# Geometry of quantum correlations

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# Quantum Systems (Q.S)

- Any Q. S. is described by a complex Hilbert space, $\mathcal{H}$. We will assume $\mathcal{H}$ is finite dimensional, that is, $\mathcal{H} \simeq \mathbb{C}^d$

→ For $v, w \in \mathcal{H}$ the inner product is $\langle v | w \rangle$.

$\langle \cdot | \cdot \rangle$ is conjugate linear in first arg. and linear in the second

→ Riesz Lemma: Any linear functional $\varphi : \mathcal{H} \to \mathbb{C}$ is given by inner product

$$\exists v_\varphi \text{ s.t. } \forall u \in \mathcal{H} \quad \varphi(u) = \langle v_\varphi | u \rangle$$

bra

→ Linear functionals from $\mathcal{H}^*$ will be denoted by $\langle \varphi |$, $\langle \psi |$, etc

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→ Vectors in $\mathcal{H}$ will be denoted by $| \varphi \rangle$, $| \psi \rangle$, etc

→ Action of $\langle \varphi | \in \mathcal{H}^*$ on $| \psi \rangle \in \mathcal{H}$ is

$$\langle \varphi | \left( | \psi \rangle \right) = \underset{\text{bra}}{\langle \varphi} \, \underset{\text{ket}}{| \psi \rangle}$$

- Pure states of Q. S.

  → In $\mathcal{H}$ we introduce equivalence relation:

$$|\psi\rangle \sim |\varphi\rangle \iff |\psi\rangle = \alpha|\varphi\rangle \qquad \alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

  → Equivalence classes of $\mathcal{H}/\sim$ are pure states of Q. S.

  → Pure states are points in $\mathbb{P}(\mathcal{H})$

  → Physicists say pure states are normalized to 1 vectors where we neglect the 'global phase factor' $e^{i\alpha}$

- Observables

  → To learn properties of Q. S. we measure values of physically relevant quantities, for example: spin, momentum, energy etc. They are called observables

  → Observables are represented by selfadjoint (Hermitian) operators on $\mathcal{H}$

$$F: \mathcal{H} \to \mathcal{H}, \quad F^* = F \quad \text{that is } \langle\varphi|F\psi\rangle = \langle F\varphi|\psi\rangle \quad \forall |\varphi\rangle, |\psi\rangle \in \mathcal{H}$$

$\sigma(F)$ - spectrum of $F$, $\quad F = \sum_{\lambda \in \sigma(F)} \lambda P_\lambda$, $\qquad \mathcal{H} = \bigoplus_{\lambda \in \sigma(F)} \mathcal{H}_\lambda \qquad \mathcal{H}_\lambda \perp \mathcal{H}_\mu$

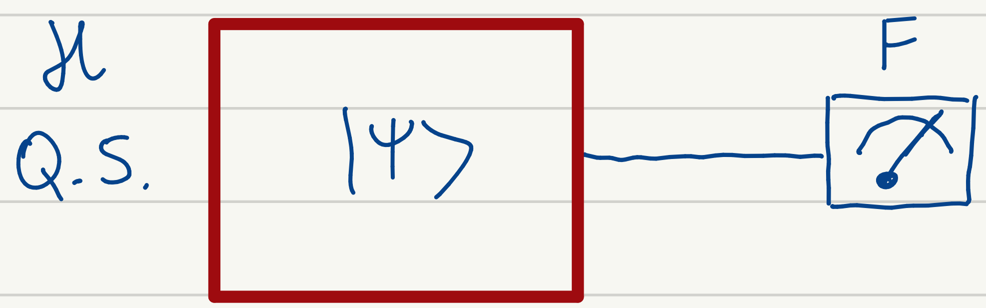
$P_\lambda$ - orthogonal projection onto $\mathcal{H}_\lambda = \mathrm{Ker}(F - \lambda I)$

## • Observables

→ To learn properties of Q.S. we measure values of physically relevant quantities, for exemple: spin, momentum, energy etc. They are called observables

→ Observables are represented by selfadjoint (Hermitian) operators on $\mathcal{H}$

→ Assume Q.S. is in a state $|\psi\rangle$ and we measure $F$




• Possible measurment outcomes are $\sigma(F)$

• The probability of getting $\lambda \in \sigma(F)$ is $\mathbb{P}_{|\psi\rangle}(\lambda) = \langle \psi | P_\lambda \psi \rangle$

