Classifying homogeneous geometric structures (Lecture 2)

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Outline

Last time: A Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P):

- ... is a "nice" soln to a Cartan equiv. problem for str. on M.
- ... is a curved deformation of $(G \to G/P, \omega_G)$.
- ... can be viewed in terms of structure equations.

Today:

1. Parabolic geometries: normalization conditions & harmonic curvature κ_H.
2. Cartan reduction method applied to 2nd order ODE.
   - Setting up structure equations.
   - Implementing the (equivariant) reduction method.
   - Interpreting the results.

Exhibiting homogeneous models

Q: How to exhibit a homogeneous model?

Example (2nd order ODE)

- Coordinate model: Let $p = y'$. Then $y'' = \exp(p)$ has syms

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_y, \quad \mathbf{X}_3 = x\partial_x + (y - x)\partial_y - \partial_p.$$

- Lie-theoretic model: $(\mathfrak{f}, \mathfrak{f}^0)$ with $\mathfrak{f}^0 = 0$, $\mathfrak{f} = \langle e_1, e_2, e_3 \rangle$ s.t.

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = -e_2 + e_3, \quad [e_2, e_3] = 0.$$

  Endow $\mathfrak{f}/\mathfrak{f}^0$ with pair of lines $\langle e_1 - e_3 \rangle \oplus \langle e_3 \rangle$. (Evaluate $E \oplus V = \langle \partial_x + p\partial_y + \exp(p)\partial_p \rangle \oplus \langle \partial_p \rangle$ at say $x = y = p = 0$.)

- Cartan-theoretic model: ?

Q: What is the curvature / holonomy? (These readily follow from the Cartan-theoretic description.)

Parabolic geometries

Parabolic subalgebras

Definition

Let $\mathfrak{g}$ be a semisimple Lie alg. A subalgebra $\mathfrak{p} \subsetneq \mathfrak{g}$ is parabolic if $\mathfrak{p} = \mathfrak{g}_{\geq 0}$ for some $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ (with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\forall i, j$).

- $\mathfrak{p} = \mathfrak{g}_0 \ltimes \mathfrak{g}_+$, $\quad \mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \times \mathfrak{g}_0^{\rm ss}$, $\quad \mathfrak{g}_+ =$ nilradical of $\mathfrak{p}$.
- $\exists$ grading element $Z \in \mathfrak{z}(\mathfrak{g}_0)$, i.e. $\mathrm{ad}_Z|_{\mathfrak{g}_j} = j\, \mathrm{id}|_{\mathfrak{g}_j}$, $\forall j \in \mathbb{Z}$.
- Defining $\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_j$, $\exists$ $\mathfrak{p}$-inv. filtration on $\mathfrak{g}$ with $\mathfrak{g}^0 = \mathfrak{p}$.
- Ndg Killing form $B(x,y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$ induces:
    - $(\mathfrak{g}_-)^* \cong \mathfrak{g}_+$ as $\mathfrak{g}_0$-modules (so $\mathfrak{g} = \mathfrak{g}_{-\nu} \oplus \ldots \oplus \mathfrak{g}_\nu$);
    - $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{g}_+$ as $\mathfrak{p}$-modules.
- Geometrically, the filtration is important! Grading is auxilliary.

Example ($\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{p} = \mathfrak{p}_{1,2}$ = (trace-free) upper triangular)

$\mathfrak{p} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \subset \mathfrak{g}$. We have $\mathfrak{g} = \overbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}^{\text{Heisenberg}} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}^{\mathfrak{p}}$, so

$\mathfrak{g} \hookrightarrow \mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$. In fact, $\mathfrak{g} \cong \mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$. (Next lecture, via $H^1(\mathfrak{g}_-, \mathfrak{g})$.)

Aside: Root space decomposition

Let $\mathfrak{g}$ be $\mathbb{C}$-semisimple, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra.

Example ($\mathfrak{g} = \mathfrak{sl}_2$)

"$\mathfrak{sl}_2$-triple": $\mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathsf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathsf{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Commutators:

$$[\mathsf{H}, \mathsf{E}] = +2\mathsf{E}, \quad [\mathsf{H}, \mathsf{F}] = -2\mathsf{F}, \quad [\mathsf{E}, \mathsf{F}] = \mathsf{H}.$$

Eigenvalues of ad_{H} wrt $(\mathsf{F}, \mathsf{H}, \mathsf{E})$-basis are $-2, 0, +2$. Here $\mathfrak{h} = \langle \mathsf{H} \rangle$.

Given $\alpha \in \mathfrak{h}^*$, define $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$.
Root system: $\Delta = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\}$. Have $\Delta = \Delta^+ \cup (-\Delta^+)$.

