 55]. Let (M, g) be a pseudo-Riem. mnfd with Lie algebra $\mathfrak{k}_{\bar{0}}$ of Killing vectors. On the bundle $\mathscr{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM)$ we have a connection D defined by

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A - R(X, \xi) \end{pmatrix}$$

for $(\xi, A) \in \Gamma(\mathscr{E}_{\bar{0}})$, where $R : \Lambda^2 TM \to \mathfrak{so}(TM)$ is Riemann curvature. A section (ξ, A) is parallel precisely if ξ is Killing vector. This allows to localise at any $x \in M$. We introduce the notation $(V, \eta) = (T_x M, g_x)$, $\mathfrak{so}(V) = \mathfrak{so}(T_x M)$ so that the fiber of $\mathscr{E}_{\bar{0}}$ at x is $\mathfrak{p}_{\bar{0}} = V \oplus \mathfrak{so}(V)$ and $\mathfrak{k}_{\bar{0}}$ is identified with (not yet $\mathbb{Z}$-graded) subspace of $\mathfrak{p}_{\bar{0}}$. For all $(\xi, A_\xi), (\zeta, A_\zeta) \in \mathfrak{k}_{\bar{0}}$ we have the bracket:

$$[(\xi, A_\xi), (\zeta, A_\zeta)] = (A_\xi \zeta - A_\zeta \xi, [A_\xi, A_\zeta] + R(\xi, \zeta)) \qquad (2)$$

Killing superalgebras as filtered deformations

Let $\mathrm{ev}_x^{\bar 0}:\mathfrak{k}_{\bar 0}\longrightarrow V$ be the composition of evaluation at x with projection onto $V=T_xM$ and set $\mathrm{Im}(\mathrm{ev}_x^{\bar 0})=V'\subseteq V$. Let also $\mathfrak{h}=\mathrm{Ker}(\mathrm{ev}_x^{\bar 0})$: it consists of the elements of $\mathfrak{k}_{\bar 0}$ of the form $(0,A)\in V\oplus\mathfrak{so}(V)$ at x and defines a subspace of $\mathfrak{so}(V)$. There is a short exact sequence of v.s.

$$0\longrightarrow\mathfrak{h}\longrightarrow\mathfrak{k}_{\bar 0}\xrightarrow{\mathrm{ev}_x^{\bar 0}}V'\longrightarrow 0$$

and hence a (non-canonical) v.s. isomorphism $\mathfrak{k}_{\bar 0}\cong\mathfrak{a}_{\bar 0}:=V'\oplus\mathfrak{h}$. Geometrically this is choosing a splitting, i.e., for every $v\in V'$ a Killing v.f. $\xi\in\mathfrak{k}_{\bar 0}$ with $\xi(x)=v$:

$$v\mapsto(v,X_v)\in\mathfrak{k}_{\bar 0}\subset V'\oplus\mathfrak{so}(V)\ ,$$

for some linear map $X:V'\to\mathfrak{so}(V)$. Translating the Lie bracket (2) of $\mathfrak{k}_{\bar 0}$ on $\mathfrak{a}_{\bar 0}$ gives

$$\begin{aligned}
[A,B]&=AB-BA & \delta(A,v)&=[A,X_v]-X_{Av}\\
[A,v]&=Av+\delta(A,v) & \alpha(v,w)&=X_vw-X_wv\\
[v,w]&=\alpha(v,w)+\rho(v,w) & \rho(v,w)&=[X_v,X_w]-X_{\alpha(v,w)}+R(v,w)
\end{aligned}$$

for $A,B\in\mathfrak{h}$, $v,w\in V'$. This shows that $\mathfrak{k}_{\bar 0}$ is a filt. def. of $\mathfrak{a}_{\bar 0}=V'\oplus\mathfrak{h}\subset\mathfrak{p}_{\bar 0}$.