• If the result is $\lambda$ the state changes to

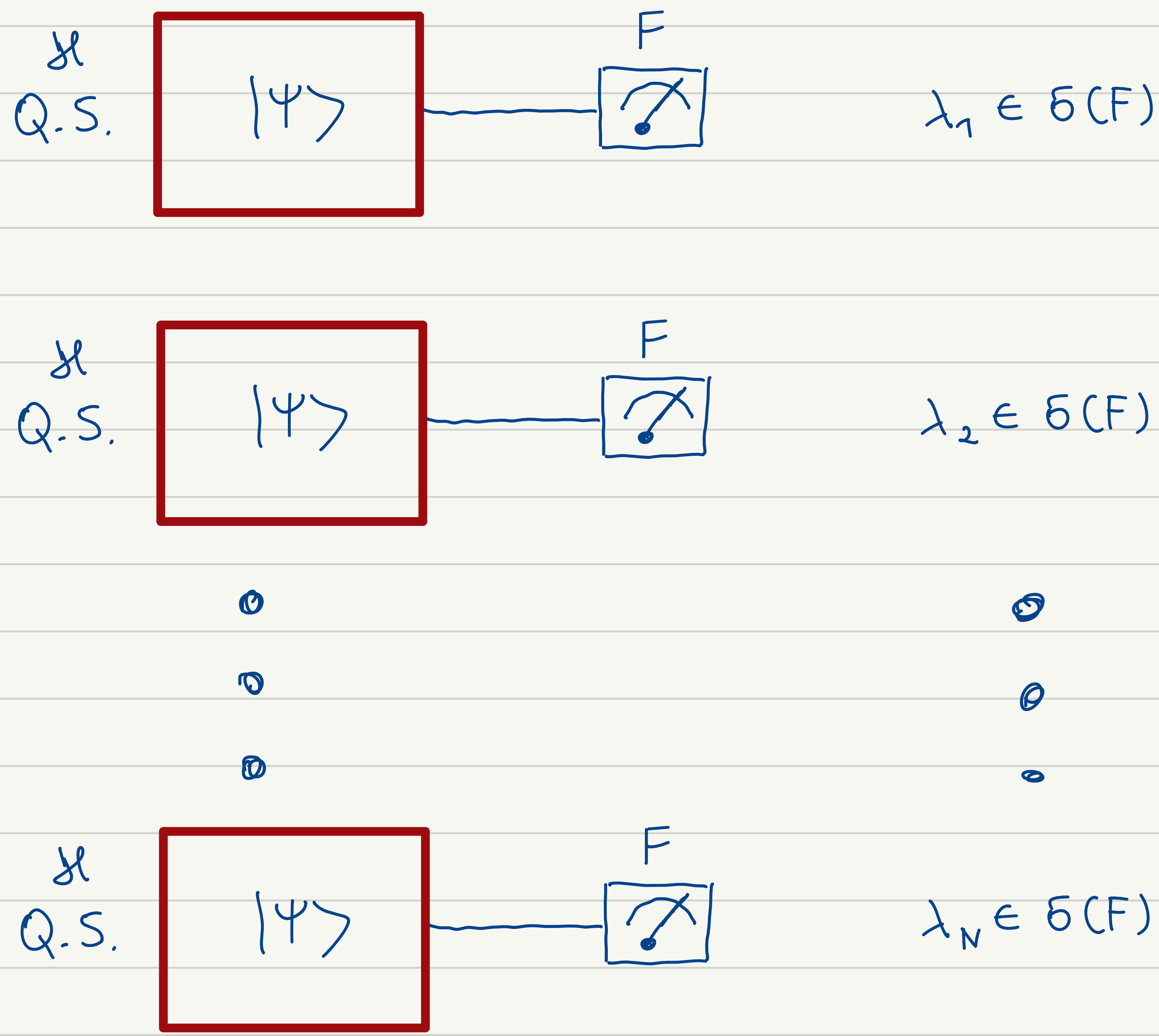
$$|\psi\rangle \longrightarrow \frac{P_\lambda |\psi\rangle}{\| P_\lambda |\psi\rangle \|} \in \mathcal{H}_\lambda$$

• The expected value of $F$: $\langle F \rangle_{|\psi\rangle} = \sum_{\lambda \in \sigma(F)} \lambda \cdot \mathbb{P}_{|\psi\rangle}(\lambda) = \langle \psi | F \psi \rangle$

- How do we interprete $\mathbb{P}_{|\psi\rangle}(\lambda)$ ?

