§6. Cartan connection from the view-point of step prolongation (2)

Def. A Cartan connection of type L/L^0: is a principal L^0-bundle $P \to M$ with $\mathfrak{l}$-valued 1-form θ on P s.t.

i) $\theta : T_z P \to \mathfrak{l}$; iso. $\forall z \in P$

ii) $R_a^* \theta = Ad(a)^{-1} \theta$, $a \in L^0$

iii) $\theta(\tilde{A}) = A$, $A \in \mathfrak{l}^0$.

$$\begin{array}{ccc} L & \theta_L & & P & \theta \\ \downarrow L^0 & & \cong & \downarrow L^0 & \\ L/L^0 & & \text{iff } K=0 & M & \end{array}$$

$$d\theta + \frac{1}{2}[\theta,\theta] = K(\theta,\theta)$$

$$K : P \to \mathrm{Hom}(\Lambda^2 \mathfrak{l}/\mathfrak{l}^0, \mathfrak{l})$$

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given a G-str.

$$\begin{array}{ccc} Q^{(0)} & \overset{?}{\rightsquigarrow} & P \quad \theta \\ \downarrow G_0 & \text{canonical} & \downarrow G^0 \\ (M,f) & & (M,f) \end{array}$$

θ ; Cartan connection of type $(\mathfrak{g}, G^0)$

where $\mathfrak{g} = \oplus\, \mathfrak{g}_p = \mathrm{Prol}(\mathfrak{g}_- \oplus \mathfrak{g}_0)$

$\mathrm{Lie\ alg}(G^0) = \mathfrak{g}^0 = \bigoplus_{p \geq 0} \mathfrak{g}_p$

(4)

Riem., conformal, CR, simple Lie alg. (Tanaka).

Condition (C) $\exists W^2 = \bigoplus_{r>0} W^2_r$: G^0-invariant

s.t. $\mathrm{Hom}(\Lambda^2 \mathfrak{g}_-, \mathfrak{g})_r = \partial \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g})_r \oplus W^2_r$, $r>0$.

Th. (M–). If the condition (C) holds, then

$$\begin{array}{ccc} Q^{(0)} & \rightsquigarrow & \mathcal{R}_{W^2}(Q^{(0)}), \ \theta_{\mathcal{R}(Q^{(0)})} \text{ Cartan connection.} \\ \downarrow & & \downarrow G^0 \\ (M, f) & & (M, f) \end{array}$$

Let $Q^{(0)} \xrightarrow{G_0} (M, f)$ be a G-structure, (5)

$W = \{ W_r^1, W_r^2 \}_{r>0}$, $Q = S_W Q^{(0)}$: W-normal step prolongation

Th. 7.1. If the structure function σ of Q is const.

then. (1) σ is const, $\sigma_s = 0$ $s < 0$ $\sigma_0 = [\, ,\,]_{\mathfrak{g}^0}$

(2) If moreover G_i are connected $(i \geq 0)$, then

$Q \to M$ is a ~~pre~~- Cartan connection,

($\theta : T.Q \to E.$ $0 \to \mathfrak{g}^0 \to E$ exact as G_0-modules

E is not necessarily Lie alg.)

(6)

Th. 7.2. If τ is trivial ($\tau = [\ ]_{\mathfrak{g}_{>i}\ \mathfrak{z}}$.) and if G_i ($i \geq 0$) are connected, then

$\mathcal{Q} = \mathcal{S}_W Q^{(0)}$ is a Cartan connection of type $(\mathfrak{g}, G^0)$.

Th. 7.3. If $W^2 = \{W_r^2\}$ satisfies (C), then

$$\mathcal{Q}_{W^2} Q^{(0)} \cong \mathcal{S}_W Q^{(0)}$$

(7)

$$(M, f) \xleftarrow{G_0} Q^{(0)} \leftarrow \cdots \leftarrow Q^{(k-1)} \xleftarrow{G_k} Q^{(k)} \leftarrow S^{(k+a)} Q^{(k)} \leftarrow S\, Q^{(k)}$$

higher order geometric str.

completed universal frame bundle

$$\begin{tikzcd}
(M, f) & P^{(0)} \arrow[l, "G_0"'] & \cdots \arrow[l] & P^{(k-1)} & P^{(k)} \arrow[l] \arrow[llll, bend left, "G^{(k)}"] & Q^{(k+a)} P^{(k)} \arrow[l] & R\, P^{(k)} \arrow[l]
\end{tikzcd}$$

higher order principal geometric str.

completed principal universal frame bundle

§ 7. Equivalence problem — the last half.

$$Q^{(0)} \xrightarrow{G_0} (M, f) : G\text{-structure}$$

$$\downarrow$$

$$\left(\mathcal{S}_W Q^{(0)}, \theta_{\mathcal{S}_W Q^{(0)}} \right) = (Q, \theta)$$

$$\theta ; T.Q \to \mathfrak{g} = \oplus \mathfrak{g}_p$$

absolute parallelism

$$d\theta + \frac{1}{2}\gamma(\theta,\theta) = 0$$

$$\gamma : Q \to \mathrm{Hom}(\Lambda^2\mathfrak{g}, \mathfrak{g})$$

Th. $(\gamma, D\gamma)$ forms a complete system of invariants of (Q, θ) and hence of $Q^{(0)}$.

$$\Theta = \sum \theta^i$$

$$d\theta^i + \frac{1}{2} \sum \gamma^k_{ij} \theta^i \wedge \theta^j$$

$$d\gamma^k_{ij;p\cdots q} = \gamma^k_{ij;p\cdots qr} \theta^r$$

We may view θ^i as $\mathbb{R}$-valued or $\mathfrak{g}_i$-valued 1-form. $\sum$ may be infinite sum.