We now add spinor fields to this construction

Killing superalgebras as filtered deformations

Let $\mathscr{E} = \mathscr{E}_{\bar{0}} \oplus \mathscr{E}_{\bar{1}}$ be the vector bundle where $\mathscr{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM)$ and $\mathscr{E}_{\bar{1}} = S(M)$. On $\mathscr{E}$ we have a connection D defined on $\mathscr{E}_{\bar{0}}$ as before and by the superconnection on $\mathscr{E}_{\bar{1}}$. Then the Killing superalgebra $\mathfrak{k}$ is given by

$$\mathfrak{k}_{\bar{0}} = \{(\xi, A) \in \Gamma(\mathscr{E}_{\bar{0}}) \mid D(\xi, A) = 0 \quad \text{and} \quad \nabla_\xi F + A(F) = 0\} ,$$
$$\mathfrak{k}_{\bar{1}} = \{\epsilon \in \Gamma(\mathscr{E}_{\bar{1}}) \mid D\epsilon = 0\} ,$$

where $A(F)$ is the natural action of $\mathfrak{so}(TM)$ on 4-forms. Again an element of $\mathfrak{k}$ is determined by the value at a point $x \in M$ of the corresponding parallel section of $\mathscr{E}$ and $\mathfrak{k}$ defines a subspace of $T_xM \oplus S(M)_x \oplus \mathfrak{so}(T_xM)$. Introduce the notation

$$(V, \eta) = (T_xM, g_x) , \qquad S = S(M)_x , \qquad \mathfrak{so}(V) = \mathfrak{so}(T_xM) ,$$

and identify $\mathfrak{k}$ with a (not yet $\mathbb{Z}$-graded) subspace of $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$. Using $\mathrm{ev}_x^{\bar{0}} : \mathfrak{k}_{\bar{0}} \longrightarrow V$, the evaluation $\mathrm{ev}_x^{\bar{1}} : \mathfrak{k}_{\bar{1}} \longrightarrow S$ of $\mathfrak{k}_{\bar{1}}$ at $x \in M$ and choosing a splitting of $\mathrm{ev}_x^{\bar{0}}$ as before, we may identify $\mathfrak{k} \cong \mathfrak{a} := V' \oplus S' \oplus \mathfrak{h}$ as v.s. and translate the Lie brackets of $\mathfrak{k}$ to $\mathfrak{a}$.

Killing superalgebras as filtered deformations

The Lie brackets of $\mathfrak{k}$ are

$$[(\xi, A_\xi), (\zeta, A_\zeta)] = (A_\xi \zeta - A_\zeta \xi, [A_\xi, A_\zeta] + R(\xi, \zeta)) \tag{3}$$

$$\begin{aligned} [(\xi, A_\xi), \epsilon] &= \nabla_\xi \epsilon + A_\xi(\epsilon) \\ &= \tfrac{1}{24}(\xi \cdot F - 3F \cdot \xi) \cdot \epsilon + A_\xi(\epsilon) \end{aligned} \tag{4}$$

$$\begin{aligned} [\epsilon, \epsilon] &= (\kappa(\epsilon, \epsilon), -\nabla \kappa(\epsilon, \epsilon)) \\ &= (\kappa(\epsilon, \epsilon), \gamma^F(\epsilon, \epsilon)) \end{aligned} \tag{5}$$

where (ξ, A_ξ), $(\zeta, A_\zeta) \in \mathfrak{k}_{\bar{0}}$, $\epsilon \in \mathfrak{k}_{\bar{1}}$ and the map $\gamma^F(\epsilon, \epsilon) \in \mathfrak{so}(TM)$ is defined by

$$\gamma^F(\epsilon, \epsilon)(X) = -\tfrac{1}{12}\kappa\big((X \cdot F - 3F \cdot X) \cdot \epsilon, \epsilon\big)$$

for all $X \in TM$. The result then follows from translating these brackets on $\mathfrak{a}$. (The red terms contribute to components of Lie bracket of positive filtration degree, independently of the choice of splitting.) ■

Observations/Questions

- *Idea*: instead of studying bkgds, we set to study filtered deformations.
- Is every filtered deformation realizable as a Killing superalgebra? There might be a counterexample....
- Any $d = 11$ Lorentzian mnfd (M, g) with *closed* $F \in \Omega^4(M)$ has an associated Killing superalgebra. Should filtered deformations be further constrained by bosonic eqs of supergravity?

