Equivalence Problems of Geometric Structures

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The lecture is based on: (1)

[1] B. Doubrov, Y. Machida, T. M____, Extrinsic geometry and linear differential equations, Sigma 17 (2021) 60 pages.

[2] J. Hong, T. M____ Prolongations, invariants, and fundamental identities, arXiv DG 22 Mar 2022, 64 pages

@ Intrinsic geometry:

$$(X, \sigma) \xrightarrow[\cong]{?\exists f} (X', \sigma')$$

@ Extrinsic geometry

?∃ (f, Λ_a)

$$\begin{tikzcd}
(X, \sigma) \arrow[r, "\varphi"] \arrow[d, "f \cong"] & (\hat{X}, \hat{\sigma}) \arrow[r, phantom, "="] \arrow[d, "\hat{f} \cong"] & L/L^0 \arrow[d, "\Lambda_a"] \\
(X', \sigma') \arrow[r, "\varphi'"] & (\hat{X}', \hat{\sigma}') \arrow[r, phantom, "="] & L/L^0
\end{tikzcd}$$

$a \in L$.

Extrinsic geometry $\simeq$ linear diff. equations

Intrinsic geometry $\sim$ non-linear diff. eqns.

Lie Klein

Cartan Ehresmann, Chern

Kuranishi Spencer

Singer-Sternberg, Guillemin, Goldschmidt

Tanaka Morimoto, Se-ashi

§1. Filtered manifold (M, F):

a diff. mfd. M equipped with a filtration $F = \{F^p\}_{p \in \mathbb{Z}}$ such that

i) $TM = F^{-\mu} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset F^{-1} \supset F^0 = 0$,

ii) $[\underline{F}^p, \underline{F}^q] \subset \underline{F}^{p+q}$, $\forall p, q \in \mathbb{Z}$

depth $\mu \geq 1$.

If $\mu = 1$, then trivial = diff. mfd itself.

$(M, f) \ni x \longmapsto \mathrm{gr}\, f_x$: a nilpotent graded Lie alg.
(symbol algebra at x)

$$\mathrm{gr}\, f_x = \bigoplus_{p<0} \mathrm{gr}_p\, f_x \,, \quad \mathrm{gr}_p\, f_x = f_x^p / f_x^{p+1}$$

$$[\mathrm{gr}_p f_x, \mathrm{gr}_q f_x] \subset \mathrm{gr}_{p+q} f_x .$$

<u>Def</u> (M, f) is of constant symbol of type $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$
if $\mathrm{gr}\, f_x \cong \mathfrak{g}_-$ as graded Lie algebra for $\forall x \in M$.

X; a (germ of) local vector field on a filtered mfd $(M, \mathcal{F})$

<u>Def</u>: weighted order of $X \leq k$ $(k \geq 0)$ if $X \in \underline{\mathcal{F}}^{-k}$.

Then we can speak of:

∘ weighted order of diff. op. ∘ weighted jet bundle.

§2 Algebraic aspects in nilpotent geometry

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a space : (M, f, σ)

a figure : $\varphi : (M, f) \to L/L^0 \subset Flag(V, \Phi) = GL(V,\Phi)/GL(V,\Phi)^0$

φ : a homomorphism of filtered mfds $\Longleftrightarrow$ φ : an osculating map

i.e. $f^p \, \varphi^i \subset \varphi^{p+i}$

where $(V, \Phi = \{\Phi^i\})$ is a filtered vector space $\varphi^i \subset V \times M$

$\downarrow$

M

In $x \in M$, $\varphi(x) = \{\varphi^i(x)\}$

Algebraic elements

Intrinsic geometry ... TGLA ($\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$

transitive: $(\mathfrak{g}\ \mathfrak{g}_-)_k = 0$ ($k \geq 0$)

Extrinsic geometry $\mathfrak{g} \subset \mathfrak{l} \subset \mathfrak{gl}(V, \phi)$

In the case $\mathfrak{l} = \mathfrak{gl}(V, \phi)$

$\mathfrak{g} \subset \mathfrak{gl}(V\phi)$ or $(\mathfrak{g}, (V, \phi))$; $\mathfrak{g}$: TGLA

(V, ϕ) : graded $\mathfrak{g}_-$-module

Transitive structures

Intrinsic geometry

$(A, \{A^p\})$: FLA s.t. $\mathrm{gr}\, A = \mathfrak{g}$ $\curvearrowleft$ $H^2_+(\mathfrak{g}_-, \mathfrak{g})$

Extrinsic geometry

$A \subset \mathfrak{l}$ s.t. $\mathrm{gr}\, A = \mathfrak{g}$ $\curvearrowleft$ $H^1_+(\mathfrak{g}_-, \mathfrak{l}/\mathfrak{g})$

filtered subalgebra

Spencer cohomology group

$\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$: GLA, $U = \bigoplus_{i \in \mathbb{Z}} U_i$: graded $\mathfrak{g}_-$-module ($\mathfrak{g}_p U_i \subset U_{p+i}$)