  $\mapsto$ Law of Large numbers: Probability is relative frequency

  $\mapsto$ We measure N-copies of Q. S. $\mathcal{H}$ prepared in a state $|\psi\rangle \in \mathcal{H}$




For any $\lambda \in \sigma(F)$
$$\frac{|\{\lambda_k \mid \lambda_k = \lambda\}|}{N} \xrightarrow[N\to\infty]{} \mathbb{P}_{|\psi\rangle}(\lambda)$$

- Free time evolution

  ⟼ Assume Q.S. is in a state $|\psi_0\rangle$ at $t = t_0$ and we do not measure anything

  ⟼ Every Q.S. has one special observable $H : \mathcal{H} \to \mathcal{H}$ called Hamiltonian

  ⟼ Free time evolution is given by time evolution operator $\mathcal{U}(t, t_0) : \mathcal{H} \to \mathcal{H}$

$$|\psi(t)\rangle = \mathcal{U}(t, t_0)\, |\psi_0\rangle$$

  that satisfies

$$i \frac{d}{dt} \mathcal{U}(t, t_0) = H\, \mathcal{U}(t, t_0), \qquad \mathcal{U}(t_0, t_0) = \mathbb{1}$$

$$\Updownarrow$$

$$i \frac{d}{dt} |\psi(t)\rangle = H\, |\psi(t)\rangle, \qquad |\psi(t_0)\rangle = |\psi_0\rangle$$

We assume a full control over Q.S. that is we can use any $H$

- $\mathcal{U}(t, t_0)$ is unitary
- $\mathcal{U}(t, t_0) = e^{i(t-t_0)H}$. Putting $t_0 = 0$ and $\mathcal{U}(t, 0) = \mathcal{U}(t) = e^{itH}$

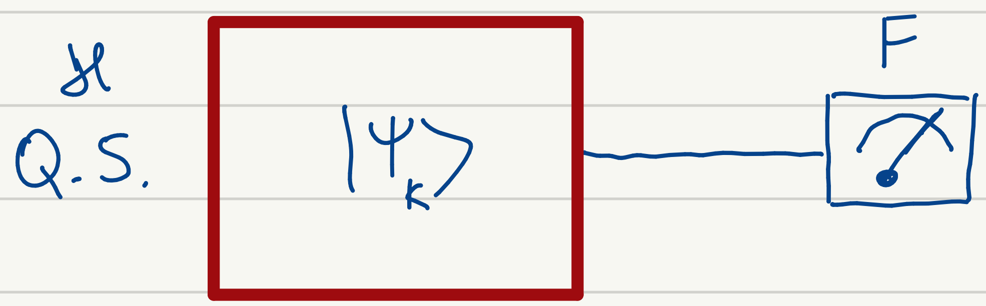
  $\mathcal{U}(t)\,\mathcal{U}(s) = \mathcal{U}(t+s)$, $\mathcal{U}(0) = \mathbb{1}$

  $\mathbb{R} \ni t \longmapsto \mathcal{U}(t) \in \mathfrak{U}(\mathcal{H})$ - continous homomorphism $(\mathbb{R}, +) \longrightarrow \mathfrak{U}(\mathcal{H})$

  $\mathcal{U}(t)$ is a 1-parameter subgroup of $\mathfrak{U}(\mathcal{H})$ generated by $H$

- Mixed states

  ↦ So far we assumed that our Q.S. is prepared in a state $|\psi\rangle \in \mathcal{H}$
  What if the preparing procedure is also probabilistic




The probability that system $\mathcal{H}$ is prepared in state $|\psi_k\rangle$ is $p_k$

↦ Possible outcomes are still $\sigma(F)$

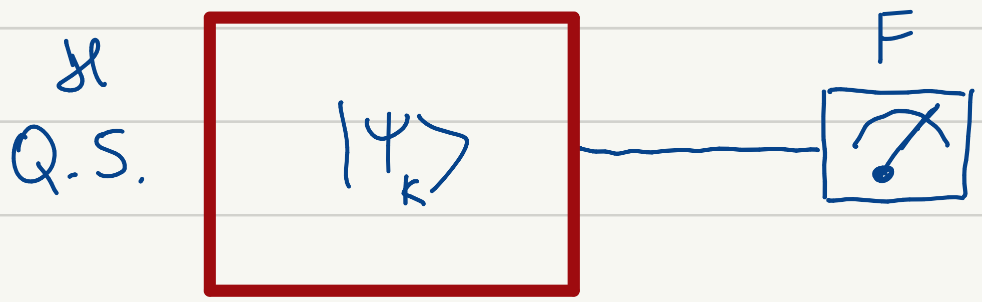
↦ But we need to modify $\mathbb{P}(\lambda)$, $\lambda \in \sigma(F)$

