An introduction to supergravity in 11 dimensions

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Partly based on joint works with P. de Medeiros and J. Figueroa-O'Farrill

Plan of the series of talks:

First part:

- $d = 11$ Supergravity
- Detour on Lie superalgebras (including the Poincaré superalgebra)
- Killing spinor equations and Killing superalgebras
- Brane solutions

Second part:

- Homogeneity theorem
- Filtered deformations
- Spencer cohomology and Killing spinors
- Maximally supersymmetric backgrounds

Third part:

- Clifford algebras
- PDEs on spinor bilinears
- Highly supersymmetric backgrounds

An incomplete history of supersymmetry

- 1960s: Is there a group larger than the Poincaré group whose irreps contain irreps of the Poincaré group with different masses and spin?
- 1967: No (Coleman–Mandula);
- 1975: Yes, more or less (Haag–Lopuszanski–Sohnius). It required the introduction of the Poincaré supergroup, whose irreps break up into particles with the same mass but different spin (bosons/fermions);
- 1976: construction of supergravity in $d = 4$ (Ferrara–Freedman–van Nieuwenhuizen);
- 1978: there exists irrep of Poincaré supergroup in $d = 11$ with field content (g, A, Ψ) (Nahm);
- 1978: the theory of 11-dimensional supergravity predicted by Nahm was constructed by Cremmer–Julia–Scherk.

An incomplete history of supersymmetry

The action functional they discovered is given by the sum

$$I = I_{EH} + I_M + I_{CS} + \cdots = \tfrac{1}{2}\int_M S \,\mathrm{dvol} + \tfrac{1}{4}\int_M F \wedge \star F + \tfrac{1}{12}\int_M F \wedge F \wedge A + O(\Psi)$$

where:

- (M, g) is an *11-dimensional Lorentzian spin manifold*;
- $F \in \Omega^4(M)$ is a *closed 4-form* on M (locally $F = dA$);
- $\Psi \in \Gamma(T^*M \otimes S(M))$ is the *gravitino* (here $S(M)$ is spinor bundle).

It is one of the crown jewels of modern theoretical physics. The action is invariant under local diffeomorphisms and also "supersymmetries", special transformations that are spinorial analogues of classical diffeomorphisms between manifolds.

An incomplete history of supersymmetry

The bosonic field equations of 11-dimensional supergravity are a very interesting system of coupled PDEs:

$$\begin{aligned} d \star F &= \tfrac{1}{2} F \wedge F && \textit{``Maxwell type eqs''} \\ \operatorname{Ric}(X,Y) &= \tfrac{1}{2} g(\imath_X F, \imath_Y F) - \tfrac{1}{6} \|F\|^2 g(X,Y) && \textit{``Einstein type eqs''} \end{aligned} \tag{1}$$

The transformation of the gravitino Ψ under a supersymmetry $\epsilon \in \Gamma(S(M))$ takes the form $\delta_\epsilon \Psi = D\epsilon + O(\Psi)$, where D is the connection on the spinor bundle given by

$$D_X \epsilon = \nabla_X \epsilon - \tfrac{1}{24}\big(X \cdot F - 3F \cdot X\big) \cdot \epsilon \tag{2}$$

Goal of these lectures

- understand these notions and their interplay;
- see the main properties of supergravity backgrounds (construction of a Lie superalgebra generated from spinor fields, structural results for highly supersymmetric backgrounds, etc...) together with the most important examples;
- along the way...a bit of spin geometry and Lie superalgebra theory (Kac's classification of simple Lie superalgebras, the Poincaré superalgebra, etc...).

Detour on Lie superalgebra theory

Def. A *Lie superalgebra* is a vector space of the form

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

endowed with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

- $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subset \mathfrak{g}_{\bar{0}}$, $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{1}}$, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{0}}$;
- for any homogeneous X, Y (i.e. with $X \in \mathfrak{g}_{\bar{i}}$, $Y \in \mathfrak{g}_{\bar{j}}$)

$$[X,Y] = -(-1)^{|X||Y|}[Y,X] \qquad \left(|X| = \text{ parity of } X = \begin{cases} 0 \\ 1 \end{cases} \right)$$

- for any homogeneous X, Y, Z

$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]]$$

Detour on Lie superalgebra theory

Equivalently, a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is the datum of:

- a *Lie algebra* $\mathfrak{g}_{\bar{0}}$;
- a *representation* $\rho : \mathfrak{g}_{\bar{0}} \to \mathfrak{gl}(\mathfrak{g}_{\bar{1}})$ of $\mathfrak{g}_{\bar{0}}$;
- a *symmetric bilinear map* $\kappa : \odot^2 \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ that is $\mathfrak{g}_{\bar{0}}$-equivariant;
- a *compatibility condition* for ρ and κ:

$$\rho\big(\kappa(X, X)\big)X = 0$$

for all $X \in \mathfrak{g}_{\bar{1}}$.