$$(\mathcal{Q}^{(0)}, x) \cong (\bar{\mathcal{Q}}^{(0)}, \bar{x})$$

$$\Updownarrow$$

Formally

$$(\mathcal{Q}, \theta, z) \cong (\bar{\mathcal{Q}}, \bar{\theta}, \bar{z}) \Longrightarrow [\![\mathcal{Q}, \theta, z]\!] \cong [\![\bar{\mathcal{Q}}, \bar{\theta}, \bar{z}]\!]$$

$$\Downarrow \qquad \Uparrow$$

$$\gamma^{k}_{ij;pq\cdots}(z) = \bar{\gamma}^{k}_{ij;pq\cdots}(\bar{z})$$

$$df = \sum f_i \theta^i, \quad df_{i\cdots j} = \sum f_{i\cdots j k}\theta^k, \tag{11}$$

$$f_{i\cdots ab\cdots k} - f_{i\cdots ba\cdots k} = P(\gamma, D\gamma)$$

$$\begin{tikzcd}
\mathbb{R}[[Q, z]] \ni f \arrow[rr, "\Phi", "\cong"'] \arrow[d, leftrightarrow] & & \bar{f} \in \mathbb{R}[[\bar{Q}, \bar{z}]], \arrow[d, leftrightarrow] \\
\mathcal{V} \ni (f(z), \cdots f_{i\cdots j}(z)\cdots) & = & (\bar{f}(\bar{z}), \cdots \bar{f}_{i\cdots j}(\bar{z})\cdots) \in \bar{\mathcal{V}}
\end{tikzcd}$$

The map $\bar{\Phi}$ thus defined gives a formal homeomorphism from (Q, z) to $(\bar{Q}, \bar{z})$ and

$$\bar{\Phi}^{*}\,\bar{\theta} = \theta.$$

Hence gives a formal isomorphism $(Q, \theta, z) \to (\bar{Q}, \bar{\theta}, \bar{z})$

<u>Th</u>: In the analytic category, if $\dim \mathfrak{f} < \infty$ then $\bar{\Phi}$ is analytic and gives a local analytic isomorphism $(Q, \theta, z) \cong (\bar{Q}, \bar{\theta}, \bar{z})$ and hence $(Q^{(0)}, x) \cong (\bar{Q}^{(0)}, \bar{x})$

Let $Q^{(0)} \xrightarrow{G_0} (M, f)$ be a G-structure, (13)

$Q^{(k)} = \mathcal{S}_{W}^{(k)} Q^{(0)}$, $Q = \mathcal{S}_{W} Q^{(0)}$.

Def. $Q^{(k)}$ involutive $\underset{\text{def.}}{\Longleftrightarrow}$ i) $H_r^2(\mathfrak{g}_-, \mathfrak{g}) = 0$

ii) $\gamma^{[k]}$: constant $\forall k$

If $Q^{(k)}$ is involutive, then γ is constant.

Theorem If $Q^{(k)}$ and $\bar{Q}^{(k)}$ are analytic and involutive and if $\gamma^{[k]} = \bar{\gamma}^{[k]}$,

then $$\left(Q^{(k)}, z^{(k)}\right) \underset{\text{loc analytic}}{\cong} \left(\bar{Q}^{(k)}, \bar{z}^{(k)}\right)$$

for $\forall\; z^{(k)} \in Q^{(k)}, \; \bar{z}^{(k)} \in \bar{Q}^{(k)}$.

(14 bis)

(0) $u : (M, \mathfrak{f}) \to (\bar{M}, \bar{\mathfrak{f}})$ is an isomorphism of $(Q^{(0)}; M, \mathfrak{f}) \to (\bar{Q}^{(0)}; \bar{M}, \bar{\mathfrak{f}})$

$\Longleftrightarrow$ $\bar{\Psi}(u) = 0$, where Ψ is a system of PDE's on $(M, \mathfrak{f})$.

(I) $\exists\ \hat{u} = (u_0, u_1, \cdots, u_k, \cdots)$ formal power series sol of Ψ s.t.

(*) $\left| (X_{i_1} \cdots X_{i_l} \hat{u})(0) \right| \leq C\, l!\, \rho^l \quad \forall (i_1 \cdots i_l) \in (1 \cdots r)^l$, $l \geq 0$.

where $\{X_1, \cdots, X_r\}$ is a local basis of $\mathfrak{f}^{-1}$

(use a nilpotent generalisation of Privalov(?) Th.)

$\boxed{\text{II}}$) If $(M, \mathcal{D})$ is bracket generating (Hörmander condition)
then u satisfying (*) is analytic.
(Use subriemannian geometry & Gabrielov's Th.)

Problems and Subjects, Wish to study.

1. Applications
   Complex geometry,
   Subriemannian geometry
2. geometric structures of non-constant symbol
   intransitive geometric structure,

3. Linear diff. eqns of infinite type.

$\cong$ Extrinsic geometry in infinite dim.

flag variety.

$(\mathfrak{g}, V)$: $\dim V = \infty$.

4. Geometry of non-linear diff. eqns.

pseudo-product str.

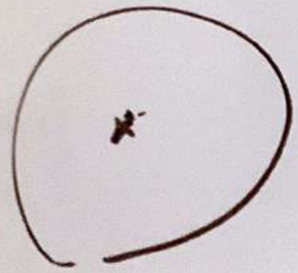
PE-structure

5. Locally around a regular point.

Global properties

- Complex geometry
  - J. Hong
  - J-M. Hwang. Ann. Math 2019
    Hirschowitz's conjecture

Singularities.



- diff. eq.
- $g : (M, I) \to Flag(V, \phi)$