↦ Total law of probability

$$\mathbb{P}(\lambda) = \sum_k \underbrace{\mathbb{P}(\lambda \mid \psi_k)}_{\substack{\text{probability that the outcome is } \lambda \\ \text{under the condition Q.S. is in the state } |\psi_k\rangle}} \cdot p_k = \sum_k \mathbb{P}_{|\psi_k\rangle}(\lambda) \cdot p_k = \sum_k \langle \psi_k | P_\lambda \psi_k \rangle \cdot p_k \overset{①}{=} \operatorname{tr}(\rho P_\lambda)$$

$$\rho = \sum_k p_k \underbrace{|\psi_k\rangle\langle\psi_k|}$$

① $\operatorname{tr}\left(\sum_k p_k |\psi_k\rangle\langle\psi_k| P_\lambda\right) = \sum_k p_k \operatorname{tr}\left(|\psi_k\rangle\langle\psi_k| P_\lambda\right) = \sum_k \langle \psi_k | P_\lambda \psi_k \rangle \cdot p_k$




The probability that system $\mathcal{H}$ is prepared in state $|\psi_k\rangle$ is $p_k$

$$\longmapsto \quad \rho = \sum_k{}' \, p_k \, |\psi_k\rangle\langle\psi_k| \quad \text{– mixed state}$$

$$\longmapsto \quad \mathbb{P}_\rho(\lambda) = \operatorname{tr}(\rho P_\lambda) = \langle P_\lambda \rangle_\rho$$

$$\longmapsto \quad \langle F \rangle_\rho = \sum_{\lambda \in \sigma(F)} \lambda \cdot \mathbb{P}_\rho(\lambda) = \sum_{\lambda \in \sigma(F)} \lambda \cdot \operatorname{tr}(\rho P_\lambda) = \operatorname{tr}(\rho F)$$

$\longmapsto$ If the result of measurement is $\lambda \in \sigma(F)$ the state is

$$|\varphi_1\rangle = \frac{P_\lambda |\psi_1\rangle}{\| P_\lambda \psi_1 \|} \quad \text{or} \quad |\varphi_2\rangle = \frac{P_\lambda |\psi_2\rangle}{\| P_\lambda \psi_2 \|} \quad \text{or} \quad \ldots \quad |\varphi_k\rangle = \frac{P_\lambda |\psi_k\rangle}{\| P_\lambda \psi_k \|} \quad \text{or} \quad \ldots$$

$$\downarrow q_1 \qquad\qquad \downarrow q_2 \qquad\qquad \downarrow q_k$$

$$q_k = \mathbb{P}(|\varphi_k\rangle \,|\, \lambda) = \frac{p_k \cdot \mathbb{P}_{|\psi_k\rangle}(\lambda)}{\mathbb{P}_\rho(\lambda)} \quad \Rightarrow \quad \sum_k{}' \, q_k \, |\varphi_k\rangle\langle\varphi_k| = \frac{P_\lambda \, \rho \, P_\lambda}{\operatorname{tr}(P_\lambda \rho)}$$

• Properties of $\rho$

$$\longmapsto \quad \operatorname{tr}(\rho) = \sum_k{}' \, p_k = 1$$

$$\longmapsto \quad \rho \geq 0, \qquad |\varphi\rangle \in \mathcal{H} \qquad \langle \varphi | \rho \varphi \rangle = \sum_k{}' \, p_k \, |\langle \varphi | \psi_k \rangle|^2 \geq 0$$

$$\longmapsto \quad \text{If } \forall F \in L(\mathcal{H}) \quad \operatorname{tr}(\rho_1 F) = \operatorname{tr}(\rho_2 F) \quad \Rightarrow \quad \rho_1 = \rho_2$$

$$\langle F \rangle_{\rho_1} = \langle F \rangle_{\rho_2}$$

- Pure state is a mixed state that is given by $|\psi\rangle\langle\psi|$ – orthogonal projection onto $\mathbb{C}|\psi\rangle$
- $\rho$ is a pure state iff $tr(\rho) = tr(\rho^2)$
- For a fixed $\rho \geq 0$, $tr(\rho) = 1$ there are many ensembles $\{|\psi_k\rangle, p_k\}_k$ for which

$$\rho = \sum_i p_k |\psi_k\rangle\langle\psi_k|$$

- A canonical choice is to use spectral decomposition of $\rho = \sum_{k=1}^{d} \lambda_k |e_k\rangle\langle e_k|$

- von Neumann entropy of state $\rho$ is

$$S(\rho) = -tr(\rho \log(\rho)) = -\sum_{k=1}^{d} \lambda_k \log \lambda_k$$

$$S(\rho) = 0 \iff \rho \text{ is a pure state}$$

- Maximally mixed state $\rho_M = \frac{1}{d}\mathbb{1}$, $S(\rho_M) = \log(d)$
- Free time evolution of $\rho$ is again unitary, $\{|\psi_k\rangle, p_k\}$

