Review: elementary Lie algebra structure theory

$\mathfrak{g}$ $\mathbb{C}$-ss, $\mathfrak{h}$ CSA, $\Delta \subset \mathfrak{h}^*$ roots, Killing form $\rightsquigarrow$ ndg $\langle \cdot, \cdot \rangle$ on $V = \mathrm{span}_{\mathbb{R}} \Delta$.

Simple roots $\{\alpha_i\}_{i=1}^{\ell} \subset \mathfrak{h}$, dual basis $\{\mathsf{Z}_i\}$, fundamental weights $\{\lambda_i\}_{i=1}^{\ell}$, i.e. $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, where $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ is the coroot of $\alpha \in \Delta$.

Cartan matrix: $c_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$. Have $\forall i \neq j$, $c_{ij} \in \mathbb{Z}_{\leq 0}$, $c_{ij} c_{ji} \in \{0, 1, 2, 3\}$.

Have basis change $\alpha_i = c_{ij} \lambda_j, \quad \lambda_i = c^{ij} \alpha_j$, where c^{ij} = inverse of c_{ij}.

Dynkin diagram: Graph with $\alpha_i \leftrightarrow$ node i; bond from i to j of multiplicity $c_{ij} c_{ji}$, directed towards the shorter root if $c_{ij} c_{ji} > 1$.

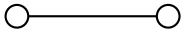
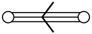
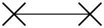
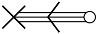
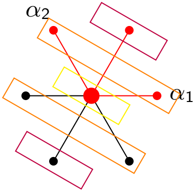
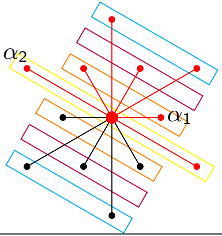
Parabolics: $\mathfrak{p} \subset \mathfrak{g} \leftrightarrow I_{\mathfrak{p}} \subset \{1, \ldots, \ell\}$. Crosses on $I_{\mathfrak{p}}$ in DD; $\mathsf{Z} := \sum_{i \in I_{\mathfrak{p}}} \mathsf{Z}_i$.

Reflection wrt $\alpha^\perp$: $\quad s_\alpha(\lambda) := \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.

Weyl group: $W \leq \mathrm{O}(V)$ is the subgroup generated by $\{s_\alpha : \alpha \in \Delta\}$.

- Δ is W-invariant.
- W is finite and generated by simple reflections $\{s_{\alpha_i}\}_{i=1}^{\ell}$.
- Any $w \in W$ is a word, e.g. $(12) := s_{\alpha_1} \circ s_{\alpha_2}$.

Our main examples

$\mathfrak{g}$ c_{ij} Dynkin diagram Highest weight λ	$\mathfrak{sl}_3$ $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$  $\alpha_1 + \alpha_2 = \lambda_1 + \lambda_2$	$\text{Lie}(G_2)$ $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$  $3\alpha_1 + 2\alpha_2 = \lambda_2$
Marked DD Grading element Z	 $Z_1 + Z_2$	 Z_1
Graded root diagram		

$\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \times \mathfrak{g}_0^{\text{ss}}$: $\boxed{\dim(\mathfrak{z}(\mathfrak{g}_0)) = \#\text{crosses}; \ \mathfrak{g}_0^{\text{ss}} \leftrightarrow \text{DD after omitting crosses}}$