# Classifying homogeneous geometric structures (Lecture 3)

Dennis The

Department of Mathematics & Statistics
UiT The Arctic University of Norway

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# Outline

Last time:

- Fund. thm. of parabolic geometries; harmonic curvature $\kappa_H$.
- Cartan reduction method. Application to 2nd order ODE.
- Cartan-theoretic descriptions. Curvature / holonomy.

Today:

- Kostant's theorem
- Algebraic models as filtered sub-deformations
- Multiply-transitive $(2, 3, 5)$-distributions

# The 1910 paper

In 1910, Élie Cartan wrote a landmark paper:

"Les systemes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre."

In this difficult paper, nowadays universally known as "the 5-variables" paper, Cartan established remarkable equivalences between:

1. contact syms of (non-Monge–Ampère) parabolic Goursat PDE in the plane, e.g. $9(u_{xx})^2 + 12(u_{yy})^2(u_{xx}u_{yy} - (u_{xy})^2) + 32(u_{xy})^3 - 36u_{xx}u_{xy}u_{yy} = 0$.
   This model (appearing in Yamaguchi 1997) was stated parametrically by Cartan in 1910.
2. contact syms of nonlinear involutive pairs of PDE in the plane, e.g. $u_{xx} = \frac{(u_{yy})^3}{3}$, $u_{xy} = \frac{(u_{yy})^2}{2}$.
3. symmetries of $(2,3,5)$-distributions, e.g. $z' = (u'')^2$ ("Hilbert–Cartan eqn").

Cartan gave a *tour-de-force* application of his equivalence method applied to $(2,3,5)$-distributions. Moreover, he classified (almost$^\dagger$) all (complex) multiply-transitive structures. Maximal sym is 14-dim, i.e. $\mathrm{Lie}(G_2)$.

†: One model was missed. Doubrov–Govorov (2013) discovered it.

# The Cartan connection

The 1910 paper has inspired and strongly influenced our notions of how we study equivalence and symmetry. Also, there are several citations in the literature to the "5-variables" paper concerning the Cartan connection constructed for $(2,3,5)$-distributions. However:

**Nurowski's (unpublished) observation (which I paraphrase here)**

*The Cartan "connection" (of 1910) is NOT a "Cartan connection".*

- The 1910 "connection" (coframing) is not (fully) equivariant.
- Failure of equivariancy is not obvious, since Cartan did not write the full structure equations for his coframing.

(⚠ There are many (inequivalent) notions of Cartan connection in the early literature. The modern Cartan connection definition is clearly stated in Kobayashi 1972.)

# The Cartan connection (continued)

Nurowski discussed with Bryant (in Oct 2013) about modifying Cartan's coframing to obtain a Cartan connection. In an illuminating circulated email, Bryant exhibited such a modification. A brief extract is:

*"My conclusion is that Cartan, at the time that he wrote his 5-variables paper, was more interested in practical calculation and exposition than he was in a theoretical development of what later became Cartan connections, and he chose his coframing solving his equivalence problem so that the formulae that he needed to write down to explain his results would be as simple as possible."*

Nevertheless, not everyone was impressed. In Tanaka (1970, p.52):

*"In his paper... E. Cartan has really carried out such reductions in a 'messy and complicated' manner."*

# Motivating questions for today

For homogeneous geometric structures, can one do classification:

1. ... more efficiently than what Cartan reduction would entail?
2. ... without setting up the full primary / secondary structure equations for the corresponding Cartan geometry?
3. ... Lie algebraically, and find their Cartan-theoretic data? (Existing classifications: coordinate & Lie-theoretic models.)

Today: The multiply-transitive $(2, 3, 5)$ classification from a modern perspective.

# Kostant's theorem

# Lie algebra cohomology

$\mathfrak{g}$ semisimple, $\mathfrak{p} \subset \mathfrak{g}$ parabolic, so $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ with $\mathfrak{p} = \mathfrak{g}_{\geq 0}$ and grading element $Z \in \mathfrak{z}(\mathfrak{g}_0)$.

Let $C^k := \bigwedge^k (\mathfrak{g}_-)^* \otimes \mathfrak{g}$. Get a complex $0 \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} C^2 \xrightarrow{\partial} \dots$ with $H^k(\mathfrak{g}_-, \mathfrak{g}) := \frac{\ker(\partial : C^k \to C^{k+1})}{\mathrm{im}(\partial : C^{k-1} \to C^k)}$. Motivations:

1. Harmonic curvature $\kappa_H$ is valued in $H^2_+(\mathfrak{g}_-, \mathfrak{g})$. (Subscript refers to the grading, i.e. Z-eigenvalue.)
2. Tanaka prolongation: Given $\mathfrak{g}$ graded, is $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{g}$? Have
   - $\partial^0 : C^0 \to C^1$ is $(\partial^0 v)(x) := [x, v] = -\mathrm{ad}_v|_{\mathfrak{g}_-}(x)$.
   - $\partial^1 : C^1 \to C^2$ is $(\partial^1 \eta)(x, y) := [x, \eta(y)] - [y, \eta(x)] - \eta([x, y])$.

$$H^1(\mathfrak{g}_-, \mathfrak{g}) = \frac{\ker(\partial^1)}{\mathrm{im}(\partial^0)} = \frac{\mathfrak{g}\text{-valued derivations on } \mathfrak{g}_-}{\langle \mathrm{ad}_v|_{\mathfrak{g}_-} : v \in \mathfrak{g} \rangle}$$

**Exercise**

(a) $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{g}$ *iff* $H^1_+(\mathfrak{g}_-, \mathfrak{g}) = 0$; (b) $\mathrm{pr}(\mathfrak{g}_-) \cong \mathfrak{g}$ *iff* $H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0$.

# Review: elementary Lie algebra structure theory

$\mathfrak{g}$ $\mathbb{C}$-ss, $\mathfrak{h}$ CSA, $\Delta \subset \mathfrak{h}^*$ roots, Killing form $\rightsquigarrow$ ndg $\langle \cdot, \cdot \rangle$ on $V = \mathrm{span}_{\mathbb{R}} \Delta$.

Simple roots $\{\alpha_i\}_{i=1}^{\ell} \subset \mathfrak{h}^*$, dual basis $\{\mathsf{Z}_i\}$, fundamental weights $\{\lambda_i\}_{i=1}^{\ell}$, i.e. $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, where $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ is the coroot of $\alpha \in \Delta$.