Spencer cohomology

Deformations of an algebraic structure are governed (at least to first order) by a cohomology theory. In this case, this is *Spencer cohomology*, a bi-graded refinement of the usual Chevalley-Eilenberg cohomology of a Lie (super)algebra and its adjoint representation to case of $\mathbb{Z}$-graded Lie (super)algebras $\mathfrak{a} = \oplus \mathfrak{a}_j$.

Let us start with Poincaré superalgebra $\mathfrak{a} = \mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$, where $\mathfrak{p}_{-2} = V$, $\mathfrak{p}_{-1} = S$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$. Let $\mathfrak{p}_- = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1}$ be the negatively graded part of $\mathfrak{p}$. The cochains of the Spencer complex of $\mathfrak{p}$ are linear maps $\Lambda^\bullet \mathfrak{p}_- \to \mathfrak{p}$, where $\Lambda^\bullet$ is meant here in the super sense. We extend degree of $\mathfrak{p}$ to the space of q-cochains $C^q(\mathfrak{p}_-, \mathfrak{p}) = \mathfrak{p} \otimes \Lambda^q \mathfrak{p}_-^*$ for all q by declaring that $\mathfrak{p}_j^*$ has degree $-j$. It follows that $C^q(\mathfrak{p}_-, \mathfrak{p}) = \oplus_{p \in \mathbb{Z}} C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$ where $C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$ is space of q-cochains with $\deg = p$. The spaces in the complexes of even cochains of small degree $\deg = p$ are given in following Table (although for $p = 4$ there are cochains also for $q = 5, 6$ which are omitted).

Even q-cochains of small degree

	q				
p	0	1	2	3	4
0	$\mathfrak{so}(V)$	$S \to S$ $V \to V$	$\odot^2 S \to V$		
2		$V \to \mathfrak{so}(V)$	$\Lambda^2 V \to V$ $V \otimes S \to S$ $\odot^2 S \to \mathfrak{so}(V)$	$\odot^3 S \to S$ $\odot^2 S \otimes V \to V$	$\odot^4 S \to V$
4			$\Lambda^2 V \to \mathfrak{so}(V)$	$\odot^2 S \otimes V \to \mathfrak{so}(V)$ $\Lambda^2 V \otimes S \to S$ $\Lambda^3 V \to V$	$\odot^4 S \to \mathfrak{so}(V)$ $\odot^3 S \otimes V \to S$

The Spencer differential

$$\partial : C^{p,q}(\mathfrak{p}_-, \mathfrak{p}) \to C^{p,q+1}(\mathfrak{p}_-, \mathfrak{p})$$

is the Chevalley–Eilenberg differential for the Lie superalgebra $\mathfrak{p}_-$ relative to its module $\mathfrak{p}$ with respect to adjoint action. We are interested in $q = 2$, $p = 2, 4$.

Even q-cochains of small degree

	q				
p	0	1	2	3	4
0	$\mathfrak{so}(V)$	$S \to S$ $V \to V$	$\odot^2 S \to V$		
2		$V \to \mathfrak{so}(V)$	$\Lambda^2 V \to V$ $V \otimes S \to S$ $\odot^2 S \to \mathfrak{so}(V)$	$\odot^3 S \to S$ $\odot^2 S \otimes V \to V$	$\odot^4 S \to V$
4			$\Lambda^2 V \to \mathfrak{so}(V)$	$\odot^2 S \otimes V \to \mathfrak{so}(V)$ $\Lambda^2 V \otimes S \to S$ $\Lambda^3 V \to V$	$\odot^4 S \to \mathfrak{so}(V)$ $\odot^3 S \otimes V \to S$

For example if $q = 1$ the Spencer differential is:

$$\partial : C^{p,1}(\mathfrak{p}_-, \mathfrak{p}) \to C^{p,2}(\mathfrak{p}_-, \mathfrak{p})$$

$$\partial\zeta(X,Y) = [X, \zeta(Y)] - (-1)^{|X||Y|}[Y, \zeta(X)] - \zeta([X,Y]) \ ,$$

where $X, Y \in \mathfrak{p}_-$ and $\zeta \in C^{p,1}(\mathfrak{p}_-, \mathfrak{p})$.