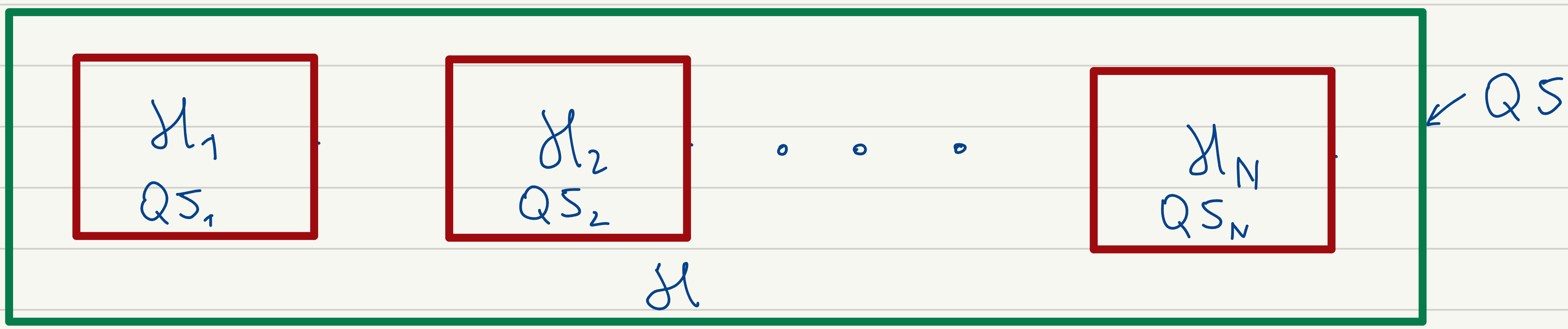
$$\rho(t) = \sum_k p_k |\psi_k(t)\rangle\langle\psi_k(t)| = U(t,t_0)^* \rho(t_0)\, U(t,t_0)$$

$|\psi_k(t)\rangle = U(t,t_0)|\psi_k(t_0)\rangle$

unitary

# Systems and subsystems

- Assume we have $N$ quantum systems described by Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_N$




→ If systems $\mathcal{H}_1, \dots, \mathcal{H}_N$ are distinguishable then

$$\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$$

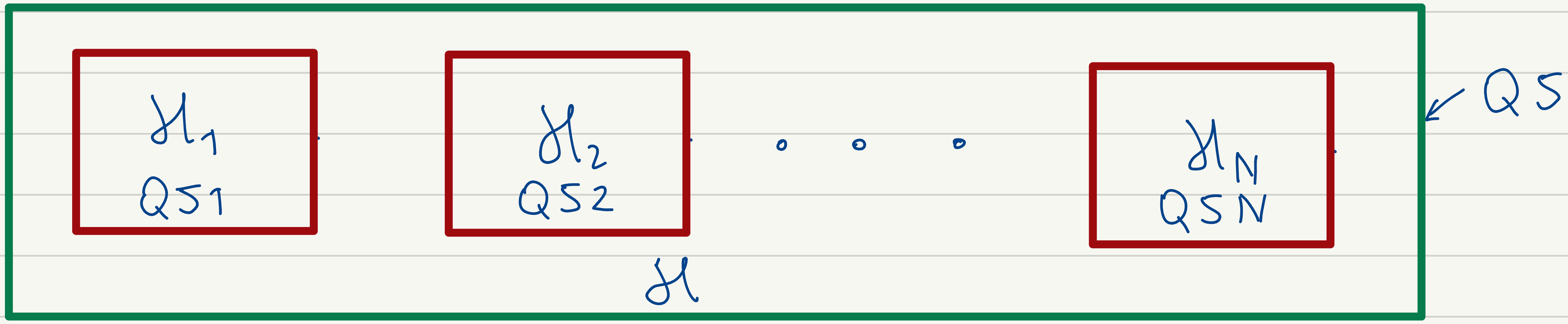
→ If systems $\mathcal{H}_1, \dots, \mathcal{H}_N$ are indistinguishable (and $\dim(\mathcal{H}_k) = d \ \forall k$)

→ Fermionic : $\mathcal{H} = \Lambda^N \mathbb{C}^d$

→ Bosonic : $\mathcal{H} = \vee^N \mathbb{C}^d$

→ We will focus mostly on distinguishable case

• Assume Q.S. is in the pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$




What are the states of $QS_1, \dots, QS_N$?