Cartan matrix: $c_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$. Have $\forall i \neq j$, $c_{ij} \in \mathbb{Z}_{\leq 0}$, $c_{ij} c_{ji} \in \{0, 1, 2, 3\}$.

Have basis change $\alpha_i = c_{ij} \lambda_j$, $\quad \lambda_i = c^{ij} \alpha_j$, where $c^{ij}$ = inverse of $c_{ij}$.

Dynkin diagram: Graph with $\alpha_i \leftrightarrow$ node $i$; bond from $i$ to $j$ of multiplicity $c_{ij} c_{ji}$, directed towards the shorter root if $c_{ij} c_{ji} > 1$.

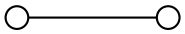
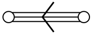
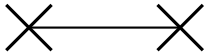
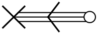
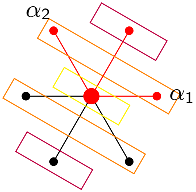
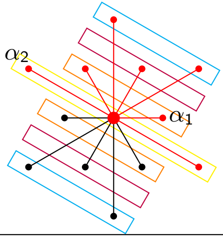
Parabolics: $\mathfrak{p} \subset \mathfrak{g} \leftrightarrow I_{\mathfrak{p}} \subset \{1, ..., \ell\}$. Crosses on $I_{\mathfrak{p}}$ in DD; $\mathsf{Z} := \sum_{i \in I_{\mathfrak{p}}} \mathsf{Z}_i$.

Reflection wrt $\alpha^\perp$: $\quad s_\alpha(\lambda) := \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.

Weyl group: $W \leq \mathrm{O}(V)$ is the subgroup generated by $\{s_\alpha : \alpha \in \Delta\}$.

- $\Delta$ is $W$-invariant.
- $W$ is finite and generated by simple reflections $\{s_{\alpha_i}\}_{i=1}^{\ell}$.
- Any $w \in W$ is a word, e.g. $(12) := s_{\alpha_1} \circ s_{\alpha_2}$.

# Our main examples

| $\mathfrak{g}$ | $\mathfrak{sl}_3$ | $\mathrm{Lie}(G_2)$ |
|---|---|---|
| $c_{ij}$ | $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ |
| Dynkin diagram |  |  |
| Highest weight $\lambda$ | $\alpha_1 + \alpha_2 = \lambda_1 + \lambda_2$ | $3\alpha_1 + 2\alpha_2 = \lambda_2$ |
| Marked DD |  |  |
| Grading element Z | $Z_1 + Z_2$ | $Z_1$ |
| Graded root diagram |   |   |

$\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \times \mathfrak{g}_0^{\mathrm{ss}}$: $\boxed{\dim(\mathfrak{z}(\mathfrak{g}_0)) = \#\text{crosses}; \ \mathfrak{g}_0^{\mathrm{ss}} \leftrightarrow \text{DD after omitting crosses}}$

(Root vectors: $e_{ij} := e_{i\alpha_1 + j\alpha_2}$, $f_{ij} := e_{-i\alpha_1 - j\alpha_2}$, e.g. use Chevalley basis.)

# Weyl group

Q: How does the simple reflection $s_{\alpha_j}$ act on $\lambda = \sum_i r_i \lambda_i$?
A: Let $b = r_j$. Add $b$ to adjacent coeffs in DD, with multiplicity if $\exists$ multiple bond directed to the adjacent node. Replace $b$ by $-b$.

**Example**

$$\overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{c}{\circ} \quad \overset{s_{\alpha_2}}{\rightsquigarrow} \quad \overset{a+b}{\circ}\!-\!\!-\!\overset{-b}{\circ}\!-\!\!-\!\overset{b+c}{\circ}$$

$$\overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\Rightarrow\overset{c}{\circ} \quad \overset{s_{\alpha_2}}{\rightsquigarrow} \quad \overset{a+b}{\circ}\!-\!\!-\!\overset{-b}{\circ}\Rightarrow\overset{2b+c}{\circ}$$

$$\overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\Leftarrow\overset{c}{\circ} \quad \overset{s_{\alpha_2}}{\rightsquigarrow} \quad \overset{a+b}{\circ}\!-\!\!-\!\overset{-b}{\circ}\Leftarrow\overset{b+c}{\circ}$$

Affine $W$-action: $w \bullet \lambda = w(\lambda + \rho) - \rho$, where $\rho := \sum_i \lambda_i$.

**Example**

$$(12) \bullet \left( \overset{0}{\circ}\Lleftarrow\overset{1}{\circ} \right) = (12)\left( \overset{1}{\circ}\Lleftarrow\overset{2}{\circ} \right) - \rho = (1)\left( \overset{7}{\circ}\Lleftarrow\overset{-2}{\circ} \right) - \rho$$

$$= \overset{-7}{\circ}\Lleftarrow\overset{5}{\circ} - \rho = \overset{-8}{\circ}\Lleftarrow\overset{4}{\circ}$$

i.e. $(12) \bullet \lambda_2 = -8\lambda_1 + 4\lambda_2 = -4\alpha_1$. Used: $(-8, 4)\left(\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right) = (-4, 0)$.

# Kostant's theorem

**Theorem (Simplified Kostant thm for $\mathfrak{g}$ $\mathbb{C}$-simple with highest weight $\lambda$)**

$H^k(\mathfrak{g}_-, \mathfrak{g}) \cong_{\mathfrak{g}_0} \bigoplus_{w \in W^{\mathfrak{p}}(k)} \mathbb{V}_{-w \bullet \lambda}$. *(Also have explicit lowest weight vectors.)*

Here, $\mathbb{V}_\mu$ is the $\mathfrak{g}_0$-irrep with lowest weight $\mu$. $W^{\mathfrak{p}}(k)$ are the length $k$ words of the Hasse subset $W^{\mathfrak{p}} \subset W$. For $k = 1$ and $k = 2$, these are easily stated:

$$W^{\mathfrak{p}}(1) = \{(j) : j \in I_{\mathfrak{p}}\}, \quad W^{\mathfrak{p}}(2) = \{(jk) : j \in I_{\mathfrak{p}} \text{ and } (k \in I_{\mathfrak{p}} \text{ or } c_{jk} \neq 0)\}.$$

For $w = (jk) \in W^{\mathfrak{p}}(2)$, lowest weight vector is $\phi_0 = e_{\alpha_j} \wedge e_{s_j(\alpha_k)} \otimes e_{-w(\lambda)}$.