Detour on Lie superalgebra theory

Ex 0. The general linear Lie superalgebra $\mathfrak{gl}(m|n)$ is defined as follows:

$$\mathbb{C}^{m|n} = \mathbb{C}^m \oplus \mathbb{C}^n \text{ (decomposition into even and odd parts)}$$

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

$$[L, L'] = L \circ L' - (-1)^{|L||L'|} L' \circ L, \text{ for example}$$

$$[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix}] = \begin{pmatrix} BC' + B'C & 0 \\ 0 & CB' + C'B \end{pmatrix}$$

The *supertrace* of L is defined as $\mathrm{str}(L) = \mathrm{tr}(A) - \mathrm{tr}(D)$ and the special linear Lie superalgebra as $\mathfrak{sl}(m|n) = \{L \in \mathfrak{gl}(m|n) \mid \mathrm{str}(L) = 0\}$. If $m = n$, then Id is central in $\mathfrak{sl}(m|n)$ and one also considers $\mathfrak{psl}(m|n) = \mathfrak{sl}(m|n)/\mathbb{C}\,\mathrm{Id}$.

Detour on Lie superalgebra theory

Ex 1. Orthosymplectic Lie superalgebra:

$\mathbb{C}^{m|n}$ together with an even non-degenerate supersymmetric bilinear form

(with, say, Gram matrix $\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & J \end{pmatrix}$)

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A^t + A = 0, D^t J + JD = 0, B^t = JC \right\}$$

Ex 2. Periplectic Lie superalgebra:

$\mathbb{C}^{m|m}$ together with an odd non-degenerate supersymmetric bilinear form

(with, say, Gram matrix $\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$)

$$\mathfrak{pe}(m) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid B^t = B, C = -C^t \right\}, \quad \mathfrak{spe}(m) = \mathfrak{pe}(m) \cap \mathfrak{sl}(m|m)$$

Ex 3. Queer Lie superalgebra:

$\mathbb{C}^{m|m}$ together with an odd complex structure (with, say, matrix $\begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$)

$$\mathfrak{q}(m) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}$$

$$\mathfrak{sq}(m) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid \mathrm{tr}(B) = 0 \right\}$$

$$\mathfrak{psq}(m) = \mathfrak{sq}(m)/\mathbb{C}\,\mathrm{Id}$$

Ex 4. Lie superalgebra of all vector fields on a purely odd supermanifold:

$$W(m) = \mathrm{Der}\ \Lambda^\bullet \mathbb{C}^m = \left\{ \sum_{\alpha=1}^{m} P_\alpha(\theta^1, \cdots, \theta^m)\partial_{\theta^\alpha} \mid P_\alpha \in \Lambda^\bullet \mathbb{C}^m \right\}$$

Detour on Lie superalgebra theory

Finite-dimensional simple (complex) Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ were classified by V. Kac in 1977 and split into two main families:

- *classical*, for which the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible;
- *Cartan Lie superalgebras* $W(m)$, $S(m)$, $\widetilde{S}(m)$, $H(m)$, analogs to simple Lie algebras of vector fields.

Classical Lie superalgebras consist in turn of the strange $\mathfrak{pe}(m)$ and $\mathfrak{psq}(m)$ and of the Lie superalgebras with a non-degenerate “Killing form”:

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	$\mathfrak{g}_{\bar{1}}$
$\mathfrak{sl}(m\|n)$ $m, n \geq 1$	$\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbb{C}$	$\left(\mathbb{C}^m \otimes (\mathbb{C}^n)^*\right) \oplus \left((\mathbb{C}^m)^* \otimes \mathbb{C}^n\right)$
$\mathfrak{osp}(m\|2n)$ $m, n \geq 1$	$\mathfrak{so}(m) \oplus \mathfrak{sp}(2n)$	$\mathbb{C}^m \otimes \mathbb{C}^{2n}$
$\mathfrak{osp}(4\|2;\alpha)$ $\alpha \neq 0, \pm 1, \infty$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
$F(3\|1)$	$\mathfrak{so}(7) \oplus \mathfrak{sl}(2)$	$\mathbb{S} \otimes \mathbb{C}^2$
$G(3)$	$G_2 \oplus \mathfrak{sl}(2)$	$\mathbb{C}^7 \otimes \mathbb{C}^2$