→ $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_N\rangle \in \mathcal{H}$ a simple tensor. Then

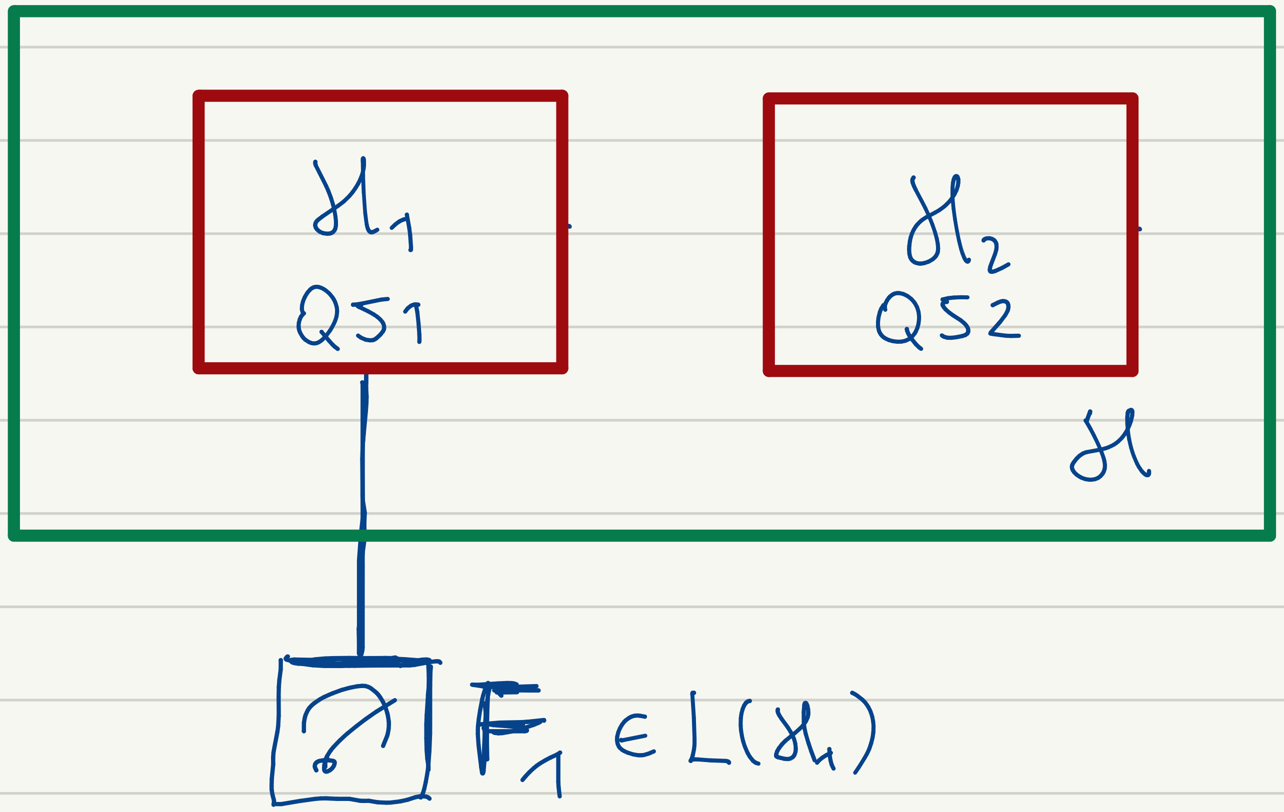
$\forall_k$ $QS_k$ is in the state $|\psi_k\rangle$