**Example ($G_2/P_1$: $\mathrm{Z} = \mathrm{Z}_1$, $W^{\mathfrak{p}}(1) = \{(1)\}$, $W^{\mathfrak{p}}(2) = \{(12)\}$)**

| Calculation | Lowest wt | Interpretation |
|---|---|---|
| $(1) \bullet \overset{0}{\times}\!\Lleftarrow\!\overset{1}{\circ} = \overset{-2}{\times}\!\Lleftarrow\!\overset{2}{\circ}$ | $2\lambda_1 - 2\lambda_2 = -2\alpha_1 - 2\alpha_2$ | $H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0$ $(\therefore \mathrm{pr}(\mathfrak{g}_-) \cong \mathfrak{g}.)$ |
| $(12) \bullet \overset{0}{\times}\!\Lleftarrow\!\overset{1}{\circ} = \overset{-8}{\times}\!\Lleftarrow\!\overset{4}{\circ}$ | $8\lambda_1 - 4\lambda_2 = +4\alpha_1$ | $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \cong S^4\mathfrak{g}_1 \cong S^4(\mathfrak{g}_{-1})^*$ |

$\therefore \kappa_H$ is a binary quartic tensor ("Cartan quartic") on $(2,3,5)$-distribution $D$.

Exercise: Do this for $SL_3/P_{1,2}$ (2nd order ODE).

Algebraic models as filtered sub-deformations

# Homogeneous Cartan geometries

A morphism from $(\mathcal{G} \to M, \omega)$ to $(\mathcal{G}' \to M, \omega')$ is a principal $P$-bundle morphism $\phi$ s.t. $\phi^*\omega' = \omega$. The geometry is homogeneous if $\exists$ Lie grp $F$ acting on $(\mathcal{G} \to M, \omega)$, inducing a transitive action on $M$. In this case:

1. $\mathcal{G} \to M$ is equivalent to $F \times_{F^0} P \to F/F^0$, $\exists$ hom. $\iota : F^0 \to P$.
2. Any $F$-invariant Cartan connection $\omega$ is determined from data at a single point $u \in \mathcal{G}$ by a linear map $\varpi : \mathfrak{f} \to \mathfrak{g}$ with:

   (C1) $\varpi|_{\mathfrak{f}^0} = \iota_* : \mathfrak{f}^0 \to \mathfrak{p}$.
   (C2) $\mathrm{Ad}_{\iota(f)} \circ \varpi = \varpi \circ \mathrm{Ad}_f$, $\forall f \in F^0$.
   (C3) $\varpi$ induces $\mathfrak{f}/\mathfrak{f}^0 \cong \mathfrak{g}/\mathfrak{p}$ as vector spaces.

   More details: see Čap–Slovak (2009), Prop.1.5.15; Hammerl (2011).

Observations:

- (C2'): $[\varpi(x), \varpi(y)] = \varpi([x,y]_{\mathfrak{f}})$, $\forall x \in \mathfrak{f}^0$, $\forall y \in \mathfrak{f}$, i.e. $\boxed{\kappa(\mathfrak{f}^0, \cdot) = 0}$.
  If $F^0$ is connected, then (C2) $\Leftrightarrow$ (C2').
- WLOG, $\varpi$ is injective. (If not: (C3) $\Rightarrow \ker(\varpi) \subset \mathfrak{f}^0$; (C2') $\Rightarrow$ $0 \neq \ker(\varpi) \subset \mathfrak{f}$ is an ideal. $\therefore$ $\mathfrak{f}/\mathfrak{f}^0$ is not "infinitesimally effective". Can always quotient by $\ker(\varpi)$ to get an equivalent description.)

# Algebraic models

Fix $\mathfrak{g}$ semisimple, $\mathfrak{p} \subset \mathfrak{g}$ parabolic. Have $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ wrt grading element $Z \in \mathfrak{z}(\mathfrak{g}_0)$, and $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$ canonical $\mathfrak{p}$-inv. filtration. Let $\mathrm{gr}_i : \mathfrak{g}^i \to \mathfrak{g}^i/\mathfrak{g}^{i+1} \cong_{\mathfrak{g}_0} \mathfrak{g}_i$ (extract leading part), so $\mathrm{gr}(\mathfrak{g}) \cong_{\mathfrak{g}_0} \mathfrak{g}$.

## Definition

*An algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ is a Lie algebra $(\mathfrak{f}, [\cdot,\cdot]_{\mathfrak{f}})$ such that:*

M1: $\mathfrak{f} \subset \mathfrak{g}$ *is a filtered subspace, with filtrands* $\mathfrak{f}^i := \mathfrak{f} \cap \mathfrak{g}^i$, *and* $\mathfrak{s} := \mathrm{gr}(\mathfrak{f})$ *satisfying* $\mathfrak{s}_- = \mathfrak{g}_-$. *(Thus,* $\mathfrak{f}/\mathfrak{f}^0 \cong \mathfrak{g}/\mathfrak{p}$.*)*

M2: $\mathfrak{f}^0$ *inserts trivially into* $\kappa(x,y) := [x,y] - [x,y]_{\mathfrak{f}}$. *(Thus,* $\kappa \in \bigwedge^2(\mathfrak{f}/\mathfrak{f}^0)^* \otimes \mathfrak{g} \cong \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$.*)*

M3: $\kappa$ *is regular and normal, i.e.* $\kappa \in \ker(\partial^*)_+$.

Given $(G, P)$, let $\mathcal{M}$ be the set of all algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$.

- $\mathcal{M}$ is partially ordered: $\mathfrak{f} \leq \mathfrak{f}'$ iff $\mathfrak{f} \hookrightarrow \mathfrak{f}'$ as Lie algs.
- $\mathcal{M}$ admits a $P$-action: i.e. $p \cdot \mathfrak{f} = \mathrm{Ad}_p \mathfrak{f}$. Classify!