Exercises on Lie superalgebra theory

The *Killing form* of $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is defined as $b(X, Y) = \mathrm{str}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. Show that:

- b is even: $b(\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}) = 0$ if $\bar{i} + \bar{j} = \bar{1}$;
- b is supersymmetric: $b(X, Y) = (-1)^{|X||Y|} b(Y, X)$;
- if $\mathfrak{g} = \mathfrak{sl}(m|n)$, then $b(X, Y) = 2(m-n)\,\mathrm{str}(X \cdot Y)$;
- if $\mathfrak{g} = \mathfrak{osp}(m|2n)$, then $b(X, Y) = (m-2n-2)\,\mathrm{str}(X \cdot Y)$;
- if $\mathfrak{g} = \mathfrak{pe}(m)$, then $b = 0$ (do <u>not</u> compute b explicitly!).

Show that:

- Levi Thm is not true in general (hint: $\mathfrak{sl}(m|m) \neq \mathfrak{psl}(m|m) \oplus \mathbb{C}\,\mathrm{Id}$);
- derivations are not all inner (hint: $\mathrm{der}(\mathfrak{psl}(m|m)) = \mathfrak{pgl}(m|m)$);
- semisimple Lie superalgebras are not necessarily direct sum of simple ideals (hint: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathbb{C}\partial_\theta \oplus \mathfrak{l} \oplus \mathfrak{l}\theta$ for $\mathfrak{l}$ simple Lie algebra).

The Poincaré superalgebra

Let (V, η) be a real d-dimensional Lorentzian vector space. The double cover $\mathrm{Spin}(V)$ of the special orthogonal group $SO(V)$ can be identified with a particular group of invertible elements in the so-called Clifford algebra $\mathcal{C}\ell(V)$. Such algebra is isomorphic to the vector space $\mathcal{C}\ell(V) = \Lambda^\bullet V = \mathbb{R} \oplus V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^d V$, but with a modified product.

$\mathcal{C}\ell(V)$ always admits a representation as a suitable matrix algebra, for example $\mathcal{C}\ell(V) \cong \mathbb{R}(32) \oplus \mathbb{R}(32)$ if $d = 11$ and $\mathcal{C}\ell(V) \cong \mathbb{R}(4)$ if $d = 4$. Since $\mathrm{Spin}(V) \subset \mathcal{C}\ell(V)$, we have a representation of $\mathrm{Spin}(V)$ (and hence of $\mathfrak{so}(V) = Lie(\mathrm{Spin}(V)) \cong \Lambda^2 V$) by means of matrices acting on the *spinor representation* S (for example, $S = \mathbb{R}^{32}$ if $d = 11$, $S = \mathbb{R}^4$ if $d = 4$).

Since $\mathcal{C}\ell(V) = \Lambda^\bullet V$, we also have that polyvectors on V correspond to matrices acting on S. The actions of such matrices are called *Clifford products between elements of $\Lambda^\bullet V$ and elements of S.*

The Poincaré superalgebra

Let $d = 11$ for concreteness. On S there is an $\mathfrak{so}(V)$-invariant symplectic form $\langle -, - \rangle$ such that $\langle v \cdot s_1, s_2 \rangle = - \langle s_1, v \cdot s_2 \rangle$ for all $v \in V$, $s_1, s_2 \in S$. The transpose of Clifford action $V \otimes S \to S$ gives a way to square spinors: a map $\kappa : \odot^2 S \to V$ known as *Dirac current*:

$$\eta(\kappa(s,s), v) = \langle s, v \cdot s \rangle \qquad v \in V \, , \;\; s \in S \tag{3}$$

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Def. The *Poincaré superalgebra* is the Lie superalgebra $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$ where

(i) $\mathfrak{p}_{\bar{0}} = \mathfrak{so}(V) \oplus V$;

(ii) $\mathfrak{p}_{\bar{1}} = S$;

(iii) the nonzero Lie brackets are:

$$[A,B] = AB - BA\,, \quad [A,s] = As\,, \quad [A,v] = Av\,, \quad [s,s] = \kappa(s,s)\,,$$

for all $A, B \in \mathfrak{so}(V)$, $s \in S$, $v \in V$.