Spencer cohomology and Killing spinors

Thm[Figueroa-O'Farrill, A.S. '17]

(i) $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$;

(ii) $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^4 V$.

We recover 4-form of $d = 11$ supergravity through cohomology. But there is more. If we look at the β-component (remember $\beta : V \otimes S \longrightarrow S$) of the normalized Spencer cocycle, we find that $\beta(v, s) = v \cdot \varphi \cdot s - 3\varphi \cdot v \cdot s$, for $\varphi \in \Lambda^4 V$.

We recognize it as the zero degree piece of the gravitino variation: in other words, it indicates what relevant superconnection and Killing spinor eqs are.

Observations

- In $d = 11$ supergravity, the relevant Killing spinor equation encodes all the information about bosonic supergravity. Indeed the *Clifford trace*

$$\sum_i e^i \cdot \mathcal{R}(e_i, -) : TM \longrightarrow \mathrm{End}(S(M))$$

  of curvature $\mathcal{R} : \Lambda^2 TM \longrightarrow \mathrm{End}(S(M))$ of the superconnection D vanishes iff $dF = 0$ and the field equations are satisfied [Gauntlett, Pakis '03]. So, in a very real sense, Spencer cohomology knows about $d = 11$ supergravity;

- We used same approach in other dimensions and looked for Killing spinor "cohomological" equations. These equations are well-suited to construct Killing superalgebras and sometimes they are new.

Maximally supersymmetric backgrounds in $d = 11$

We classified maximally supersymmetric filtered deformations in $d = 11$: filtered deformations $\mathfrak{g}$ of subalgebras $\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0$ of $\mathfrak{p}$ with $\mathfrak{a}_{-2} = V$, $\mathfrak{a}_{-1} = S$ and $\mathfrak{a}_0 = \mathfrak{h}$ a subalgebra of $\mathfrak{so}(V)$. The fact that $\mathfrak{a}_{-1} = S$ means we have maximal supersymmetry, whereas the fact that $V' = V$ (which is forced) means we are describing (locally) homogeneous geometries. We bootstrapped the computation of $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ from that of $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ and obtained the following results.

Thm[Figueroa-O'Farrill, A.S. '17] The *maximally supersymmetric filt. def.* are:

(i) $\mathfrak{p}$ itself for Minkowski spacetime $\mathbb{M}^{1,10}$ with $F = 0$;

(ii) $\mathrm{osp}(8|4)$ for $AdS_4 \times S^7$ with $F \propto \mathrm{dvol}_{AdS_4}$ and $S_{S^7} = -\frac{7}{8} S_{AdS_4}$ [Freund, Rubin '80];

(iii) $\mathrm{osp}(6,2|4)$ for $S^4 \times AdS_7$ with $F \propto d\,\mathrm{vol}_{S^4}$ and $S_{AdS_7} = -\frac{7}{8} S_{S^4}$ [Pilch, van Nieuwenhuizen, Townsend '84];

(iv) Killing superalgebra of pp-wave with F lightlike [Kowalski-Glikman '84].

In all cases $\mathfrak{h} = \mathfrak{so}(V) \cap \mathrm{stab}(\varphi)$ where $\varphi = F|_x \in \Lambda^4 V$.

