Applications of Tractor Calculus in General Relativity

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Almost Einstein equation

Let $(M, \mathbf{c})$ be a Lorentzian $n-$dimensional manifold with a conformal class of metrics $\mathbf{c}$ of signature $(n-1,1)$.

Let $\hat{\sigma} \in \Gamma(\mathcal{E}[1])$ and

$$\nabla^g_a \nabla^g_b \hat{\sigma} + P^g_{ab} \hat{\sigma} + \mathbf{g}_{ab} \rho = 0, \qquad (1)$$

or with the use of the *scale tractor* $I_{\hat{\sigma}} := \frac{1}{n} D\hat{\sigma}$,

$$\nabla^{\mathcal{T}}_a I_{\hat{\sigma}} = 0. \qquad (2)$$

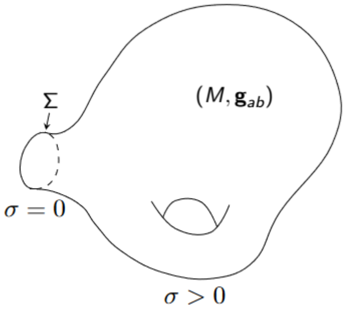
We have $I_{\hat{\sigma}} \to \hat{\sigma} = h(X, I_{\hat{\sigma}})$ and

$$h(I_{\hat{\sigma}}, I_{\hat{\sigma}}) = \mathbf{g}^{ab} \nabla^g_a \hat{\sigma} \nabla^g_b \hat{\sigma} - \frac{2}{n} \hat{\sigma} \left(\Delta^g + \mathrm{tr} P^g \right) \hat{\sigma} \qquad (3)$$

gives the conformally covariant notion of the scalar curvature.

Geometry of the scale

Let $(M, \mathbf{g}_{ab})$ be a manifold with the spacelike boundary Σ:



Let $\sigma \in \Gamma(\mathcal{E}[1])$ be a defining density of Σ:

$$\Sigma : \sigma = 0, \quad \nabla^{g}_{a}\sigma\big|_{\Sigma} \neq 0. \tag{4}$$

Define

$$n_a := \nabla^{g}_{a}\sigma \tag{5}$$

as the extension of the normal vector of Σ to M.

Induced conformal metric on Σ:

$$\overline{\mathbf{g}}_{ab} \overset{\Sigma}{:=} \mathbf{g}_{ab} - \frac{n_a n_b}{\mathbf{g}(n, n)}. \tag{6}$$

Normal tractor

Let

$$N := I_\sigma\big|_\Sigma = \begin{pmatrix} 0 \\ n_a \\ -H^g \end{pmatrix} \tag{7}$$

where H^g is the mean curvature of Σ.

Singular Yamabe equation

A rescaled scale

$$\sigma \to f\sigma, \quad f > 0 \tag{8}$$

is also a defining density of Σ.

Singular Yamabe equation

To remove this ambiguity solve the *singular Yamabe problem*: find $f \in C^{\infty}_{+}(M)$ such that the singular metric

$$g^{0}_{ab} := \frac{\mathbf{g}_{ab}}{(f\sigma)^2} \tag{9}$$

has constant scalar curvature on $M \setminus \Sigma$, i.e.

$$R^{g^0} = n(n-1)\,. \tag{10}$$

Conformally covariant version of this equation

$$h(I_{f\sigma}, I_{f\sigma}) = -1 \tag{11}$$

implies

$$\mathbf{g}(\widehat{n}, \widehat{n}) = -1 + \mathcal{O}(f\sigma) \tag{12}$$

where $\widehat{n}_a := \nabla^{g}_{a}(f\sigma)$ is the extension of the unit normal vector of Σ to M.