Killing superalgebras

Let (M, g, F) be Lorentzian mnfd (M, g), $\dim M = 11$, with closed $F \in \Omega^4(M)$ and endowed with a spinor bundle $S(M) \longrightarrow M$ (the fiber $S(M)_x \cong S = \mathbb{R}^{32}$). The bosonic equations of supergravity are two coupled PDEs [Cremmer-Julia-Scherk '78]:

$$\left.\begin{aligned} \mathrm{Ric}(X,Y) &= \tfrac{1}{2} g(i_X F, i_Y F) - \tfrac{1}{6} g(X,Y)|F|^2 \\ d * F &= \tfrac{1}{2} F \wedge F \end{aligned}\right\} \quad (*)$$

Supersymmetry transf. $\delta_\epsilon \Psi = D\epsilon + O(\Psi)$ of the gravitino Ψ gives the so-called *superconnection* on $S(M)$:

$$D_X \epsilon = \nabla_X \epsilon - \tfrac{1}{24}\big(X \cdot F - 3F \cdot X\big) \cdot \epsilon \ ,$$

for all v.f. X and sections ϵ of $S(M)$.

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for all v.f. X and sections ϵ of $S(M)$.

Def. A *symmetry* of a solution of $(*)$ is a pair (ξ, ϵ) given by

(i) a Killing vector field for g preserving F, *i.e.*, a v.f. ξ s.t $\mathcal{L}_\xi g = \mathcal{L}_\xi F = 0$;

(ii) a (generalized) Killing spinor, *i.e.*, a section ϵ of $S(M)$ s.t. $D\epsilon = 0$.

Killing superalgebras

Thm[Figueroa-O'Farrill, Meessen, Philip '05] The $\mathbb{Z}_2$-graded v.s. $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of symmetries of (M, g, F) has a natural structure of Lie superalgebra, called the *Killing superalgebra*.

Ex. (M, g) Minkowski, $F = 0$, $D = \nabla$ then $\mathfrak{k}_{\bar{1}} \cong S$, $\mathfrak{k}_{\bar{0}} \cong \mathfrak{so}(V) \oplus V$ and $\mathfrak{k} = \mathfrak{p}$.

Idea of the proof.

We will use that the L-C connection is compatible with symplectic form on $S(M)$ and Clifford multiplication, and employ combinatorial identities for the Clifford algebra together with PDEs associated to differential forms. Now

$$\mathfrak{k}_{\bar{0}} = \{\xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0\}$$

is clearly a Lie algebra, and we have a putative bracket $[\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}] \subset \mathfrak{k}_{\bar{0}}$ given by the Dirac current. What about the action of $\mathfrak{k}_{\bar{0}}$ on $\mathfrak{k}_{\bar{1}}$?

Idea of the proof – continued

Kosmann's *spinorial Lie derivative* is defined for all Killing vector fields ξ and spinor fields ϵ by

$$\mathcal{L}_\xi \epsilon = \nabla_\xi \epsilon + A_\xi(\epsilon) \ ,$$

where $A_\xi \in \Gamma(\mathfrak{so}(TM))$ is the tensor defined by $A_\xi(X) = -\nabla_X \xi$. (As kind of motivation, note $\mathcal{L}_\xi X = \nabla_\xi X + A_\xi(X) = \nabla_\xi X - \nabla_X \xi = [\xi, X]$.)

Exercise:

- $[\mathcal{L}_\xi, \mathcal{L}_\eta]\epsilon = \mathcal{L}_{[\xi,\eta]}\epsilon$
- $\mathcal{L}_\xi(X \cdot \epsilon) = [\xi, X] \cdot \epsilon + X \cdot \mathcal{L}_\xi \epsilon$
- $\mathcal{L}_\xi(f\epsilon) = \xi(f)\epsilon + f\mathcal{L}_\xi \epsilon$
- $[\mathcal{L}_\xi, \nabla_X]\epsilon = \nabla_{[\xi,X]}\epsilon$

It follows that $\mathfrak{k}_{\bar{0}}$ acts on $\mathfrak{k}_{\bar{1}}$ via the spinorial Lie derivative:

Idea of the proof – continued

for all $\xi \in \mathfrak{k}_{\bar{0}}$, $\epsilon \in \mathfrak{k}_{\bar{1}}$ and $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
\nabla_X(\mathcal{L}_\xi \epsilon) &= \mathcal{L}_\xi(\nabla_X \epsilon) - \nabla_{[\xi,X]}\epsilon \\
&= \mathcal{L}_\xi\Big(\tfrac{1}{24}(X\cdot F - 3F\cdot X)\,\cdot\,\epsilon\Big) - \tfrac{1}{24}\Big([\xi,X]\cdot F - 3F\cdot[\xi,X]\Big)\,\cdot\,\epsilon \\
&= \tfrac{1}{24}(X\cdot \mathcal{L}_\xi F - 3\mathcal{L}_\xi F\cdot X)\,\cdot\,\epsilon + \tfrac{1}{24}(X\cdot F - 3F\cdot X)\,\cdot\,\mathcal{L}_\xi\epsilon \\
&= \tfrac{1}{24}(X\cdot F - 3F\cdot X)\,\cdot\,\mathcal{L}_\xi\epsilon\ .
\end{aligned}$$

It is also easy to see that the Dirac current is equivariant:

$$\begin{aligned}
g\big(\mathcal{L}_\xi\kappa(\epsilon,\epsilon), X\big) &= \xi\big(g(\kappa(\epsilon,\epsilon),X)\big) - g\big(\kappa(\epsilon,\epsilon),\mathcal{L}_\xi X\big) \\
&= \xi\big(\langle\epsilon, X\cdot\epsilon\rangle\big) - \langle\epsilon, \mathcal{L}_\xi X\cdot\epsilon\rangle \\
&= \langle\mathcal{L}_\xi\epsilon, X\cdot\epsilon\rangle + \langle\epsilon, X\cdot\mathcal{L}_\xi\epsilon\rangle \\
&= g\big(\kappa(\mathcal{L}_\xi\epsilon,\epsilon),X\big) + g\big(\kappa(\epsilon,\mathcal{L}_\xi\epsilon),X\big)\ .
\end{aligned}$$

It remains to show that κ takes values in $\mathfrak{k}_{\bar{0}}$ and the odd-odd-odd identity.

Idea of the proof – continued

First of all, for $\xi = \kappa(\epsilon, \epsilon)$, we have

$$
\begin{aligned}
g\big(\nabla_X \xi, Y\big) &= g\big(\nabla_X \kappa(\epsilon,\epsilon), Y\big) = 2g\big(\kappa(\nabla_X \epsilon, \epsilon), Y\big) \\
&= \tfrac{2}{24} g\big(\kappa(X \cdot F \cdot \epsilon, \epsilon), Y\big) - 3\tfrac{2}{24} g\big(\kappa(F \cdot X \cdot \epsilon, \epsilon), Y\big) \\
&= \tfrac{2}{24} \langle X \cdot F \cdot \epsilon, Y \cdot \epsilon \rangle - 3\tfrac{2}{24} \langle F \cdot X \cdot \epsilon, Y \cdot \epsilon \rangle \\
&= -\tfrac{2}{24} \langle Y \cdot X \cdot F \cdot \epsilon, \epsilon \rangle + 3\tfrac{2}{24} \langle Y \cdot F \cdot X \cdot \epsilon, \epsilon \rangle \\
&= -\tfrac{2}{24} \langle \imath_Y \imath_X F \cdot \epsilon, \epsilon \rangle - \tfrac{2}{24} \langle Y \wedge X \wedge F \cdot \epsilon, \epsilon \rangle \\
&\quad - 3\tfrac{2}{24} \langle \imath_Y \imath_X F \cdot \epsilon, \epsilon \rangle + 3\tfrac{2}{24} \langle Y \wedge F \wedge X \cdot \epsilon, \epsilon \rangle
\end{aligned}
$$

where we used $\odot^2 S \cong \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V$ and $\Lambda^2 S \cong \Lambda^0 V \oplus \Lambda^3 V \oplus \Lambda^4 V$. The expression is clearly skew in X and Y, hence $\nabla \xi$ is a section of $\mathfrak{so}(TM)$ and ξ a Killing vector field! The rest is the difficult part of the proof.

Idea of the proof – continued

To prove that $\xi = \kappa(\epsilon, \epsilon)$ preserves also F, we shall consider the 2-form $\omega^{(2)} \in \Omega^2(M)$ constructed quadratically out of ϵ:

$$\omega^{(2)}(X, Y) = \langle \epsilon, X \wedge Y \cdot \epsilon \rangle \ .$$

Since ϵ is a Killing spinor, the 2-form satisfies some interesting PDEs (we will be back on this in the next lectures). One of them is $d\omega^{(2)} = -\imath_\xi F$. Then

$$\mathcal{L}_\xi F = d\imath_\xi F + \imath_\xi dF = -d(d\omega^{(2)}) = 0 \ ,$$

