# Symmetry computations in Maple

Author: Dennis The
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This worksheet is a supplement to the lecture "Classifying homogeneous geometric structures (Lecture 1) " (http://cft.edu.pl/grieg/activities.html).

In this worksheet, we give several examples of how to use Maple to compute symmetries of various differential geometric structures.

```
> restart: with(DifferentialGeometry): with(GroupActions): with
  (LieAlgebras): with(JetCalculus):
  sym:=v->InfinitesimalSymmetriesOfGeometricObjectFields(v,output=
  "list"):
```

## Metrics

```
M > DGsetup([x,y],M);
```

*Manifold: M* **(1.1)**

```
M > g:=evalDG(dx &s dx + dy &s dy);
```

$$g := dx \otimes dx + dy \otimes dy \tag{1.2}$$

```
M > S:=sym(g);
    nops(%);
```

$$S := \left[ -y\,\partial_x + x\,\partial_y, \partial_y, \partial_x \right]$$

$$3 \tag{1.3}$$

## Conformal structures

```
M > DGsetup([x,y,z,w],M);
```

*Manifold: M* **(2.1)**

```
M > c:=[evalDG(dy &s dy + dz &s dz + dw &s dx + y^2*dw &s dw)];
```

$$c := \left[ \frac{1}{2}\, dx \otimes dw + dy \otimes dy + dz \otimes dz + \frac{1}{2}\, dw \otimes dx + y^2\, dw \otimes dw \right] \tag{2.2}$$

```
M > S:=sym([c]);
    nops(%);
```

$$S := \left[ x\,\partial_x + \frac{y}{2}\,\partial_y + \frac{z}{2}\,\partial_z, \partial_w, -2\,z\,\partial_x + w\,\partial_z, \partial_z, 2\,e^{-w}\,y\,\partial_x + e^{-w}\,\partial_y, -2\,e^{w}\,y\,\partial_x + e^{w}\,\partial_y, \partial_x \right]$$

$$7 \tag{2.3}$$

```
M > LieAlgebraData(S,alg):
    DGsetup(%);
    MultiplicationTable();
```

*Lie algebra: alg*

| alg | e1 | e2 | e3 | e4 | e5 | e6 | e7 |
|---|---|---|---|---|---|---|---|
| e1 | $0$ | $0$ | $-\frac{1}{2}\,e3$ | $-\frac{1}{2}\,e4$ | $-\frac{1}{2}\,e5$ | $-\frac{1}{2}\,e6$ | $-e7$ |
| e2 | $0$ | $0$ | $e4$ | $0$ | $-e5$ | $e6$ | $0$ |
| e3 | $\frac{1}{2}\,e3$ | $-e4$ | $0$ | $2\,e7$ | $0$ | $0$ | $0$ |
| e4 | $\frac{1}{2}\,e4$ | $0$ | $-2\,e7$ | $0$ | $0$ | $0$ | $0$ |
| e5 | $\frac{1}{2}\,e5$ | $e5$ | $0$ | $0$ | $0$ | $-4\,e7$ | $0$ |
| e6 | $\frac{1}{2}\,e6$ | $-e6$ | $0$ | $0$ | $4\,e7$ | $0$ | $0$ |
| e7 | $e7$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

**(2.4)**

```
alg > Series("Derived");
      Series("Lower");
```

$$\left[[e1, e2, e3, e4, e5, e6, e7], \left[-\frac{1}{2}\,e3, -\frac{1}{2}\,e4, -\frac{1}{2}\,e5, -\frac{1}{2}\,e6, -e7\right], \left[\frac{1}{2}\,e7\right], [\,]\right]$$

$$\left[[e1, e2, e3, e4, e5, e6, e7], \left[-\frac{1}{2}\,e3, -\frac{1}{2}\,e4, -\frac{1}{2}\,e5, -\frac{1}{2}\,e6, -e7\right], \left[-\frac{1}{4}\,e3, \frac{1}{2}\,e4, -e7, -\frac{1}{4}\,e5, -\frac{1}{4}\,e6\right]\right] \tag{2.5}$$

```
alg > Nilradical();
      Radical();
```