Then $H^p_r(\mathfrak{g}_-, U)$: the cohomology gp associated with

$$0 \to U_r \xrightarrow{\partial} \mathrm{Hom}(\mathfrak{g}_-, U)_r \xrightarrow{\partial} \cdots \to \mathrm{Hom}(\Lambda^p \mathfrak{g}_-, U)_r \to \cdots$$

<u>Theorem</u> (Finitude). Assume U transitive (i.e., $U^{\mathfrak{g}_-}_{\geq 0} = 0$)

$\exists r_0$ s.t. $H^p_r(\mathfrak{g}, U) = 0$ for $^\forall r \geq r_0$, $^\forall p$

§3. G-structures (geometric structures of order 0)

Assume that:

(M, f) is of type $\mathfrak{g}_-$.

For $x \in M$,

$$\mathcal{S}^{(0)}(M, f) = \{ z : \mathfrak{g}_- \to gr f_x \}$$

GLA iso

$$\begin{array}{c} \mathcal{S}^{(0)}(M, f) \\ \downarrow \; G_0(\mathfrak{g}_-) \\ (M, f) \end{array}$$

$$G_0(\mathfrak{g}_-) = \mathrm{Der}_0(\mathfrak{g}_-)$$

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$$\begin{array}{ccc} \mathcal{S}^{(0)}(M, f) & \supset & Q^{(0)} \\ \downarrow G_0(\mathfrak{g}_-) & & \downarrow G_0 \\ (M, f) & & (M, f) \end{array}$$

G-str. in the sense of nilpotent geometry

$$(\mathrm{EP})_0 : \begin{array}{ccc} Q^{(0)} & \xrightarrow[\cong]{?} & Q'^{(0)} \\ \downarrow G_0 & & \downarrow G_0 \\ (M, f) & \to & (M', f') \end{array}$$

§4. W-normal step prolongation

0. Start: $Q^{(0)} \subset \mathcal{S}^{(0)}(M, \mathfrak{f})$: G-structure

$$\begin{array}{c} Q^{(0)} \\ \downarrow G_0 \\ (M, \mathfrak{f}) \end{array}$$

1. Take the prolongation $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of $\mathfrak{g}_- \oplus \mathfrak{g}_0$;
the maximal TGLA extending $\mathfrak{g}_- \oplus \mathfrak{g}_0$, i.e., for $k > 0$,

$$\mathfrak{g}_k = \{ \alpha \in \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}_- + \cdots + \mathfrak{g}_{k-1})_k \; ; \; \text{for } \forall u, v \in \mathfrak{g}_-, \; [\alpha(u), v] + [u, \alpha(v)] - \alpha[u, v] = 0 \}$$

(13)

2. Choose and fix: $W = \{W_r^1, W_r^2\}_{r>0}$ s.t.

$$\mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g})_r = \partial \mathfrak{g}_r \oplus W_r^1, \quad \mathrm{Hom}(\Lambda^2\mathfrak{g}_-, \mathfrak{g})_r = \partial \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g})_r \oplus W_r^2$$

$$0 \to \mathfrak{g}_r \xrightarrow{\partial} \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g})_r \xrightarrow{\partial} \mathrm{Hom}(\Lambda^2\mathfrak{g}_-, \mathfrak{g})_r \xrightarrow{\partial}$$

3. Construct: $(Q^{(\infty)}, \theta^{(\infty)}, \gamma^{(\infty)}) = (Q, \theta, \gamma)$

$$(M, f) \xleftarrow{G_0} Q^{(0)} \leftarrow \cdots \leftarrow Q^{(k-1)} \xleftarrow{G_k} Q^{(k)} \leftarrow \cdots \leftarrow Q^{(\infty)}$$

$$\theta : T.Q \xrightarrow[\cong]{} \mathfrak{g}, \quad d\theta + \frac{1}{2}\gamma(\theta, \theta) = 0, \quad \gamma : Q \to \mathrm{Hom}(\Lambda^2\mathfrak{g}, \mathfrak{g}).$$

$$
\begin{tikzcd}
(\mathcal{S}^{(\infty)}(M,f), \Theta) \arrow[dd] & (\mathcal{S}^{(\infty)}(Q^{(0)}), \Theta) \arrow[d] & (\mathcal{S}^{(\infty)}(Q^{(1)}), \Theta) \arrow[d] & (Q^{\infty}, \theta) \arrow[dl, dotted] \\
 & \mathcal{S}^{(1)}(Q^{(0)}) \arrow[d] \arrow[r, phantom, "\supset"] & Q^{(1)} \arrow[dl, "\varphi_1"] & \\
\mathcal{S}^{(0)}(M,f) \arrow[d] \arrow[r, phantom, "\supset"] & Q^{(0)} \arrow[dl, "\varphi_0"] & & \\
(M,f) & & &
\end{tikzcd}
$$