- Root space decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.
- Borel subalg $\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$.
- Defn: Any subalgebra $\mathfrak{p} \subsetneq \mathfrak{g}$ containing $\mathfrak{b}$ is parabolic.
  (Equivalent to previous defn.)

Aside: Lie algebra gradings

Example (Root space decomp: $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{h}$ = trace-free diagonal)

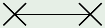
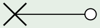
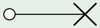
Let $\epsilon_i(\text{diag}(a_1, a_2, a_3)) = a_i$. Simple rts: $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$.

$\alpha \in \Delta$	α_1	α_2	$\alpha_1 + \alpha_2$	$-\alpha_1$	$-\alpha_2$	$-\alpha_1 - \alpha_2$
Basis for $\mathfrak{g}_\alpha$	E_{12}	E_{23}	E_{13}	E_{21}	E_{32}	E_{31}
Alt. notation	e_{10}	e_{01}	e_{11}	f_{10}	f_{01}	f_{11}

$\ell := \text{rank}(\mathfrak{g})$. Simple roots $\{\alpha_i\}_{i=1}^{\ell} \subset \mathfrak{h}^*$, dual basis $\{Z_i\}_{i=1}^{\ell} \subset \mathfrak{h}$.
FACT: $\mathfrak{p} \subset \mathfrak{g} \leftrightarrow I_{\mathfrak{p}} \subset \{1, ..., \ell\}$. Grading element $Z := \sum_{i \in I_{\mathfrak{p}}} Z_i$.

Example (Parabolics in $\mathfrak{sl}_3$)

$Z_1 = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, $Z_2 = \text{diag}(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$.

Z	$Z_1 + Z_2$	Z_1	Z_2
$\mathfrak{p}$	$\mathfrak{p}_{1,2} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$	$\mathfrak{p}_1 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\mathfrak{p}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$
marked Dynkin diagram			

Parabolic geometries

G: ss Lie grp, P: parabolic subgrp, i.e. parabolic $\mathfrak{p} = \mathrm{Lie}(P) \subset \mathfrak{g}$.

A parabolic geom. is a Cartan geom. $(\mathcal{G} \to M, \omega)$ of type (G, P):

- $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear iso. $\forall u \in \mathcal{G}$;
- $R_p^*\omega = \mathrm{Ad}_{p^{-1}} \circ \omega$, $\forall p \in P$;
- $\omega(\widetilde{X}) = X$, $\forall X \in \mathfrak{p}$, where $\widetilde{X}$ = fund. vertical v.f. for X.

Curvature $K = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}; \mathfrak{g})$. Curvature function

$$\kappa : \mathcal{G} \to \bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{g}.$$

Definition

1. *regular* if $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$, $\forall i, j$. ($\iff$ κ is valued in the subspace of $\bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ on which Z has pos. eigenvalues.)
2. *normal* if $\partial^*\kappa = 0$, where $\partial^* : \bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{g} \to \mathfrak{g}_+ \otimes \mathfrak{g}$ is def. by

$$\partial^*(X \wedge Y \otimes v) = -Y \otimes [X, v] + X \otimes [Y, v] - [X, Y] \otimes v.$$

$\therefore \kappa \in \ker(\partial^*)_+ = \{\phi \in \ker(\partial^*) : \mathrm{Z}(\phi) > 0\}$ ("curvature module").

Fundamental theorem of parabolic geometries

Theorem (Tanaka, Morimoto, Čap–Schichl)

There is an equivalence of categories:

$$\left\{ \begin{array}{c} \textit{regular, normal} \\ \textit{parabolic geometry of type } (G,P) \\ (\mathcal{G} \to M, \omega) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{“underlying geometric} \\ \textit{structure” on } M \end{array} \right\}$$

Example (Underlying structures of parabolic geometries)

Conformal, projective, $(2,3,5)$, CR, 2nd order ODE systems, ...

The formulation above is a paraphrasing. More precisely:

- If $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{g}$, then “underlying str.” is a filtered G_0-str.
- If $\mathrm{pr}(\mathfrak{g}_-) \cong \mathfrak{g}$, then “underlying str.” is a distribution.
- Notable exceptions when $\mathfrak{g}$ is simple:
    - projective: $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$, $\mathfrak{p} = \mathfrak{stab}([e_1])$.
    - contact projective: $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$, $\mathfrak{p} = \mathfrak{stab}([e_1])$.