since F is closed. It remains the odd-odd-odd identity, which amounts to the vanishing of $\mathcal{L}_\xi \epsilon = \nabla_\xi \epsilon + A_\xi(\epsilon) = \frac{1}{24}\big(\xi \cdot F - 3F \cdot \xi\big)\epsilon + A_\xi(\epsilon)$. Now

$$\begin{aligned} A_\xi(X) &= -\nabla_X \xi = -\nabla_X \big(\kappa(\epsilon, \epsilon)\big) = -2\kappa(\nabla_X \epsilon, \epsilon) \\ &= -\tfrac{1}{12}\kappa(\big(X \cdot F - 3F \cdot X\big)\epsilon, \epsilon) \ , \end{aligned}$$

so that A_ξ and $\mathcal{L}_\xi \epsilon$ depend algebraically on ϵ. One then checks $\mathcal{L}_\xi \epsilon = 0$ algebraically. ■

Examples of supergravity backgrounds

The (elementary) brane solutions are described on

$$M = \mathbb{R}^{1,p} \times \mathbb{R}_{>0} \times S^m \qquad (m = 9-p)$$

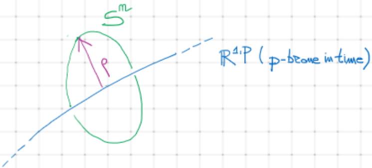
$\mathbb{R}^{1,p}$: coordinates (including time) of p-dimensional "black" object, called brane.

$\mathbb{R}_{>0}$: Distance ρ from the brane.

with a metric of the form $g = H(\rho)^{\alpha} g_{\mathbb{R}^{1,p}} - H(\rho)^{\beta}\left(d\rho^2 + \rho^2 g_{S^m}\right)$, where H is a harmonic function on $\mathbb{R}^{10-p} \cong \mathbb{R}_{>0} \times S^m$ depending only on the distance from the brane

$$H(\rho) = a + \frac{b}{\rho^{8-p}} \qquad (a, b \in \mathbb{R}_{\geq 0})$$

and F or $\ast F$ is $-dH^{-1} \wedge dvol_{\mathbb{R}^{1,p}}$. Hence $p=2$ or $p=5$, for real numbers $\alpha, \beta \in \mathbb{R}$ to be determined via the equations of supergravity.



M2 Brane solutions. (i.e. $p=2$) In this case $\alpha=-2/3$, $\beta=\frac{1}{3}$ with $H(\rho)=a+\frac{b}{\rho^6}$ $\Longrightarrow$

$$g=\left(a+\frac{b}{\rho^6}\right)^{-2/3} g_{\mathbb{R}^{1,2}}-\left(a+\frac{b}{\rho^6}\right)^{\frac{1}{3}}\left(d\rho^2+\rho^2 g_{S^7}\right)$$

and $F=-dH^{-1}\wedge d\mathrm{vol}_{\mathbb{R}^{1,2}}$. (This brane has an electric charge, given by integrating $\star F$ on S^7 and sending $\rho\mapsto+\infty$.)

M5 Brane solutions. (i.e. $p=5$) In this case $\alpha=-\frac{1}{3}$, $\beta=\frac{2}{3}$ with $H(\rho)=a+\frac{b}{\rho^3}$ $\Longrightarrow$

$$g=\left(a+\frac{b}{\rho^3}\right)^{-\frac{1}{3}} g_{\mathbb{R}^{1,5}}-\left(a+\frac{b}{\rho^3}\right)^{\frac{2}{3}}\left(d\rho^2+\rho^2 g_{S^4}\right)$$

and $\star F=-dH^{-1}\wedge d\mathrm{vol}_{\mathbb{R}^{1,5}}$. (This brane has a magnetic charge, given by integrating F on S^4 and sending $\rho\mapsto+\infty$.)

Roughly speaking, the brane is located at $\rho=0$, which can be thought as a horizon. Physically:

- Limit $a\longrightarrow 0$ ("$\rho\mapsto 0$") is the near horizon geometry ($AdS_4\times S^7$ for $p=2$, $AdS_7\times S^4$ for $p=5$);
- Limit $b\longrightarrow 0$ ("$\rho\mapsto+\infty$") is the limit at infinity (Minkowski spacetime $\mathbb{M}^{1,10}$ in both cases).

The brane solutions were discovered by Duff&Stelle in '91 and Guven in '92. Generically they admit 16 Killing spinors but for the above special limits there are many more....

Thanks!