# Necessary constraints

**Definition (Extrinsic Tanaka prolongation)**

*Let $\mathfrak{g}$ be a graded Lie alg with $\mathfrak{g}_{-1}$ generating $\mathfrak{g}_-$. Given $\phi$ in a $\mathfrak{g}_0$-rep, let $\mathfrak{a} := \mathfrak{a}^\phi \subset \mathfrak{g}$ be the graded Lie subalg with $\mathfrak{a}_{\leq 0} := \mathfrak{g}_- \oplus \mathfrak{ann}(\phi)$ and*

$$\mathfrak{a}_k := \{x \in \mathfrak{g}_k : [x, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{k-1}\}, \quad \forall k > 0.$$

**Proposition**

*Let $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ be an algebraic model. Then*

1. $(\mathfrak{f}, [\cdot,\cdot]_\mathfrak{f})$ *is a filtered Lie alg, and* $\mathfrak{s} = \mathrm{gr}(\mathfrak{f}) \subset \mathfrak{g}$ *is a graded Lie subalg.*
2. $\mathfrak{f}^0 \cdot \kappa = 0$, *i.e.* $[z, \kappa(x,y)] = \kappa([z,x],y) + \kappa(x,[z,y])$, $\forall x, y \in \mathfrak{f}$, $\forall z \in \mathfrak{f}^0$.
3. $\mathfrak{s} \subset \mathfrak{a}^{\kappa_H}$, *i.e.* $\mathfrak{f}$ *is a "filtered sub-deformation" of* $\mathfrak{a}^{\kappa_H}$.

**Proof.**

1. Recall $\mathfrak{f}^i := \mathfrak{f} \cap \mathfrak{g}^i$. Hence, $[\mathfrak{f}^i, \mathfrak{f}^j]_\mathfrak{f} \subset \mathfrak{f}^{i+j}$ follows from regularity.
2. Use Jacobi identity for $[\cdot,\cdot]_\mathfrak{f} = [\cdot,\cdot] - \kappa(\cdot,\cdot)$.
3. $\partial^*$ is $\mathfrak{p}$-equiv., so $\mathrm{im}(\partial^*)$ is $\mathfrak{p}$-inv. Then (2) $\Rightarrow \mathfrak{f}^0 \cdot \kappa_H = 0$, so $\mathfrak{s}_0 \cdot \kappa_H = 0$, since $\mathfrak{g}_+$ is trivial on $H_2(\mathfrak{g}_+, \mathfrak{g})$. For $k > 0$, $[\mathfrak{s}_k, \mathfrak{g}_{-1}] = [\mathfrak{s}_k, \mathfrak{s}_{-1}] \subset \mathfrak{s}_{k-1}$. Let $\mathfrak{a} := \mathfrak{a}^{\kappa_H}$, so $\mathfrak{s}_0 \subset \mathfrak{a}_0 := \mathfrak{ann}(\kappa_H)$. Inductively, $\mathfrak{s}_k \subset \mathfrak{a}_k$, $\forall k > 0$. $\square$

# Prolongation-rigidity for $G_2/P_1$

$$\mathfrak{g} = \mathrm{Lie}(G_2) = \mathfrak{g}_{-3} \oplus \ldots \oplus \mathfrak{g}_3, \quad \mathfrak{p} = \mathfrak{g}_{\geq 0}, \quad H^2_+(\mathfrak{g}_-, \mathfrak{g}) \cong S^4\mathfrak{g}_1.$$

| weight | $4\alpha_1$ | $4\alpha_1 + \alpha_2$ | $4\alpha_1 + 2\alpha_2$ | $4\alpha_1 + 3\alpha_2$ | $4\alpha_1 + 4\alpha_2$ |
|---|---|---|---|---|---|
| wt vec. $\phi$ | $y^4$ | $xy^3$ | $x^2y^2$ | $x^3y$ | $x^4$ |

($\{x, y\}$ is the std basis for the std $\mathfrak{sl}_2$-rep, while $\mathsf{Z} = \mathsf{Z}_1$ acts on $H^2_+$ by $+4$.)

$$\boxed{G_2/P_1 : \; 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g}) \quad \Rightarrow \quad \mathfrak{a}^\phi_+ = 0.}$$

Kruglikov–T. (2014) ("prolongation-rigidity")

(⚠ Not all $G/P$ are prolongation-rigid, e.g. $A_3/P_{1,2}$.)

## Example ($\phi = y^4$)

Let $\mathfrak{a} := \mathfrak{a}^\phi_+$, so $\mathfrak{a}_0 = \mathfrak{ann}(\phi) = \langle \mathsf{Z}_2, e_{-\alpha_2} \rangle$. $\boxed{\text{Claim: } \mathfrak{a}_+ = 0}$.

Let $x = ae_{\alpha_1} + be_{\alpha_1+\alpha_2} \in \mathfrak{g}_1$. Note $[x, e_{-\alpha_1}] = ah_{\alpha_1} + bce_{\alpha_2}$:

- $c \neq 0$. (Use $\dim(\mathfrak{g}_\alpha) = 1$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ when $\alpha, \beta, \alpha + \beta \in \Delta$.)
- $h_{\alpha_1} := [e_{\alpha_1}, e_{-\alpha_1}] = 2\mathsf{Z}_1 - 3\mathsf{Z}_2$. (Chevalley basis $\Rightarrow h_{\alpha_i}(e_{\alpha_j}) = c_{ji}e_{\alpha_j}$)

Then $[x, \mathfrak{g}_{-1}] \subset \mathfrak{a}_0$ forces $x = 0$, so $\mathfrak{a}_1 = 0$ and hence $\mathfrak{a}_+ = 0$.

# Root types for Cartan's quartic

Any binary quartic can be classified by its root type. One can normalize such quartics using the $G_0 \cong \mathrm{GL}_2(\mathbb{C})$ action.

| Root type | Normal form $\phi$ | $\mathfrak{ann}(\phi)$ | $\dim(\mathfrak{a}^\phi)$ |
|---|---|---|---|
| N | $y^4$ | $Z_2,\ e_{-\alpha_2}$ | 7 |
| III | $xy^3$ | $Z_1 - 4Z_2$ | 6 |
| D | $x^2y^2$ | $Z_1 - 2Z_2$ | 6 |
| II | $x^2y(x-y)$ | 0 | 5 |
| I | $xy(x-y)(x-ky)$ | 0 | 5 |

For non-flat homogeneous $(2,3,5)$-distributions, $\boxed{\mathfrak{s} \subset \mathfrak{a}^{\kappa_H}}$ implies $\boxed{\text{sym dim} \leq 7}$. (Type-specific bounds in the second-last column.)

Multiply-transitive models: potentially only in types N, III, D.

Note: Submaximal sym dim $\boxed{\mathfrak{S} = 7}$: Cartan 1910 (locally constant type assumed), Kruglikov–T. 2014 (no assumptions).