§5. Fundamental identities.

$$\begin{cases} \partial \kappa_{[k]} = \bar{\Psi}_{I}\left(\kappa_{[i]}, \tau_{[i]} ; i<k\right) \\ \partial \tau_{[k]} = \bar{\Psi}_{II}\left(\kappa_{[i]}, \tau_{[i]}, \sigma_{[i]} ; i<k\right) \\ \partial \sigma_{[k]} = \bar{\Psi}_{III}\left(\tau_{[i]}, \sigma_{[i]} ; i<k\right) \end{cases}$$

where

$\kappa = \gamma_{I}$: $Hom(\Lambda^2 \mathfrak{g}_-, \mathfrak{g})$-component

$\tau = \gamma_{II}$: $Hom(\mathfrak{g}_{\geq 0} \otimes \mathfrak{g}_-, \mathfrak{g})$ " $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_{\geq 0}$.

$\sigma = \gamma_{III}$: $Hom(\Lambda^2 \mathfrak{g}_{\geq 0}, \mathfrak{g})$- "

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$$\gamma : Q \to \mathrm{Hom}(\Lambda^2 \mathfrak{g}, \mathfrak{g})$$

Write $\gamma = \sum \gamma_{ij}^k$, where $\gamma_{ij}^k : Q \to \mathrm{Hom}(\mathfrak{g}_i \otimes \mathfrak{g}_j, \mathfrak{g}_k)$

$\deg \gamma_{ij}^k = k - i - j \qquad \widetilde{\deg}\, \gamma = k - \tilde{i} - \tilde{j}$

$(\tilde{i} = \mathrm{Min}\{-1, i\})$

$$\gamma_r = \sum_{\deg = r} \gamma_{ij}^k \qquad \gamma_{[s]} = \sum_{\widetilde{\deg} = s} \gamma_{ij}^k$$

Theorem 4.1 (Fundamental identities).

(1) *For $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$, and $Z \in \mathfrak{g}_z$, where $x, y, z < 0$, and for a nonnegative integer k, we have*

$$(\partial\kappa_{[k]})(X, Y, Z)$$

$$= \mathfrak{S}_{X,Y,Z}\left\{\sum_{\substack{d_1+d_2=k\\ d_1,d_2>0}} \left\{\kappa_{[d_1]}(\kappa_{[d_2]}(X,Y)_-, Z) + \tau_{(d_1)[k+x+y+1]}(\kappa_{[d_2]}(X,Y)_+, Z)\right\} - D_X\kappa_{[k+x]}(Y,Z)\right\}.$$

(2) *For $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$, and $A \in \mathfrak{g}_a$, where $x, y < 0$ and $a \geq 0$, and for $k = d+a+1$, where d is a nonnegative integer, we have*

(1) *For $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$, and $Z \in \mathfrak{g}_z$, where $x, y, z < 0$, and for a nonnegative integer k, we have*

$$
\begin{aligned}
&(\partial\kappa_{[k]})(X, Y, Z) \\
= \ & \mathfrak{S}_{X,Y,Z}\left\{\sum_{\substack{d_1+d_2=k \\ d_1,d_2>0}} \left\{\kappa_{[d_1]}(\kappa_{[d_2]}(X,Y)_-, Z) + \tau_{(d_1)[k+x+y+1]}(\kappa_{[d_2]}(X,Y)_+, Z)\right\} - D_X\kappa_{[k+x]}(Y,Z)\right\}.
\end{aligned}
$$

(2) *For $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$, and $A \in \mathfrak{g}_a$, where $x, y < 0$ and $a \geq 0$, and for $k = d+a+1$, where d is a nonnegative integer, we have*

$$
\begin{aligned}
&(\partial\tau_{(d)[k]}(A, \cdot))(X, Y) \\
= \ & (D_A\kappa_{[k-1]} + \rho(A)\kappa_{[d]})(X, Y) - \mathscr{A}_{X,Y}\tau_{(d)[k+x]}([A, X]_+, Y) + [\kappa_{[d]}(X, Y)_+, A] - \sigma_0(\kappa_{[d]}(X, Y)_+, A) \\
& - \sum_{\substack{\delta_1+\delta_2=d \\ -a\leq\delta_1<d \text{ and } \delta_1\neq 0 \\ 0<\delta_2\leq d+a=k-1}} \left\{\tau_{(d_1)[k-1]}(\kappa_{[\delta_2]}(X, Y)_-, A) + \sigma_{(\delta_1)[k+x+y+1]}(\kappa_{[\delta_2]}(X, Y)_+, A)\right\} \\
& + \mathscr{A}_{X,Y}\sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left\{\kappa_{[d_1]}(\tau_{(d_2)[k-d_1]}(A, X)_-, Y) + \tau_{(d_1)[k+x]}(\tau_{(d_2)[k-d_1]}(A, X)_+, Y)\right\} \\
& + \mathscr{A}_{X,Y}D_Y\tau_{(d+y)[k+y]}(A, X).
\end{aligned}
$$