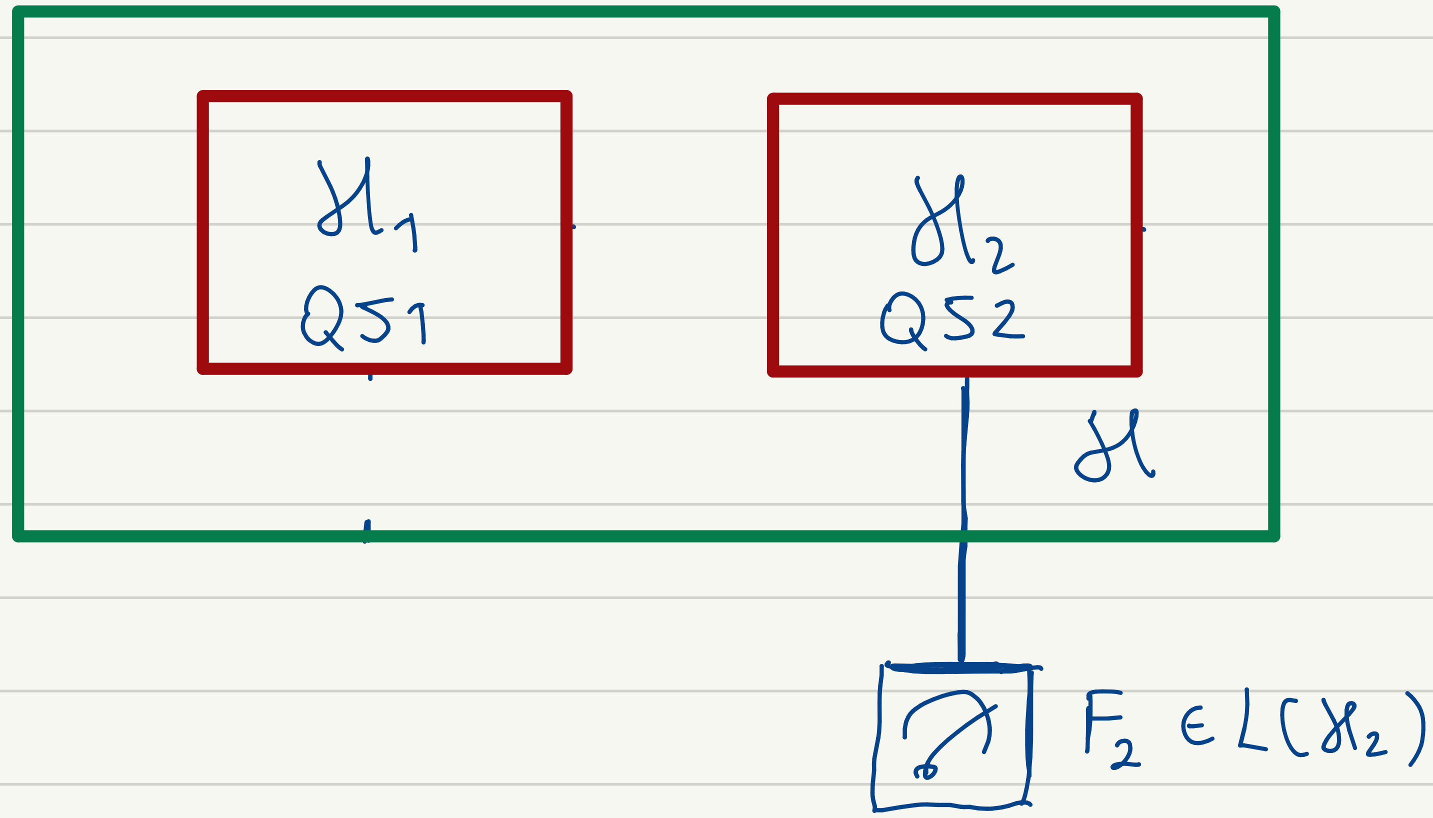
We say $|\psi\rangle$ is separable or not entangled

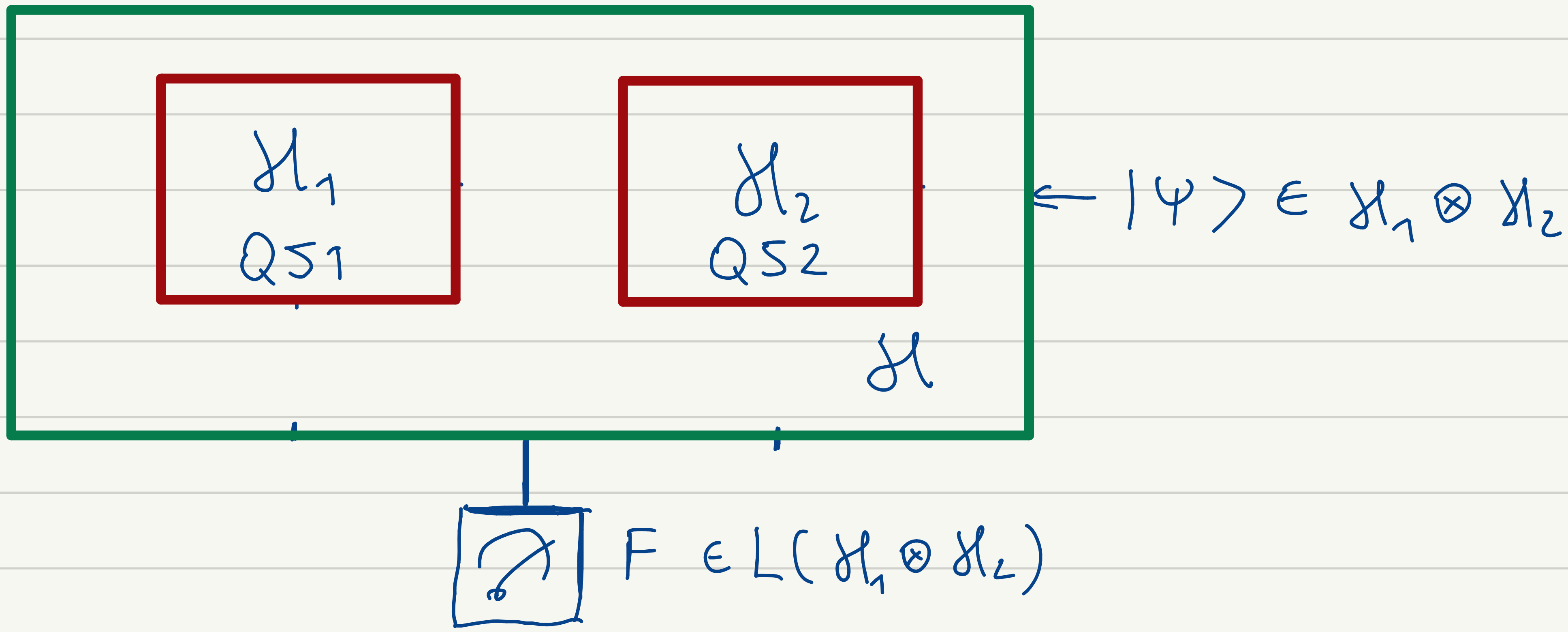
→ In general $(N=2)$ $\mathcal{H}_1 = \mathrm{Span}\{|e_1\rangle, \dots, |e_d\rangle\}$, $\mathcal{H}_2 = \mathrm{Span}\{|f_1\rangle, \dots, |f_d\rangle\}$