Multiply-transitive $(2,3,5)$-distributions

(Fund. thm of parabolic geometries $\Rightarrow$ regular, normal Cartan geometry of type $(G,P) = (G_2, P_1)$)

# Type III

**Proposition (Cartan 1910)**

$\nexists$ *multiply-transitive, type* III $(2,3,5)$*-distribution.*

Kruglikov–T. (2014): Efficient proof that avoids Cartan reduction.

Here is our proof (with minor modifications). Assume $(\mathfrak{f};\mathfrak{g},\mathfrak{p})$ is a type III algebraic model with $\dim(\mathfrak{f}^0) > 0$. WLOG, use $G_0$-action s.t. $\kappa_H$ corresponds to $xy^3$, with weight $4\alpha_1 + \alpha_2$.

$$\begin{aligned}\mathfrak{s} = \mathrm{gr}(\mathfrak{f}) = \mathfrak{a}^{\kappa_H} &= \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a}_0 \\ &= \langle f_{31}, f_{32}\rangle \oplus \langle f_{21}\rangle \oplus \langle f_{10}, f_{11}\rangle \oplus \langle T := Z_1 - 4Z_2\rangle\end{aligned}$$

Step #1: Using the $P$-action to normalize $\mathfrak{f}$, we may assume $T \in \mathfrak{f}^0$.

Pf: Wrt basis $\{e_{10}, e_{11}, e_{21}, e_{31}, e_{32}\}$, $\mathrm{ad}_T|_{\mathfrak{g}_+} = \mathrm{diag}(1,-3,-2,-1,-5)$.
Let $\widetilde{T} \in \mathfrak{f}^0$, $\mathrm{gr}_0(\widetilde{T}) = T$, i.e. $\widetilde{T} = T + c_{10}e_{10} + \ldots + c_{32}e_{32}$. Use $P$-action:

$$\mathrm{Ad}_{\exp(te_{10})}\widetilde{T} = \exp(t\,\mathrm{ad}_{e_{10}})\widetilde{T} = \widetilde{T} + t[e_{10},\widetilde{T}] + \ldots = T + (c_{10} - t)e_{10} + \ldots$$

Set $t = c_{10}$, so WLOG $c_{10} = 0$. Continuing, we may assume $T \in \mathfrak{f}^0$.
Pf #2: No multiple of $4\alpha_1 + \alpha_2$ is contained in the roots of $\mathfrak{g}_+$.

# Type III - continued

$\boxed{\text{Step \#2: Show } \kappa = 0.}$

Wrt basis $\{f_{10}, f_{11}, f_{21}, f_{31}, f_{32}\}$, $\mathrm{ad}_T|_{\mathfrak{g}_-} = \mathrm{diag}(-1, 3, 2, 1, 5)$. Let $X_1 \in \mathfrak{f}^{-1}$ with $\mathrm{gr}_{-1}(X_1) = f_{10}$, etc. Then in the basis $\{T, X_1, X_2, X_3, X_4, X_5\}$, we have $\mathrm{ad}_T|_{\mathfrak{f}} = \mathrm{diag}(0, -1, 3, 2, 1, 5)$.

Note $[\cdot, \cdot]_{\mathfrak{f}}$ is $\mathrm{ad}_T$-equivariant, so (rescaling basis elements if necessary) the only non-trivial brackets beyond the $[T, \cdot]_{\mathfrak{f}} = [T, \cdot]$ are:

$$\begin{aligned} &[X_1, X_2]_{\mathfrak{f}} = X_3, \quad [X_1, X_3]_{\mathfrak{f}} = X_4, \quad [X_2, X_3]_{\mathfrak{f}} = X_5, \\ &[X_1, X_4]_{\mathfrak{f}} = aT, \quad [X_3, X_4]_{\mathfrak{f}} = bX_2. \end{aligned}$$

Let $\mathrm{Jac}_{\mathfrak{f}}$ be the Jacobiator. Then

$$0 = Jac_{\mathfrak{f}}(X_1, X_4, X_5) = -5aX_5, \quad 0 = Jac_{\mathfrak{f}}(X_1, X_3, X_4) = (2a - b)X_3.$$

$\therefore \kappa = 0$, so $\kappa_H = 0$. Contradiction to type III.

# Type $\mathrm{N}$

All multiply-transitive type $\mathrm{N}$ models have either 6 or 7 syms.

- $\mathrm{N.7}$: $z' = (y'')^a$ for $a \in \mathbb{C} \backslash \{-1, -\frac{1}{3}, \frac{2}{3}, 2\}$; also, $z' = \log(y'')$. See Cartan (1910). Also see Doubrov–Kruglikov (2014).
- $\mathrm{N.6}$: $z' = y + (y'')^{1/3}$, see Doubrov–Govorov (2013).

Let's focus on $\mathrm{N.7}$. Via his reduction method, Cartan arrives at:

$$\left\{\begin{aligned}
\omega_1' &= 2\,\omega_1\varpi_1 + \omega_2\varpi_2 + \omega_3\omega_4,\\
\omega_2' &= \omega_2\varpi_1 + \omega_3\omega_5,\\
\omega_3' &= \mathrm{I}\,\omega_2\omega_5 + \omega_3\varpi_1 + \omega_4\omega_5,\\
\omega_4' &= \frac{4}{3}\mathrm{I}\,\omega_3\omega_5 + \omega_4\varpi_1 + \omega_5\varpi_2,\\
\omega_5' &= 0,\\
\varpi_1' &= 0,\\
\varpi_2' &= \varpi_2\varpi_1 - \mathrm{I}\,\omega_4\omega_5 + \omega_2\omega_3.
\end{aligned}\right.$$

He also gave "embedding data" (cf. Lecture 2). But remember: The Cartan "connection" is NOT a "Cartan connection". Let's find the Cartan-theoretic description, i.e. an algebraic model $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$.