$d_1,d_2>0$

$$+ \mathscr{A}_{X,Y} D_Y \tau_{(d+y)[k+y]}(A, X).$$

(3) *For $A_a \in \mathfrak{g}_a$, $B \in \mathfrak{g}_b$, $X \in \mathfrak{g}_x$, where $a, b \geq 0$ and $x < 0$, and for $k = d+a+b+2$, where d is an integer, we have*

$$\begin{aligned}
&[\sigma_{(d)[k]}(A, B), X] \\
= \; & \mathscr{A}_{A,B} \sum_{\substack{d_1+d_2=d \\ 0 \leq d_2 \leq d+a}} \{\sigma_{(d_1)[k+x-d_2]}(A, \tau_{(d_2)[k-1-(d_1+a)]}(B, X)_+) \\
&\qquad\qquad + \tau_{(d_1)[k-1-(d_2+b)]}(A, \tau_{(d_2)[k-1-(d_1+a)]}(B, X)_-)\} \\
&+ \sum_{\substack{\delta_1+\delta_2=d \\ 0<\delta_1 \leq d+\min(a,b)}} \tau_{(\delta_1)[k-1]}(X, \sigma_{(\delta_2)[k-\delta_1]}(A, B)) \\
&+ D_X \sigma_{(d+x)[k+x]}(A, B) + \mathscr{A}_{A,B} D_A \tau_{(d+a)[k-1]}(B, X).
\end{aligned}$$

(4) *For $A \in \mathfrak{g}_a$, $B \in \mathfrak{g}_b$, $C \in \mathfrak{g}_c$, where $a, b, c \geq 0$, and for $k = d + a + b + c + 3$, where d is an integer, we have*

$$\mathfrak{S}_{A,B,C} \left\{ \sum_{\substack{d_1+d_2=d \\ -\min\{a,b\} \leq d_2 \leq d+c}} \sigma_{(d_1)[k-1]}(\sigma_{(d_2)[k-1-c-d_1]}(A, B), C) - D_C \sigma_{(d+c)[k-1]}(A, B) \right\} = 0.$$

Fundamental identities in lower orders: (1)

(1) $\partial \kappa_1 = 0$

(2) $\partial \kappa_2 = \kappa_1 \circ \kappa_1 - \sum_{p<0} D_{\pi_p} \kappa_{2+p}$

(3) $\partial \tau_{[2]}(A_0, \cdot) = D_{A_0} \kappa_1 + \rho(A_0) \kappa_1$, $A_0 \in \mathfrak{g}_0$

The other identities for order ≤ 2 are trivial.

Recall that: $\kappa_{[k]} = \kappa_k$, $\tau_{[k]}(A, \cdot)$: $Hom(\mathfrak{g}_-, \mathfrak{g})_{k-1}$ - valued.

$$\tau_{[1]}(A_0, x) = [A_0, x], \text{ for } A_0 \in \mathfrak{g}_0,\ x \in \mathfrak{g}_-.$$

<u>Theorem 1</u> The equivalence problem of $Q^{(0)}$ reduces to that of $(\mathcal{S}_W Q^{(0)}, \theta)$, where $\mathcal{S}_W Q^{(0)} = Q^{(\infty)}$.

Set $I = \{ r > 0 \,;\, H^2_r(\mathfrak{g}_-, \mathfrak{g}) \neq 0 \}$. Then

<u>Theorem 2.</u> $\{ \kappa_i, D^\ell \kappa_i \}_{i \in I,\, \ell \geq 0}$ form a fundamental system of invariants.

Theorem 3. Let $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$ be a nilpotent graded Lie algebra and G_0 a Lie subgroup of $G_0(\mathfrak{g}_-)(= \mathrm{Aut}(\mathfrak{g}_-))$. Let $\mathfrak{g}$ be the prolongation of $\mathfrak{g}_- \oplus \mathfrak{g}_0$. If $H^2_+(\mathfrak{g}_-, \mathfrak{g}) = 0$, then all G_0-structure $Q^{(0)} \xrightarrow{G_0} (M, f)$ on a filtered manifold of type $\mathfrak{g}_-$ are all locally isomorphic in the C^∞ (resp C^ω)-category if $\dim \mathfrak{g} < \infty$ (resp. $\dim \mathfrak{g} = \infty$).