Kostant's harmonic theory

Recall: $(\mathfrak{g}_-)^* \cong_{\mathfrak{g}_0} (\mathfrak{g}/\mathfrak{p})^* \cong_{\mathfrak{p}} \mathfrak{g}_+$. Codomain of κ can be viewed in:

$$\begin{array}{ccccc} & & \bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} & & \\ & & \wr\| & & \\ \cdots & \xleftarrow{\partial^*} & \bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{g} & \xleftarrow{\partial^*} & \cdots \\ & & \wr\| & & \\ \cdots & \xrightarrow{\partial} & \bigwedge^2 (\mathfrak{g}_-)^* \otimes \mathfrak{g} & \xrightarrow{\partial} & \cdots \end{array}$$

Kostant Laplacian: $\boxed{\square = \partial\partial^* + \partial^*\partial}$ Algebraic Hodge decomp:

$$\bigwedge^2 (\mathfrak{g}_-)^* \otimes \mathfrak{g} = \overbrace{\mathrm{im}(\partial^*) \oplus \underbrace{\mathrm{ker}(\square)}_{}}^{\mathrm{ker}(\partial^*)} \!\!\!\!\!\!\!\!\underbrace{\phantom{\mathrm{ker}(\square)} \oplus \mathrm{im}(\partial)}_{\mathrm{ker}(\partial)},$$

$$H_2(\mathfrak{g}_+, \mathfrak{g}) := \frac{\mathrm{ker}(\partial^*)}{\mathrm{im}(\partial^*)} \cong_{\mathfrak{g}_0} \mathrm{ker}(\square) \cong_{\mathfrak{g}_0} \frac{\mathrm{ker}(\partial)}{\mathrm{im}(\partial)} =: H^2(\mathfrak{g}_-, \mathfrak{g})$$

homology | cohomology!

Harmonic (homological) curvature

For (regular / normal) parabolic geometries, define

$$\kappa_H : \mathcal{G} \to \frac{\ker(\partial^*)}{\mathrm{im}(\partial^*)} =: H_2(\mathfrak{g}_+, \mathfrak{g}), \quad \kappa_H := \kappa \mod \mathrm{im}(\partial^*).$$

Theorem (Harmonic curvature completely obstructs flatness)

$\kappa_H = 0$ *iff* $\kappa = 0$ *iff (locally) equivalent to* $(G \to G/P, \omega_G)$.

Examples (Harmonic curvature)

- conformal geometry: Weyl ($n \geq 4$) or Cotton ($n = 3$).
- $(2, 3, 5)$-distributions: binary quartic.
- 2nd order ODE: Tresse (relative) invariants I_1, I_2.

$H_2(\mathfrak{g}_+, \mathfrak{g})$ is a completely reducible $\mathfrak{p}$-rep, i.e. $\mathfrak{g}_+$ acts trivially.
$\therefore$ Study $H_2(\mathfrak{g}_+, \mathfrak{g}) \cong_{\mathfrak{g}_0} H^2(\mathfrak{g}_-, \mathfrak{g})$. (Use Kostant's thm - next lecture.)

Cartan reduction applied to 2nd order ODE

Cartan reduction - starting point

We'll describe a "top-down" method. (Need not be "homog.", but we need "locally const type" for bundle reductions.)

Input: (G, P). Fund. thm. $\Rightarrow$ abstract str. eqns for $(\mathcal{G} \to M, \omega)$, i.e. $d\omega = -\frac{1}{2}[\omega, \omega] + K$. Since ω is P-equivariant, then so is K.

Cartan reduction is a combination of: (i) orbit normalization, and (ii) integrability conditions ($d^2 = 0$). Key ideas:

- Use κ (& derivatives) to reduce str grp P. (Start with κ_H.)
- Reductions $\mathcal{H} \hookrightarrow \mathcal{G}$ (also subject to str grp action) are equipped with coframings, and sym. dim. $\leq \dim(\mathcal{H})$.
- Case branching is based on different orbit types.
  - Organize based on κ_H type, sym dim, further invariants, ...
  - Some branches are extraneous due to $d^2 = 0$.
- Output: List of homogeneous models. For each:
  1. Adapted coframings with constant structure functions.
  2. Embedding data, i.e. how does the original ω restrict?

Integration is only used on a given output ($\rightsquigarrow$ coordinate model).

2nd order ODE

Tresse (1896): For $y'' = f(x,y,y')$, $\exists$ two key relative invariants

$$I_1 = (D_x)^2(f_{pp}) - f_p D_x(f_{pp}) - 4D_x(f_{yp}) + 4f_p f_{yp} - 3f_y f_{pp} + 6f_{yy}, \quad I_2 = f_{pppp},$$

where $D_x := \partial_x + p\partial_y + f\partial_p$. Homogeneous classification (over $\mathbb{C}$):

Sym dim	Coordinate models ($p = y'$)
8	$y'' = 0$
3	$y'' = p^a$, $y'' = \frac{p(1-p^2)+c(1-p^2)^{3/2}}{x}$, $y'' = \exp(p)$, $y'' = (xp-y)^3$

The parameters a, c have exclusions and redundancies. (Later.)

