# Tanaka Prolongation in Maple

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Date: 26 January 2021

This worksheet is a supplement to the lecture "Classifying homogeneous geometric structures (Lecture 1) " (http://cft.edu.pl/grieg/activities.html).

In this worksheet, we illustrate how to compute the Tanaka prolongation pr(m,g[0]) for various geometric structures.

```
> restart: with(DifferentialGeometry): with(LieAlgebras):
```

## Metrics

```
> # Dimension 2

  StrEq := [
  [R,x1]=x2, [R,x2]=-x1],
  [x1,x2,R];
  LD := LieAlgebraData(StrEq, alg, grading = [-1,-1,0]):
  DGsetup(LD);

  infolevel[TanakaProlongation] := 2:
  prLD := TanakaProlongation(alg, 5, pralg):
```

$$StrEq := [[R, x1] = x2, [R, x2] = -x1], [x1, x2, R]$$

*Lie algebra: alg*

```
m:
   [[e1, e2], [e3]]
   [-1, 0]
```

```
> # Dimension 3

  StrEq := [
  [R12,x1]=x2, [R12,x2]=-x1,
  [R23,x2]=x3, [R23,x3]=-x2,
  [R31,x3]=x1, [R31,x1]=-x3],
  [x1,x2,x3,R12,R23,R31];
  LD := LieAlgebraData(StrEq, alg1, grading = [-1,-1,-1,0,0,0]):
  DGsetup(LD);

  infolevel[TanakaProlongation] := 2:
  prLD := TanakaProlongation(alg1, 5, pralg1):
```

$$StrEq := [[R12, x1] = x2, [R12, x2] = -x1, [R23, x2] = x3, [R23, x3] = -x2, [R31, x3] = x1, [R31, x1] = -x3], [x1, x2, x3, R12, R23, R31]$$

*Lie algebra: alg1*

```
m:
   [[e1, e2, e3], [e4, e5, e6]]
   [-1, 0]
```

## Conformal structures

```
> # Dimension 3
```

```
  StrEq := [
  [R12,x1]=x2, [R12,x2]=-x1,
  [R23,x2]=x3, [R23,x3]=-x2,
  [R31,x3]=x1, [R31,x1]=-x3,
  [D,x1]=x1, [D,x2]=x2, [D,x3]=x3],
  [x1,x2,x3,R12,R23,R31,D];
  LD := LieAlgebraData(StrEq, alg, grading = [-1,-1,-1,0,0,0,0]):
  DGsetup(LD);

  infolevel[TanakaProlongation] := 2:
  prLD := TanakaProlongation(alg, 5, pralg):
```

$$StrEq := [[R12, x1] = x2, [R12, x2] = -x1, [R23, x2] = x3, [R23, x3] = -x2, [R31, x3] = x1, [R31, x1] = -x3, [D, x1] = x1, [D, x2] = x2, [D, x3] = x3], [x1, x2, x3, R12, R23, R31, D]$$

*Lie algebra: alg*

```
m:
   [[e1, e2, e3], [e4, e5, e6, e7]]
   [-1, 0]
The Tanaka prolongation at order 1 is:
   [[e1, e2, e3], [e4, e5, e6, e7], [e8, e9, e10]]
   [-1, 0, 1]
```

## (2,3,5)-distributions

```
> # g[-1]=<x1,x2>, g[-2]=<x3>, g[-3]=<x4,x5> (g[0]= graded
  derivations of g[-])

  StrEq := [
  [x1,x2]=x3, [x1,x3]=x4, [x2,x3]=x5],
  [x5,x4,x3,x2,x1];
  LD := LieAlgebraData(StrEq, alg, grading = [-3,-3,-2,-1,-1]):
  DGsetup(LD);

  infolevel[TanakaProlongation] := 2:
  prLD := TanakaProlongation(alg, 5, pralg):
```

$$StrEq := [[x1, x2] = x3, [x1, x3] = x4, [x2, x3] = x5], [x5, x4, x3, x2, x1]$$

*Lie algebra: alg*

```
m:
   [[e1, e2], [e3], [e4, e5]]
   [-3, -2, -1]
The Tanaka prolongation at order 0 is:
   [[e1, e2], [e3], [e4, e5], [e6, e7, e8, e9]]
   [-3, -2, -1, 0]
The Tanaka prolongation at order 1 is:
   [[e1, e2], [e3], [e4, e5], [e6, e7, e8, e9], [e10, e11]]
   [-3, -2, -1, 0, 1]
The Tanaka prolongation at order 2 is:
   [[e1, e2], [e3], [e4, e5], [e6, e7, e8, e9], [e10, e11],
[e12]]
   [-3, -2, -1, 0, 1, 2]
The Tanaka prolongation at order 3 is:
   [[e1, e2], [e3], [e4, e5], [e6, e7, e8, e9], [e10, e11],
[e12], [e13, e14]]
   [-3, -2, -1, 0, 1, 2, 3]
```

# 2nd order ODE

```
> # g[-1]=<x1,x2>, g[-2]=<x3>, g[0]=<T1,T2>

  StrEq := [
  [x1,x2]=x3,
  [T1,x1]=x1, [T1,x3]=x3,
  [T2,x2]=x2, [T2,x3]=x3],
  [x3,x2,x1,T1,T2];
  LD := LieAlgebraData(StrEq, alg, grading = [-2,-1,-1,0,0]):
  DGsetup(LD);

  infolevel[TanakaProlongation] := 2:
  prLD := TanakaProlongation(alg, 5, pralg):
```

$$StrEq := [[x1, x2] = x3, [T1, x1] = x1, [T1, x3] = x3, [T2, x2] = x2, [T2, x3] = x3], [x3, x2, x1, T1, T2]$$

*Lie algebra: alg*

```
m:
   [[e1], [e2, e3], [e4, e5]]
   [-2, -1, 0]
The Tanaka prolongation at order 1 is:
   [[e1], [e2, e3], [e4, e5], [e6, e7]]
   [-2, -1, 0, 1]
The Tanaka prolongation at order 2 is:
   [[e1], [e2, e3], [e4, e5], [e6, e7], [e8]]
   [-2, -1, 0, 1, 2]
```