$$[e3, e4, e5, e6, e7]$$

$$[e7, e6, e5, e4, e3, e2, e1] \tag{2.6}$$

```
alg > Center();
```

$$[\,] \tag{2.7}$$

# 2nd order ODE

```
M > DGsetup([x,y,p],M);
```

$$\text{Manifold: } M \tag{3.1}$$

```
M > f:=exp(p):
    E:=evalDG([D_x+p*D_y+f*D_p]);
    V:=evalDG([D_p]);
    ODE:=[E,V];
```

$$E := \left[\partial_x + p\,\partial_y + e^p\,\partial_p\right]$$

$$V := \left[\partial_p\right]$$

$$ODE := \left[\left[\partial_x + p\,\partial_y + e^p\,\partial_p\right], \left[\partial_p\right]\right] \tag{3.2}$$

```
M > S:=sym(ODE);
```

```
nops(%);
```

$$S := \left[ -x\,\partial_x + (x-y)\,\partial_y + \partial_p, \partial_y, \partial_x \right]$$

$$3 \tag{3.3}$$

```
M > LieAlgebraData(S,alg):
    DGsetup(%);
    MultiplicationTable();
```

*Lie algebra: alg*

| alg | e1 | e2 | e3 |
|---|---|---|---|
| e1 | 0 | e2 | − e2 + e3 |
| e2 | − e2 | 0 | 0 |
| e3 | e2 − e3 | 0 | 0 |

(3.4)

```
M > DGsetup([x],[y],J,2);

    X:=evalDG([D_x,D_y[],x*D_x+(y[]-x)*D_y[]]);

    # These were the point symmetries given above.
    # (Rather, their projections to (x,y)-space.)
    # We've written them here on the zeroth jet space J^0(R,R).
```

*Jet Bundle: J*

$$X := \left[ \partial_x, \partial_{y[\,]}, x\,\partial_x - \left(-y[\,] + x\right)\partial_{y[\,]} \right] \tag{3.5}$$

```
J > # Let's prolong the symmetries up to the second jet spacce
    J^2(R,R).

    X2:=map(v->Prolong(v,2),X);

    # These prolonged point symmetries are tangent vector fields
    to y[1,1]=exp(y[1]).  Let's check:

    [seq(LieDerivative(X2[i],y[1,1]-exp(y[1])),i=1..3)];
    eval(%,y[1,1]=exp(y[1]));
```

$$X2 := \left[ \partial_x, \partial_{y[\,]}, x\,\partial_x - \left(-y[\,] + x\right)\partial_{y[\,]} - \partial_{y[1]} - y_{1,1}\,\partial_{y[1,1]} \right]$$

$$\left[ 0, 0, \mathrm{e}^{y_1} - y_{1,1} \right]$$

$$[0, 0, 0] \tag{3.6}$$

# (2,3,5)-distributions

```
M > DGsetup([x,y,p,q,z],M);
```

*Manifold: M* (4.1)

```
M > # Hilbert-Cartan

    f:=q^2:
    dist:=evalDG([D_x+p*D_y+q*D_p+f*D_z,D_q]);

    LieBracket(dist[1],dist[2]);
    LieBracket(dist[1],%);
    LieBracket(dist[2],%%);
```

$$dist := \left[\partial_x + p\,\partial_y + q\,\partial_p + q^2\,\partial_z, \partial_q\right]$$
$$-\partial_p - 2\,q\,\partial_z$$
$$\partial_y$$
$$-2\,\partial_z \qquad \textbf{(4.2)}$$

```
M > S:=sym([dist]);
    nops(%);
    pt:=[x=0,y=0,p=0,q=1,z=0]:
    IsotropyFiltration(S,pt):
    map(nops,%);
```