Why adopt the Cartan approach? Is it overkill?

- Reasoning: This geometry is low-dim. One could obtain the list via classification of Lie alg of vector fields in the plane, i.e. prolong these to $J^2(\mathbb{C},\mathbb{C})$ and find their invariant ODEs.
- Limitations of v.f. approach to classification:
  - extraneous Lie algebras (need to solve PDE for each).
  - v.f. classification is incomplete in dim ≥ 3.
  - exclusions / redundancies? e.g. $y'' = p^3$ has 8-dim sym.
  - what is the curvature / holonomy of these models?

Parabolic geometry associated to 2nd order ODE

Fund. Thm. $\Rightarrow$ any 2nd order ODE corresponds to a regular, normal parabolic geometry $(\mathcal{G}^8 \to M^3, \omega)$ of type $(SL_3, P = P_{1,2})$.

$$\omega = \begin{pmatrix} \frac{2\zeta_1+\zeta_2}{3} & \tau_1 & \tau_3 \\ \varpi_1 & \frac{-\zeta_1+\zeta_2}{3} & \tau_2 \\ \varpi_3 & \varpi_2 & \frac{-\zeta_1-2\zeta_2}{3} \end{pmatrix} \quad \in \quad \Omega^1(\mathcal{G}; \mathfrak{sl}_3).$$

- Initial structure group dim $= \dim(P) = 5$; sym dim ≤ 8.
- Grading element: $Z = Z_1 + Z_2$. Bi-grading element: (Z_1, Z_2).

- ω is an abstract coframing; ϖ_i are semi-basic forms.
- Regular / normal $\Rightarrow \kappa \in \ker(\partial^*)_+$.

- Since $\mathfrak{g} = \mathfrak{sl}_3$ is a matrix Lie algebra, then $\frac{1}{2}[\omega, \omega] = \omega \wedge \omega$, i.e. matrix multiply & wedge the 1-form components.
- Useful later: given fund.v.f. $\widetilde{Z_1}$, we have $\omega(\widetilde{Z_1}) = Z_1$, so $\zeta_1(\widetilde{Z_1}) = 1$, $\zeta_2(\widetilde{Z_1}) = 0$, $\tau_1(\widetilde{Z_1}) = 0$, etc.

Structure equations

Curvature module $\ker(\partial^*)_+$ is 4-dim:

Hom.	Bi-hom.	2-chain in $\bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{g}$	2-cochain in $\bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$
$+4$	$(3,1)$	$e_{10} \wedge e_{11} \otimes e_{10}$	$f_{10}^* \wedge f_{11}^* \otimes e_{10}$
	$(1,3)$	$e_{01} \wedge e_{11} \otimes e_{01}$	$f_{01}^* \wedge f_{11}^* \otimes e_{01}$
$+5$	$(3,2)$	$e_{10} \wedge e_{11} \otimes e_{11}$	$f_{10}^* \wedge f_{11}^* \otimes e_{11}$
	$(2,3)$	$e_{01} \wedge e_{11} \otimes e_{11}$	$f_{01}^* \wedge f_{11}^* \otimes e_{11}$

Killing form $\Rightarrow (f_{10}^*, f_{01}^*, f_{11}^*) \mapsto (6e_{10}, 6e_{01}, 6e_{11})$.

"Primary" structure equations: $\boxed{d\omega = -\omega \wedge \omega + K}$

$$d\tau_3 = -(\zeta_1 + \zeta_2) \wedge \tau_3 - \tau_1 \wedge \tau_2 + B_1 \varpi_1 \wedge \varpi_3 + B_2 \varpi_2 \wedge \varpi_3$$
$$d\tau_2 = -\zeta_2 \wedge \tau_2 + \tau_3 \wedge \varpi_1 + A_2 \varpi_2 \wedge \varpi_3$$
$$d\tau_1 = -\zeta_1 \wedge \tau_1 - \tau_3 \wedge \varpi_2 + A_1 \varpi_1 \wedge \varpi_3$$
$$d\zeta_2 = \tau_1 \wedge \varpi_1 - 2\tau_2 \wedge \varpi_2 - \tau_3 \wedge \varpi_3$$
$$d\zeta_1 = -2\tau_1 \wedge \varpi_1 + \tau_2 \wedge \varpi_2 - \tau_3 \wedge \varpi_3$$
$$d\varpi_1 = \zeta_1 \wedge \varpi_1 - \tau_2 \wedge \varpi_3$$
$$d\varpi_2 = \zeta_2 \wedge \varpi_2 + \tau_1 \wedge \varpi_3$$
$$d\varpi_3 = \varpi_1 \wedge \varpi_2 + (\zeta_1 + \zeta_2) \wedge \varpi_3$$

$K = 0$: Maurer–Cartan str eqns for $\mathfrak{g} = \mathfrak{sl}_3$.