# Algebraic model for N.7

Using the $G_0$-action, WLOG $\kappa_H \leftrightarrow y^4$. Weight: $+4\alpha_1$.

$$\mathfrak{s} = \mathrm{gr}(\mathfrak{f}) = \mathfrak{a}^{\kappa_H} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a}_0 = \langle f_{31}, f_{32} \rangle \oplus \langle f_{21} \rangle \oplus \langle f_{10}, f_{11} \rangle \oplus \langle Z_2, f_{01} \rangle$$

**Theorem**

*Any N.7 algebraic model is P-equivalent to $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ given below for some $c \in \mathbb{C}$. These are classified by the essential invariant $c^2$.*

$$\mathfrak{f}: \quad \begin{array}{llll} S = Z_2, & X_1 = f_{10} + c e_{10}, & \multirow{2}{*}{$X_3 = f_{21},$} & X_4 = f_{31}, \\ N = f_{01}, & X_2 = f_{11}, & & X_5 = f_{32}. \end{array}$$

$$\kappa = f_{10}^* \wedge f_{31}^* \otimes f_{01}$$

| $[\cdot,\cdot]_{\mathfrak{f}}$ | $S$ | $N$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|---|---|---|---|---|---|---|---|
| $S$ | $\cdot$ | $-N$ | $\cdot$ | $-X_2$ | $-X_3$ | $-X_4$ | $-2X_5$ |
| $N$ | | $\cdot$ | $X_2$ | $\cdot$ | $\cdot$ | $-X_5$ | $\cdot$ |
| $X_1$ | | | $\cdot$ | $-3cN - 2X_3$ | $-2cX_2 + 3X_4$ | $-N + cX_3$ | $\cdot$ |
| $X_2$ | | | | $\cdot$ | $-3X_5$ | $\cdot$ | $\cdot$ |
| $X_3$ | | | | | $\cdot$ | $\cdot$ | $\cdot$ |
| $X_4$ | | | | | | $\cdot$ | $\cdot$ |
| $X_5$ | | | | | | | $\cdot$ |

- Let $S, N \in \mathfrak{f}^0$, $\mathrm{gr}_0(S) = Z_2$, $\mathrm{gr}_0(N) = f_{01}$. $Z_2(\alpha) \neq 0$, $\forall \alpha \in \Delta(\mathfrak{g}_+) \backslash \{\alpha_1\}$, so normalize $S = Z_2 + a e_{10}$, which reduces $P_+ = \exp(\mathfrak{g}_+)$ to $\exp(\mathfrak{g}_{\alpha_1})$.
- Let $X \in \mathfrak{f}^{-1}$, $\mathrm{gr}_{-1}(X) = f_{10}$. WLOG, $X \equiv f_{10} + b e_{01} \mod \mathfrak{g}_+$. (Use $S, N$; use $\exp(\mathfrak{g}_{\alpha_1})$ to remove $h_{10} = 2Z_1 - 3Z_2$ component.) Since $\kappa(S, \cdot) = 0$:

$$[S, X]_{\mathfrak{f}} = [S, X] \equiv a h_{10} + b e_{01} \mod \mathfrak{g}_+.$$

- $[S,X]_{\mathfrak{f}} \in \mathfrak{f}^0$, but $\mathrm{gr}_0([S,X]_{\mathfrak{f}}) \cap \mathfrak{a}_0 = 0$, so $a = b = 0$. Also, $[S,X]_{\mathfrak{f}} \in \mathfrak{f}$ forces $[S,X]_{\mathfrak{f}} = 0$. $\therefore$ $\boxed{S = \mathsf{Z}_2 \in \mathfrak{f}^0}$ and $\boxed{X = f_{10} + c e_{10}}$, $c \in \mathbb{C}$.
- Write the deformation "tails" as $\mathfrak{d} \in \mathfrak{a}^* \otimes \mathfrak{g}/\mathfrak{a}$, e.g. $X_\alpha = f_\alpha + \mathfrak{d}(f_\alpha)$, and $[S, X_\alpha]_{\mathfrak{f}} = [S, X_\alpha] = [S, f_\alpha] + [S, \mathfrak{d}(f_\alpha)]$. For $\alpha \in \Delta(\mathfrak{g}_+) \backslash \{\alpha_1\}$, $[S, X_\alpha] \neq 0$, so $\mathfrak{d}([S, f_\alpha]) = [S, \mathfrak{d}(f_\alpha)]$. Similar for $N$. $\therefore$ $\boxed{\mathfrak{d} \text{ is } S\text{-invariant}}$.
- $\mathfrak{a}^* \otimes \mathfrak{g}/\mathfrak{a}$ is an $\mathfrak{h}$-module, all weights are $\geq 0$. $\boxed{\text{Want: multiples of } \alpha_1}$. $\Rightarrow$ weight vectors: $\mathsf{Z}_2^* \otimes e_{10}$ (removed above), $f_{10}^* \otimes e_{10}$ (corresponds to $c$).

Now investigate curvature $\kappa$. Since $\mathsf{Z}_2 \in \mathfrak{f}^0$, then $\mathsf{Z}_2 \cdot \kappa = 0$. Key facts:

- Parabolic geometries $\Rightarrow$ lowest degree part of $\kappa$ is harmonic.
- $G_2/P_1$ geometry is torsion-free, i.e. $\kappa \in \bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{p}$.

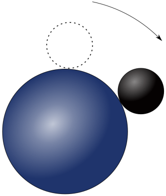
Want weights $\boxed{\sigma = r\alpha_1}$ in $\bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{p}$ with $\boxed{r = \mathsf{Z}_1(\sigma) \geq 4}$ (by Kostant).

- $r = 4$: Kostant $\Rightarrow$ weight vector $\phi_0 = f_{10}^* \wedge f_{31}^* \otimes f_{01}$.
- $r \geq 5$: Write $\sigma = \alpha + \beta + \gamma$ with $\alpha, \beta \in \Delta(\mathfrak{g}_+)$ distinct, $\gamma \in \Delta(\mathfrak{p}) \cup \{0\}$.
  - $\gamma \geq 0$: $\alpha, \beta$ are multiples of $\alpha_1$, so $\alpha = \beta = \alpha_1$ (contradiction).
  - $\gamma \in \Delta^-(\mathfrak{p})$: $\gamma = -\alpha_2$ and $\mathsf{Z}_1(\alpha + \beta) \geq 5$, so $\alpha, \beta \in \Delta(\mathfrak{g}_{\geq 2})$ $= \{2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$. Then $\mathsf{Z}_2(\alpha + \beta) \geq 2$, so $\mathsf{Z}_2(\sigma) \geq 1$ (contradiction).

# Type D

Cartan (1910): Classified type D models over $\mathbb{C}$. By our earlier calculation, all type D homogeneous structures have sym dim $\leq 6$.

The list over $\mathbb{R}$ (Willse 2019) contains many interesting distributions including those describing "rolling without twisting or slipping" involving surfaces of constant curvature: 2-sphere, Euclidean & hyperbolic planes.