Maximally supersymmetric backgrounds in $d = 4$

Thm[de Medeiros, Figueroa-O'Farrill, A.S. '16] If $d = 4$ then $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$ and

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^0 V \oplus \Lambda^4 V \oplus \Lambda^1 V .$$

The Spencer cocycle indicates Killing spinor eqs for Lorentzian 4-dimensional spin manifolds: $D_X \epsilon = \nabla_X \varepsilon - X \cdot (a + b) \cdot \epsilon + \frac{1}{2}(X \cdot \varphi + 3\varphi \cdot X) \cdot \mathrm{vol} \cdot \epsilon = 0$, where $a \in \Omega^0(M)$, $b \in \Omega^4(M)$, $\varphi \in \Omega^1(M)$ are the auxiliary fields in $d = 4$ supergravity. The *maximally supersymmetric filt. def.* split into three families:

(i) $\mathfrak{p}$ itself for Minkowski spacetime;

(ii) In this case $\mathfrak{h} = \mathfrak{so}(V)$, there exist $a, b \in \mathbb{R}$ with $a^2 + b^2 > 0$ and associated homogeneous Lorentzian mnfd $(M = G_{\bar{0}}/H, g)$, $Lie(G) = \mathfrak{g}_{\bar{0}}$, $Lie(H) = \mathfrak{h}$ is locally isometric to AdS_4;

(iii) In this case there is nonzero $\varphi \in V$, $\mathfrak{h}$ is a subalgebra of $\mathfrak{so}(V) \cap \mathrm{stab}(\varphi)$ and the geometry is that of a Lie group with bi-invariant metric:

- If φ is spacelike then (M, g) is locally isometric to $\mathrm{AdS}_3 \times \mathbb{R}$;
- If φ is timelike then (M, g) is locally isometric to $\mathbb{R} \times S^3$;
- If φ is lightlike then we have the so-called Nappi-Witten group NW_4.

Maximally supersymmetric backgrounds in $d = 6$

Thm[de Medeiros, Figueroa-O'Farrill, A.S. '18] If $d = 6$ then $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$ and

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^3 V \oplus (V \otimes \mathfrak{sp}(1)) \ .$$

The Spencer cocycle indicates *new* Killing spinor equations for Lorentzian 6-dimensional spin manifolds:

$$D_X \epsilon = \nabla_X \varepsilon - \iota_X H \cdot \varepsilon + 3\varphi(X) \cdot \varepsilon - X \wedge \varphi \cdot \varepsilon = 0 \ ,$$

where $H \in \Omega^3(M)$, $\varphi \in \Omega^1(M; \mathfrak{sp}(1))$. The *maximally supersymmetric filt. def.* are given by three families:

(i) Minkowski spacetime with $H = \varphi = 0$;

(ii) If $\varphi \neq 0$ and $H = 0$ we have $\mathrm{AdS}_5 \times \mathbb{R}$, $\mathbb{R} \times S^5$ and a conformally flat Lorentzian symmetric plane wave;

(iii) If $H \neq 0$ and $\varphi = 0$ then H is parallel Cartan 3-form of a 6-dimensional Lie group with bi-invariant Lorentzian metric. If H is in addition *self-dual*, these are the maximally susy bkgds of $d = 6$ supergravity.

Exercises on Riemannian geometry

Let $\xi \in \mathfrak{X}(M)$ and $A_\xi \in \Gamma(\mathrm{End}(TM))$ be given by $A_\xi(X) = -\nabla_X \xi$. Show that:

- ξ is a Killing vector field iff $A_\xi \in \Gamma(\mathfrak{so}(TM))$;
- in this case then A_ξ satisfies the Killing's identity $\nabla_X A_\xi = R(X, \xi)$.

If ξ is a Killing vector field, then we have Kosmann's spinorial Lie derivative $\mathcal{L}_\xi \epsilon = \nabla_\xi \epsilon + A_\xi(\epsilon)$ acting on any spinor field $\epsilon \in \Gamma(S(M))$. Show that:

- $[\mathcal{L}_\xi, \mathcal{L}_\eta]\epsilon = \mathcal{L}_{[\xi,\eta]}\epsilon$
- $\mathcal{L}_\xi(X \cdot \epsilon) = [\xi, X] \cdot \epsilon + X \cdot \mathcal{L}_\xi \epsilon$
- $\mathcal{L}_\xi(f\epsilon) = \xi(f)\epsilon + f\mathcal{L}_\xi \epsilon$
- $[\mathcal{L}_\xi, \nabla_X]\epsilon = \nabla_{[\xi,X]}\epsilon$

Thanks!