Equivariancy: vertical variations

Since ω is P-equivariant, so is K: $\boxed{R_p^* K = \mathrm{Ad}_{p^{-1}} \circ K}$, $\forall p \in P$.

Infinitesimally, $\boxed{\mathcal{L}_{\widetilde{X}} K = -\mathrm{ad}_X \circ K}$, $\forall X \in \mathfrak{p}$.

"Secondary" str. eqns (Exercise: Derive these in two ways.):

$$\begin{aligned} d(A_1) &= -(3\zeta_1 + \zeta_2)A_1 + \alpha_1 \\ d(A_2) &= -(\zeta_1 + 3\zeta_2)A_2 + \alpha_2 \\ d(B_1) &= -(3\zeta_1 + 2\zeta_2)B_1 + \tau_2 A_1 + \beta_1 \\ d(B_2) &= -(2\zeta_1 + 3\zeta_2)B_2 - \tau_1 A_2 + \beta_2 \end{aligned}$$

where $\alpha_i = a_{ij}\varpi_j$ and $\beta_i = b_{ij}\varpi_j$ are semi-basic forms. Comments:

- Cartan would write $\delta(A_1) = -(3\zeta_1 + \zeta_2)A_1$ ("vertical variation").
- I've abused notation with α_i earlier. (Not simple roots above.)

Bianchi identities ($d^2 = 0$, given primary / secondary str. eqns.):

$$a_{21} = -B_2, \quad a_{12} = B_1, \quad b_{12} = b_{21}.$$

This completes the setup of the structure equations.

Cartan reduction - first steps

Goal: Find non-flat homogeneous models.

Note A_1, A_2, B_1, B_2 are undetermined functions on $\mathcal{G}$. Start with A_1, A_2, i.e. harmonic curvature κ_H. (Note: Work over $\mathbb{C}$.)

$$\delta(A_1) = -(3\zeta_1 + \zeta_2)A_1, \quad \delta(A_2) = -(\zeta_1 + 3\zeta_2)A_2.$$

Coeffs of ζ_1 and ζ_2 $\quad\rightsquigarrow\quad$ $\mathfrak{p}$-action on (A_1, A_2)-space:

$$3A_1 \frac{\partial}{\partial A_1} + A_2 \frac{\partial}{\partial A_2}, \quad A_1 \frac{\partial}{\partial A_1} + 3A_2 \frac{\partial}{\partial A_2}.$$

(More precisely: Evaluate 1-forms on fund. v.f. $\widetilde{Z_1}$ and $\widetilde{Z_2}$.)

Have linearly indep. (complex) scalings, so we case-split:

1. $A_1 = A_2 = 0$: by Bianchi identities, we get the flat model;
2. $A_1 \neq 0$, $A_2 = 0$ (or dually, $A_1 = 0$, $A_2 \neq 0$);
3. $A_1 A_2 \neq 0$. (We'll illustrate key ideas in this branch.)

$A_1 A_2 \neq 0$ case

$$d(A_1) = -(3\zeta_1 + \zeta_2)A_1 + \alpha_1$$
$$d(A_2) = -(\zeta_1 + 3\zeta_2)A_2 + \alpha_2$$

Given $A_1 A_2 \neq 0$ (assumed locally true), we now normalize:

- $\boxed{A_1 = A_2 = 1}$. Get sub-bundle $\mathcal{H} \hookrightarrow \mathcal{G}$. Fibre dim $= 3$; sym dim $\leq \mathrm{dim}(\mathcal{H}) = 6$.
- On $\mathcal{H}$, $\boxed{\zeta_1 = \frac{3\alpha_1 - \alpha_2}{8}, \zeta_2 = \frac{3\alpha_2 - \alpha_1}{8}}$ are semi-basic. (Implicitly, I will henceforth omit "pullback" notation for forms.)

$$\begin{cases} d(B_1) = -(3\zeta_1 + 2\zeta_2)B_1 + \tau_2 A_1 + \beta_1 \\ d(B_2) = -(2\zeta_1 + 3\zeta_2)B_2 - \tau_1 A_2 + \beta_2 \end{cases} \overset{\text{on } \mathcal{H}}{\Rightarrow} \begin{cases} \delta(B_1) = \tau_2 \\ \delta(B_2) = -\tau_1 \end{cases}$$

On $\mathcal{H}$, coeffs of $\tau_1, \tau_2 \rightsquigarrow$ translations $\partial_{B_2}, -\partial_{B_1}$ on (B_1, B_2)-space, so normalize to get a subbundle $\mathcal{H}' \hookrightarrow \mathcal{H}$:

- $\boxed{B_1 = B_2 = 0}$. Moreover, $\boxed{\tau_1 = \beta_2, \tau_2 = -\beta_1}$ are semi-basic.
- Fibre dim. $= 1$; sym dim $\leq \mathrm{dim}(\mathcal{H}') = 4$.