**Theorem (Exceptionality of the 3 : 1 ratio)**

*Consider two 2-spheres in $\mathbb{R}^3$ with ratio of radii $\rho > 1$. Let $(M^5, D_\rho)$ be the $(2,3,5)$-distribution associated to their rolling on each other without twisting or slipping. Then $D_\rho$ has symmetry algebra:*
(i) $\mathfrak{so}(3) \times \mathfrak{so}(3)$ *when* $\rho \neq 3$; (ii) $\mathfrak{g}_2^{\text{split}}$ *when* $\rho = 3$.

Origins: Bryant (unpublished), Zelenko (2006), Bor–Montgomery (2009), Baez–Huerta (2014).

# Origins of the $3:1$ ratio

Baez–Huerta (2014): Viewed it via "rolling spinor on a projective plane" & incidence geometry of $G_2$.

Zelenko (2006): Computed Cartan quartic via abnormal extremals. It vanishes when $\rho = 3$.

Bor–Montgomery (2009): Lie-theoretic data & embedding into $\mathfrak{g}_2^{\mathrm{split}}$ when $\rho = 3$. Also: *"This theorem was communicated to us by Robert Bryant for whom it is in essence contained in E. Cartan's notoriously difficult "Five Variables Paper" of 1910. R. Bryant wrote to us:*

*'Cartan himself gave a geometric description of the flat $G_2$-structure as the differential system that describes space curves of constant torsion 2 or $\frac{1}{2}$ in the standard unit 3-sphere. (See the concluding remarks of Section 53 in Paragraph XI in the Five Variables Paper.) One can easily transform the rolling balls problem (for arbitrary ratios of radii) into the problem of curves in the 3-sphere of constant torsion and, in this guise, one can recover the 3:1 or 1:3 ratio as Cartan's torsion 2 or $\frac{1}{2}$ with a minimum of fuss. Thus, one could say that Cartan's calculation essentially covers the rolling ball case.' "*

Cartan (1910): *"Dans le cas où cette torsion est égale à 2 ou $\frac{1}{2}$, le système admet un group à quatorze paramètres; dans les autres cas il admet le groupe à six paramètres des mouvements de l'espace considéré."*

# Lie-theoretic description

Bor–Montgomery (2009): Gave Lie-theoretic data $\mathfrak{f}_{\mathbb{R}}/\mathfrak{f}^0_{\mathbb{R}}$ associated to $(M^5, D_\rho)$. They showed $\mathfrak{f}_{\mathbb{R}} \hookrightarrow \mathfrak{g}_2^{\text{split}}$ when $\rho = 3$.

- Let $\mathfrak{so}(3) = (\mathbb{R}^3, \times)$: $\quad [\mathbf{i},\mathbf{j}] = \mathbf{k},\ [\mathbf{j},\mathbf{k}] = \mathbf{i},\ [\mathbf{k},\mathbf{i}] = \mathbf{j}$.
- Let $\mathfrak{f}_{\mathbb{R}} := \mathfrak{so}(3) \times \mathfrak{so}(3)$, $\mathfrak{f}^0_{\mathbb{R}} := \langle (\mathbf{k},\mathbf{k}) \rangle$, & for $\rho \in \mathbb{R}^\times$,

$$\mathfrak{f}^{-1}_{\mathbb{R}} := \langle (\mathbf{i}, -\rho\mathbf{i}), (\mathbf{j}, -\rho\mathbf{j}), (\mathbf{k},\mathbf{k}) \rangle \qquad (\mathfrak{f}^0_{\mathbb{R}}\text{-invariant})$$

- Swapping $\mathfrak{so}(3)$ factors induces $\rho \mapsto \frac{1}{\rho}$.
- $(\mathbf{i},\mathbf{j},\mathbf{k}) \mapsto (\pm\mathbf{i}, \mp\mathbf{j}, -\mathbf{k})$ are two involutions of $\mathfrak{so}(3)$. Using one on each $\mathfrak{so}(3)$ factor induces $\rho \mapsto -\rho$.

Exclude $\rho = 1$ ($\mathfrak{f}^{-1}_{\mathbb{R}}$ is a subalgebra), so $\boxed{\rho > 1}$ suffices. Get weak derived flag $\mathfrak{f}_{\mathbb{R}} = \mathfrak{f}^{-3}_{\mathbb{R}} \supset \mathfrak{f}^{-2}_{\mathbb{R}} \supset \mathfrak{f}^{-1}_{\mathbb{R}} \supset \mathfrak{f}^0_{\mathbb{R}}$. Define an adapted basis:

$$\widetilde{T} = (\mathbf{k},\mathbf{k}), \quad \widetilde{X}_1 = (\mathbf{i}, -\rho\mathbf{i}), \quad \widetilde{X}_2 = (\mathbf{j}, -\rho\mathbf{j}),$$
$$\widetilde{X}_3 = (\mathbf{k}, \rho^2\mathbf{k}), \quad \widetilde{X}_4 = (\mathbf{j}, -\rho^3\mathbf{j}), \quad \widetilde{X}_5 = (\mathbf{i}, -\rho^3\mathbf{i}).$$

I'll now outline a Cartan-theoretic proof of exceptionality of $\rho = 3$.

**Theorem (T. 2021)**

*Any* D.6 *algebraic model is P-equivalent to one given below (denoted (†)), where* $a \in \mathbb{C}$, $b \in \mathbb{C}^\times$. *These are classified by* $\mathcal{I} := \frac{a^2}{b} \in \mathbb{C}$, *so WLOG,* $b = 1$.