$$S := \left[\left(-\frac{3\,q\,y}{2} + p^2\right)\partial_x + \left(-\frac{3}{2}\,y\,p\,q + \frac{3}{4}\,y\,z + \frac{2}{3}\,p^3\right)\partial_y + \left(-\frac{3\,q^2\,y}{4} + \frac{3\,p\,z}{4}\right)\partial_p - \left(\frac{1}{2}\,p\,q^2 - \frac{3}{4}\,q\,z\right)\partial_q - \left(\frac{y\,q^3}{2} - \frac{3\,z^2}{4}\right)\partial_z, \frac{x^2}{4}\,\partial_y + \frac{x}{2}\,\partial_p + \frac{1}{2}\,\partial_q + p\,\partial_z, \partial_z, x\,\partial_y + \partial_p, \partial_y, \partial_x, -\left(-3\,y\,x + 2\,p\,x^2 - \frac{1}{2}\,q\,x^3\right)\partial_x - \left(-3\,y^2 - \frac{1}{2}\,p\,q\,x^3 + p^2\,x^2 + \frac{1}{4}\,x^3\,z\right)\partial_y - \left(-3\,y\,p - \frac{1}{4}\,q^2\,x^3 + p^2\,x + \frac{3}{4}\,x^2\,z\right)\partial_p + \left(\frac{1}{2}\,q^2\,x^2 - q\,p\,x + 2\,p^2 - \frac{3}{2}\,x\,z\right)\partial_q + \left(3\,y\,z + \frac{1}{6}\,q^3\,x^3 - 3\,p\,x\,z + \frac{4}{3}\,p^3\right)\partial_z, -\left(-\frac{3}{2}\,q\,x^2 + 4\,p\,x - 3\,y\right)\partial_x - \left(-\frac{3}{2}\,x^2\,p\,q + 2\,p^2\,x + \frac{3}{4}\,x^2\,z\right)\partial_y - \left(-\frac{3}{4}\,q^2\,x^2 + p^2 + \frac{3}{2}\,x\,z\right)\partial_p - \left(-q^2\,x + p\,q + \frac{3}{2}\,z\right)\partial_q - \left(-\frac{q^3\,x^2}{2} + 3\,p\,z\right)\partial_z, \left(2\,p - \frac{3\,q\,x}{2}\right)\partial_x + \left(-\frac{3}{2}\,q\,p\,x + p^2 + \frac{3}{4}\,x\,z\right)\partial_y - \left(\frac{3\,q^2\,x}{4} - \frac{3\,z}{4}\right)\partial_p - \frac{q^2}{2}\,\partial_q - \frac{q^3\,x}{2}\,\partial_z, x^2\,\partial_x + 3\,y\,x\,\partial_y + \left(p\,x + 3\,y\right)\partial_p + \left(-q\,x + 4\,p\right)\partial_q + 4\,p^2\,\partial_z, -y\,\partial_y - p\,\partial_p - q\,\partial_q - 2\,z\,\partial_z, -q\,\partial_x - \left(p\,q - \frac{z}{2}\right)\partial_y - \frac{q^2}{2}\,\partial_p - \frac{q^3}{3}\,\partial_z, -\frac{x^3}{6}\,\partial_y - \frac{x^2}{2}\,\partial_p - x\,\partial_q - \left(2\,p\,x - 2\,y\right)\partial_z, -x\,\partial_x - 2\,y\,\partial_y - p\,\partial_p - z\,\partial_z\right]$$

$$14$$
$$[9, 2, 0] \qquad \textbf{(4.3)}$$

```
M > # f=exp(q)

    f:=exp(q):
    dist:=evalDG([D_x+p*D_y+q*D_p+f*D_z,D_q]);
    S:=sym([dist]);
    nops(%);
    pt:=[x=0,y=0,p=0,q=0,z=0]:
    IsotropyFiltration(S,pt):
```

```
map(nops,%);
```

$$dist := \left[\partial_x + p\,\partial_y + q\,\partial_p + \mathrm{e}^q\,\partial_z,\ \partial_q\right]$$

$$S := \left[-\mathrm{e}^q\,\partial_x - \left(\mathrm{e}^q\,p - z\right)\partial_y - \mathrm{e}^q\,(q-1)\,\partial_p - \frac{\mathrm{e}^{2\,q}}{2}\,\partial_z,\ -x\,\partial_x + \left(\frac{x^2}{2} - 2\,y\right)\partial_y - \left(p - x\right)\partial_p + \partial_q,\ \frac{x^2}{2}\,\partial_y + x\,\partial_p + \partial_q + z\,\partial_z,\ \partial_z,\ x\,\partial_y + \partial_p,\ \partial_y,\ \partial_x\right]$$

$$7$$

$$[2, 0] \tag{4.4}$$