$A_1 A_2 \neq 0$ continued...

Q: On $\mathcal{H}'$, can we normalize the residual 1-dim str grp?

On $\mathcal{H}'$, evaluate $d\omega$ in two ways, e.g. for τ_1, we have:

1. $d\tau_1 = -\zeta_1 \wedge \tau_1 - \tau_3 \wedge \varpi_2 + A_1 \varpi_1 \wedge \varpi_3$; impose relations.
2. Substitute $\tau_1 = \beta_2 = b_{2j}\varpi_j$. Take d.

Let *MESS* be the difference above, which must equal zero (on $\mathcal{H}'$).

Let $\varpi_{123} := \varpi_1 \wedge \varpi_2 \wedge \varpi_3$. We find that:

$$0 = MESS \wedge \varpi_1 \wedge \varpi_3 = -\varpi_{123} \wedge (d(b_{22}) + \tau_3).$$

Get a translation $-\partial_{b_{22}}$ on b_{22}-space. Normalize to get $\mathcal{H}'' \hookrightarrow \mathcal{H}'$ via $b_{22} = 0$ (and $\tau_3 = T_{3j}\varpi_j$). On $\mathcal{H}''$, we have $(\varpi_1, \varpi_2, \varpi_3)$ and 0-dim str. grp. Conclusion: sym.dim ≤ 3 for $A_1 A_2 \neq 0$ branch.

Now impose compatibility conditions as above and $d^2 = 0$. Get a homogeneous model if all str. coeffs are constant wrt ϖ_i, e.g. T_{3j}. In this case, Lie's 3rd thm $\Rightarrow \mathcal{H}''$ is locally a Lie group F.

Interpreting the output of Cartan reduction

One branch of the Cartan reduction yields a 1-parameter family:
For $\rho \in \mathbb{C}^\times$, we have an adapted coframing $(\varpi_1, \varpi_2, \varpi_3)$ with:

$$(*): \quad \begin{cases} d\varpi_1 &= \left(2\rho\varpi_1 + \frac{1}{3\rho}\varpi_2\right) \wedge \varpi_3 \\ d\varpi_2 &= \left(\frac{1}{3\rho}\varpi_1 - 2\rho\varpi_2\right) \wedge \varpi_3 \\ d\varpi_3 &= \varpi_1 \wedge \varpi_2 \end{cases}$$

- Residual str. grp: $(\varpi_1, \varpi_2, \varpi_3) \mapsto (\sigma\varpi_1, \frac{1}{\sigma^3}\varpi_2, \frac{1}{\sigma^2}\varpi_3)$, where $\sigma^8 = 1$, which induces $\rho \mapsto \rho\sigma^2$. Essential parameter: ρ^4.
- Contact distribution $C = \ker\{\varpi_3\} = E \oplus V$, where $E = \ker\{\varpi_2, \varpi_3\}$, $V = \ker\{\varpi_1, \varpi_3\}$.
- "Self-dual": $E \leftrightarrow V$ via $(\varpi_1, \varpi_2, \varpi_3) \mapsto (-i\varpi_2, i\varpi_1, -\varpi_3)$.
- All str. fcns are constant, so Lie's 3rd thm $\Rightarrow \{\varpi_i\}$ is a basis for the left-inv. 1-forms on a (local) Lie group F. What is $\mathfrak{f}$?
  - Killing form is ndg iff $\rho^4 \neq -\frac{1}{36}$ iff $\mathfrak{f} \cong \mathfrak{sl}_2$.
  - When $\rho^4 = -\frac{1}{36}$, $\mathfrak{f}$ is solvable, with derived series $= [3, 2, 0]$.

Interpreting the structure equations

Cartan reduction involves only structure group normalization & exterior differentiation. Integration is not involved in this process.

To interpret the str eqns $(*)$ in classical terms, we need to integrate them, e.g. find $\varpi_1, \varpi_2, \varpi_3$ in coords (x, y, p) s.t.

1. $\varpi_3 = \langle dy - pdx \rangle, \langle \varpi_2, \varpi_3 \rangle = \langle dy - pdx,\, dp - fdx \rangle,\ \langle \varpi_1, \varpi_3 \rangle = \langle dx, dy \rangle$, i.e. $E = \langle \partial_x + p\partial_y + f\partial_p \rangle,\ V = \langle \partial_p \rangle$, for some function $f = f(x, y, p)$.
2. the structure equations $(*)$ are satisfied.