$$|\psi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} c_{ij} \, |e_i\rangle \otimes |f_j\rangle$$

→ To determine states of $QS_1$ and $QS_2$ we consider observables




$$L(\mathcal{H}_1) \hookrightarrow L(\mathcal{H}_1 \otimes \mathcal{H}_2) \qquad F_1 \longmapsto F_1 \otimes \mathbb{1}$$
$$L(\mathcal{H}_2) \hookrightarrow L(\mathcal{H}_1 \otimes \mathcal{H}_2) \qquad F_2 \longmapsto \mathbb{1} \otimes F_2$$

A state of a system is determined by all expected values of observables.

$$\forall F_1 \in L(\mathcal{H}_1) \quad \langle F_1 \rangle_{\rho_1} = \langle F_1 \otimes \mathbb{1} \rangle_{|\psi\rangle} \quad , \quad tr(\rho_1 F_1) = \langle \psi | F_1 \otimes \mathbb{1}\, \psi \rangle \qquad \forall F_1 \in L(\mathcal{H}_1)$$
$$\forall F_2 \in L(\mathcal{H}_2) \quad \langle F_2 \rangle_{\rho_2} = \langle \mathbb{1} \otimes F_2 \rangle_{|\psi\rangle} \quad , \quad tr(\rho_2 F_2) = \langle \psi | \mathbb{1} \otimes F_2\, \psi \rangle \qquad \forall F_2 \in L(\mathcal{H}_2)$$

- $\rho_1, \rho_2$ are called reduced states of $QS_1$ and $QS_2$
- If $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$ then

$$\langle \Psi | F_1 \otimes \mathbb{1}\, \Psi\rangle = \langle \Psi_1 | F_1 \Psi_1\rangle \underbrace{\langle \Psi_2 | \Psi_2\rangle}_{1}$$
$$\langle \Psi | \mathbb{1} \otimes F_2 \Psi\rangle = \langle \Psi_2 | F_2 \Psi_2\rangle \underbrace{\langle \Psi_1 | \Psi_1\rangle}_{1}$$
$$\Rightarrow \quad \rho_1 = |\Psi_1\rangle\langle\Psi_1|, \quad \rho_2 = |\Psi_2\rangle\langle\Psi_2| \quad \text{Pure states}$$

- In general $|\Psi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} c_{ij} |e_i\rangle \otimes |f_j\rangle$

$$\rho_1 = tr_2(|\Psi\rangle\langle\Psi|) = \sum_{i,j,k} c_{ik} \overline{c_{jk}}\, |e_i\rangle\langle e_j|$$
$$\rho_2 = tr_1(|\Psi\rangle\langle\Psi|) = \sum_{i,j,k} c_{ki} \overline{c_{kj}}\, |f_i\rangle\langle f_j|$$

- Entanglement entropy:

$$S_E(\Psi) = S(\rho_1(\Psi)) + S(\rho_2(\Psi)) = -2 \sum_{i=1}^{d} p_i \log(p_i) \qquad p_i\text{'s} \leftarrow \text{eigenvalues of } \rho_1/\rho_2$$

$|\Psi\rangle$ – Separable state $\Rightarrow S_E(|\Psi\rangle) = 0$

$|\Psi\rangle = \frac{1}{\sqrt{d}}\left(|e_1\rangle \otimes |f_1\rangle + \dots + |e_d\rangle \otimes |f_d\rangle\right) \Rightarrow \rho_1 = \rho_2 = \frac{1}{d}\mathbb{1} \qquad S_E = 2\log(d) \leftarrow$ maximal