$$\mathfrak{f}: \quad \begin{array}{lll} T = h_{01} = -Z_1 + 2Z_2, & X_2 = f_{11} + a e_{10} + b e_{31}, & X_4 = f_{31} + b e_{11} + a(a^2 + \frac{b}{3}) e_{32} \\ X_1 = f_{10} + a e_{11} + b e_{32}, & X_3 = f_{21} + (a^2 + b) e_{21}, & X_5 = f_{32} + b e_{10} + a(a^2 + \frac{b}{3}) e_{31} \end{array}$$

$$\kappa = [\cdot,\cdot] - [\cdot,\cdot]_{\mathfrak{f}} = 4b\,\kappa_4 + \frac{4ab}{3}\,\kappa_6 + 2a^2 b\,\kappa_8$$

$$\begin{cases} \kappa_4 = f_{11}^* \wedge f_{32}^* \otimes f_{01} + (-f_{10}^* \wedge f_{32}^* + f_{11}^* \wedge f_{31}^*) \otimes h_{01} + f_{10}^* \wedge f_{31}^* \otimes e_{01} \\ \kappa_6 = f_{11}^* \wedge f_{31}^* \otimes e_{21} - f_{10}^* \wedge f_{32}^* \otimes e_{21} + f_{31}^* \wedge f_{32}^* \otimes h_{01} \\ \qquad + 2 f_{21}^* \wedge f_{31}^* \otimes e_{11} - 2 f_{21}^* \wedge f_{32}^* \otimes e_{10} \quad \in \mathrm{im}(\partial^*) \\ \kappa_8 = f_{21}^* \wedge f_{32}^* \otimes e_{31} - f_{21}^* \wedge f_{31}^* \otimes e_{32} + f_{31}^* \wedge f_{32}^* \otimes e_{21} \quad \in \mathrm{im}(\partial^*) \end{cases}$$

| $[\cdot,\cdot]_{\mathfrak{f}}$ | $T$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|---|---|---|---|---|---|---|
| $T$ | $\cdot$ | $+X_1$ | $-X_2$ | $\cdot$ | $+X_4$ | $-X_5$ |
| $X_1$ | | $\cdot$ | $-2X_3 + 3aT$ | $+3X_4 + 2aX_1$ | $\cdot$ | $-aX_3 + 6bT$ |
| $X_2$ | | | $\cdot$ | $-3X_5 - 2aX_2$ | $+aX_3 - 6bT$ | $\cdot$ |
| $X_3$ | | | | $\cdot$ | $-(a^2 + 3b)X_1$ | $(a^2 + 3b)X_2$ |
| $X_4$ | | | | | $\cdot$ | $-2bX_3 + a(a^2 - b)T$ |
| $X_5$ | | | | | | $\cdot$ |

Let $\mathfrak{f} := \mathfrak{f}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ (filtered). $\boxed{\text{Goal: Match it to the D.6 eqns } (\dagger).}$ There, $\mathrm{ad}_T$ has (double) eigenvalues $0, \pm 1$, while $\mathrm{ad}_{\widetilde{T}}$ has $0, \pm i$, so define $T := i\widetilde{T}$. The general filtration-adapted, $\mathrm{ad}_T$-eigenbasis extending $T$ is:

$$X_1 := s_1(\widetilde{X}_1 + i\widetilde{X}_2), \quad X_2 := s_2(\widetilde{X}_1 - i\widetilde{X}_2), \qquad X_3 := s_3\widetilde{X}_3 + t_1 i\widetilde{T}, \qquad X_4 := s_4(\widetilde{X}_4 - i\widetilde{X}_5) + t_2(\widetilde{X}_1 + i\widetilde{X}_2), \quad X_5 := s_5(\widetilde{X}_4 + i\widetilde{X}_5) + t_3(\widetilde{X}_1 - i\widetilde{X}_2).$$

Now impose the $\mathfrak{f}$-brackets from $(\dagger)$:

$$0 = [X_1, X_2]_{\mathfrak{f}} + 2X_3 - 3aT, \quad 0 = [X_1, X_5]_{\mathfrak{f}} + aX_3 - 6bT,$$
$$0 = [X_1, X_3]_{\mathfrak{f}} - 3X_4 - 2aX_1, \quad 0 = [X_3, X_4]_{\mathfrak{f}} + (a^2 + 3b)X_1.$$

One finds $s_i, t_j$ from these, as well as $\boxed{b = \frac{a^2(\rho+3)(\rho-3)(\rho+\frac{1}{3})(\rho-\frac{1}{3})}{4(\rho^2+1)^2}}$.

1. $b \neq 0$: WLOG, $b = 1$, so $\boxed{\mathcal{I} = a^2 = \frac{4(\rho^2+1)^2}{(\rho+3)(\rho-3)(\rho+\frac{1}{3})(\rho-\frac{1}{3})}}$.
2. $b = 0$: Have $s_i \neq 0 \Rightarrow a \neq 0$, so $\rho \in \{\pm 3, \pm\frac{1}{3}\}$ and $\kappa = 0$, so $\mathfrak{f} \hookrightarrow \mathrm{Lie}(G_2)$, and hence $\mathfrak{f}_{\mathbb{R}} \hookrightarrow \mathfrak{g}_2^{\mathrm{split}}$. $\square$

# Holonomy

In Lecture 2, we saw how to compute (infinitesimal) holonomy. For the algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$ shown today, we easily find:

**Theorem**

$$\mathfrak{hol} \cong \begin{cases} \textit{5-dim Heisenberg}, & \text{N.7;} \\ \mathfrak{g} = \mathrm{Lie}(G_2), & \text{D.6, } \mathcal{I} \neq 0; \\ \mathfrak{sl}_3, & \text{D.6, } \mathcal{I} = 0. \end{cases}$$

See also Willse (2014, 2017) & Sagerschnig–Willse (2016, 2017) for related (conformal) holonomy results concerning the Nurowski (2004) conformal structure for $(2, 3, 5)$-distributions (and associated ambient metric).

# Perspectives & outlook

These lectures have been an invitation to Cartan / parabolic geometry.

- 20th century: A canonical connection is the goal / endpoint of many articles. It is the "solution in the sense of Élie Cartan".
- 21st century: Cartan perspective is the input / starting point for further geometric investigations. (Symmetry classification, emphasized here, is merely one application among many.)
- Presented two natural approaches to symmetry classification via:
  - Cartan reduction method;
  - $P$-orbits of algebraic models $(\mathfrak{f}; \mathfrak{g}, \mathfrak{p})$.
- Parabolic setting: (i) Kostant's thm is crucial, (ii) Tanaka prolongation $\Rightarrow$ "graded" geometric approximation, i.e. $\boxed{\mathfrak{s} \subset \mathfrak{a}^{\kappa_H}}$.
- Contrasted coordinate / Lie-theoretic vs. Cartan-theoretic model descriptions. (The latter is highly lacking in the literature.)

Concerning 1910, I often think of a quote from Bryant (in Sharpe 1997):

*"You read the introduction to a paper of Cartan and you understand nothing. Then you read the rest of the paper and still you understand nothing. Then you go back and read the introduction again and there begins to be the faint glimmer of something very interesting."*