(Above, $\langle \cdots \rangle$ refers to arbitrary linear combinations with function coeffs.)

Let's outline some details in the exceptional $\boxed{\rho^4 = -\frac{1}{36}}$ case.
What 2nd order ODE does this correspond to?

Example of integrating structure equations

Set $\boxed{\rho := \frac{\exp(i\pi/4)}{\sqrt{6}}}$, $\{w_i\}$ & $\{\varpi_i\}$ dual. Then $\mathfrak{f}^{(1)} = \mathrm{span}_{\mathbb{C}}\{w_1 - iw_2, w_3\}$, $\mathfrak{f}^{(2)} = 0$, and $0 \neq \mathrm{ad}_{w_1}|_{\mathfrak{f}^{(1)}}$ trace-free & diagonalizable. Define $(w_1', w_2', w_3') = (aw_1, w_1 - iw_2 + bw_3, w_1 - iw_2 - bw_3)$. We find $\exists a, b$ s.t.

$$\begin{cases} [w_1', w_2'] = w_2' \\ [w_1', w_3'] = -w_3' \\ [w_2', w_3'] = 0 \end{cases} \iff \begin{cases} d\varpi_1' = 0 \\ d\varpi_2' = -\varpi_1' \wedge \varpi_2' \\ d\varpi_3' = +\varpi_1' \wedge \varpi_3' \end{cases}$$

Poincaré lemma $\Rightarrow \varpi_1' = dz_1$. Other eqns: $0 = d(e^{z_1}\varpi_2') = d(e^{-z_1}\varpi_3')$, so $\varpi_2' = e^{-z_1}dz_2$, $\varpi_3' = e^{z_1}dz_3$. Passing back to ϖ_i, we get:

$$\begin{cases} \varpi_1 = \frac{3^{1/4}}{2}(-1+i)^{3/2}dz_1 + e^{-z_1}dz_2 + e^{z_1}dz_3 \\ \varpi_2 = -ie^{-z_1}dz_2 - ie^{z_1}dz_3 \\ \varpi_3 = \frac{3^{1/4}}{\sqrt{-1+i}}(e^{-z_1}dz_2 - e^{z_1}dz_3) \end{cases} \Rightarrow \begin{cases} E = \langle \partial_{z_1} \rangle \\ V = \langle \partial_{z_3} + e^{2z_1}\partial_{z_2} + \frac{2(1+i)}{3\sqrt{-1+i}} e^{z_1} 3^{3/4} \partial_{z_1} \rangle \end{cases}$$

Set $(x, y, p) := (cz_3, cz_2, e^{2z_1})$. Choose c s.t. $V = \langle \partial_x + p\partial_y + p^{3/2}\partial_p \rangle$ and $E = \langle \partial_p \rangle$. By self-duality, this 2nd order ODE is $\boxed{y'' = (y')^{3/2}}$.

Exercise

Find syms, pick a basepoint, align Lie alg str to the reduced str eqns.

$y'' = (y')^{3/2}$ and the rest of the family

The transformation $(x,y) \stackrel{\phi}{\longmapsto} (y,-x)$ prolongs to $J^2(\mathbb{C},\mathbb{C})$ as $(x,y,y',y'') \stackrel{\phi^{(2)}}{\longmapsto} (y,-x,-\frac{1}{y'},\frac{y''}{(y')^3})$, and induces the equivalence:

$$y'' = (y')^a \quad \stackrel{\phi^{(2)}}{\longmapsto} \quad y'' = (y')^{3-a}.$$

Note that ϕ has order 4, but this induces a $\mathbb{Z}_2$-action $a \mapsto 3-a$. The value $\boxed{a = \frac{3}{2}}$ is the unique fixed point. The family $y'' = (y')^a$ has:

- 3-dim symmetry iff $a \in \mathbb{C}\backslash\{0,1,2,3\}$. (Use Tresse's I_1, I_2.) In this case, $\mathfrak{f}$ is solvable (so $y'' = (y')^a$ is not the ODE for $\rho^4 \neq -\frac{1}{36}$) with:

$$[v_1,v_2] = (a-1)v_2, \quad [v_1,v_3] = (a-2)v_3, \quad [v_2,v_3] = 0.$$

- essential parameter $a(3-a)$.

Exercise

Relate syms of $y'' = \frac{p(1-p^2)+c(1-p^2)^{3/2}}{x}$ *to* $(*)$. *Show* $\boxed{\frac{1}{\rho^4} = -36(1+\frac{1}{c^2})}$. *In particular,* $c \neq 0$, *with redundancy* $c \mapsto -c$. *Essential parameter:* c^2.

Sample output 2: Embedding data

$$(**):\quad \begin{cases} \zeta_1 = -\rho\varpi_3, \quad \zeta_2 = 2\rho\varpi_3, \\ \tau_1 = \frac{1}{3\rho}\varpi_1, \quad \tau_2 = -\rho\varpi_1 - \frac{1}{3\rho}\varpi_2, \quad \tau_3 = \frac{1}{9\rho^2}\varpi_3 \end{cases}$$

1. Get a linear map $\bar{\omega} : \mathfrak{f} \hookrightarrow \mathfrak{sl}_3$ given by:

$$aw_1 + bw_2 + cw_3 \quad \mapsto \quad \begin{pmatrix} 0 & \frac{a}{3\rho} & \frac{c}{9\rho^2} \\ a & \rho c & -\rho a - \frac{b}{3\rho} \\ c & b & -\rho c \end{pmatrix}.$$

(Observe: $\mathfrak{f}$ is a filtered deformation of $\mathfrak{g}_-$.)

2. Cartan curvature κ obstructs $\bar{\omega}$ being a Lie alg morphism:
   - $\kappa(x,y) = [\bar{\omega}(x), \bar{\omega}(y)]_{\mathfrak{sl}_3} - \bar{\omega}([x,y]_{\mathfrak{f}})$.
   - $\kappa(w_1, w_2) = 0$, $\kappa(w_1, w_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\kappa(w_2, w_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Rmk: The invariant ρ arises in the embedding, not in κ.

Cartan reduction output: $\boxed{(\mathfrak{f}, [\cdot,\cdot]_{\mathfrak{f}}) \;\&\; \bar{\omega} : \mathfrak{f} \to \mathfrak{g}}$. (More next time.)

Application: Holonomy

On the principal G-bundle $\widehat{\mathcal{G}} = \mathcal{G} \times_P G$, ω equivariantly extends to a principal connection $\widehat{\omega} \in \Omega^1(\widehat{\mathcal{G}}; \mathfrak{g})$. Get a notion of holonomy.

For homogeneous Cartan geometries $(\mathcal{G} \to M, \omega)$, invariant ω are encoded by data $(\mathfrak{f}, [\cdot,\cdot]_\mathfrak{f})$ and $\bar{\omega} : \mathfrak{f} \to \mathfrak{g}$. (More next time.) Use it to compute holonomy (à la Ambrose–Singer, see e.g. Hammerl 2011):

Recall: $\kappa(\cdot,\cdot) = [\bar{\omega}(\cdot), \bar{\omega}(\cdot)]_\mathfrak{g} - \bar{\omega}([\cdot,\cdot]_\mathfrak{f})$. Define subspaces of $\mathfrak{g}$ by:

$$\mathfrak{hol}^0(\bar{\omega}) := \langle \kappa(x,y) : x, y \in \mathfrak{f} \rangle,$$
$$\mathfrak{hol}^i(\bar{\omega}) := \mathfrak{hol}^{i-1}(\bar{\omega}) + [\bar{\omega}(\mathfrak{f}), \mathfrak{hol}^{i-1}(\bar{\omega})], \; i \geq 1.$$

The holonomy is obtained at stabilization, i.e. $\mathfrak{hol}(\bar{\omega}) := \mathfrak{hol}^\infty(\bar{\omega})$.

Theorem

For the 2nd order ODE encoded by $(*)$ and $(**)$, $\mathfrak{hol}(\bar{\omega}) = \mathfrak{sl}_3$.

This is obtained by simple direct computation: $\mathfrak{hol}^0 = \langle e_{10}, e_{01} \rangle$, $\mathfrak{hol}^1 = \mathfrak{hol}^0 \oplus \langle e_{11}, f_{10}, f_{01} \rangle \oplus \mathfrak{h}$, and $\mathfrak{hol}^\infty = \mathfrak{hol}^2 = \mathfrak{sl}_3$.

Summary

- "Regular/normal" parabolic geometries $\rightsquigarrow$ structure eqns.
- Cartan reduction is a "top-down" classification method.
  - Strengths: systematic, conceptual approach is indep. of dim, coordinate-free, uses only d and orbit normalization, get structural information: e.g. curvature / holonomy.
  - Limitations: can have heavy branching, introducing coordinates on Cartan reduction output requires effort (integration), data management is an issue (many relations arise).
- 2nd order ODE: submax sym dim 3 with simply-transitive models, i.e. trivial isotropy $\mathfrak{f}^0$. For many other structures, e.g. $(2,3,5)$, $\exists$ multiply-transitive ($\mathfrak{f}^0 \neq 0$) non-flat models. This aids the search for models, e.g. $\mathfrak{f}^0 \cdot \kappa = 0$ (next lecture).