Singular Yamabe equation

Q: Can we solve singular Yamabe equation? Formally yes, up to order n; First step:

$$\sigma_0 \longrightarrow I^2_{\sigma_0} = -1 + \mathcal{O}\left(\sigma_0\right) \tag{13}$$

Then

$$\sigma_1 = \sigma_0 + \alpha\sigma_0^2 \longrightarrow I^2_{\sigma_1} = -1 + \mathcal{O}\left(\sigma_1^2\right) \tag{14}$$

$$\vdots$$

Last step:

$$\sigma_{n-1} \longrightarrow I^2_{\sigma_{n-1}} = -1 + \mathcal{O}\left(\sigma^n_{n-1}\right) \tag{15}$$

Singular Yamabe equation

Let σ be the singular Yamabe scale, i.e.

$$I_\sigma^2 = -1 + \mathcal{O}\left(\sigma^n\right) \tag{16}$$

Define the extension of the normal tractor of the boundary Σ as

$$N^e := I_\sigma \tag{17}$$

Then

$$h\left(N^e, N^e\right) = -1 + \mathcal{O}\left(\sigma^n\right) \tag{18}$$

We can use jets of N^e (jets of σ) to study the geometry of the embedded hypersurface Σ:

- S. Blitz, A. R. Gover and A. Waldron, *Conformal Fundamental Forms and the Asymptotically Poincaré–Einstein Condition*, arXiv:2107.10381, 2021.

First conformal fundamental form – induced conf. metric

$$\overline{\mathbf{g}}_{ab} \stackrel{\Sigma}{=} \mathbf{g}_{ab} + \widehat{n}_a \widehat{n}_b. \tag{19}$$

Second conformal fundamental form

We have

$$\overline{\mathbf{g}}_a{}^b \nabla^{\mathcal{T}}_b N^e \big|_\Sigma = \begin{pmatrix} 0 \\ \mathring{K}_{ab} \\ -\frac{1}{n-2} \overline{\nabla}^{\overline{g}}_b \mathring{K}_c{}^b \end{pmatrix} \tag{20}$$

where $\mathring{K}_{ab}$ is the trace-free second fundamental form (extrinsic curvature) of Σ – the second conformal fundamental form.

Higher order fundamental forms

Example – surface in $\mathbb{R}^3$

Third fundamental form $K^{(3)}_{ab}$ can be defined as

$$K^{(3)}_{ab} := \top \left(\nabla_{\hat{n}} \nabla_a \hat{n}_b \right) \tag{21}$$

Extension of $\mathring{K}_{ab}$

Let

$$E_{ab} := q^* \left(\nabla_a^{\mathcal{T}} N^e \right) = \nabla_a^g \widehat{n}_b + \sigma P_{ab} + \rho \mathbf{g}_{ab} \tag{22}$$

where q^* extracts the 'tensorial' slot in the conformally covariant way – $E_{ab} \in \Gamma \left(\mathcal{E}_{(ab)_0}[1] \right)$.

Higher order conformal fundamental forms

Let

$$\mathring{K}_{ab}^{(i)} := \mathring{\top} \delta^i E_{ab} \tag{23}$$

where δ is constructed with the use of $h(N^e, D)$, i.e.

$$\delta = \nabla_n + \ldots \tag{24}$$

and $\mathring{\top}$ denotes the trace-free projection on ∂M.

The leading terms in $\mathring{K}_{ab}^{(3)}$, $\mathring{K}_{ab}^{(4)}$ and $\mathring{K}_{ab}^{(5)}$ are projections of Weyl, Cotton and Bach tensors.

Einstein field equations with positive Λ

Let $(\widetilde{M}, \widetilde{g}_{ab})$ be a solution of the Einstein field equations,

$$\widetilde{R}_{ab} - \frac{1}{2}\widetilde{R}\widetilde{g}_{ab} + \Lambda\widetilde{g}_{ab} = \widetilde{T}_{ab}, \tag{25}$$

where $\Lambda > 0$ is the cosmological constant and $\widetilde{T}_{ab}$ is the physical stress-energy tensor.

Asymptotically de Sitter spacetime

Conformal extension (M, g_{ab}) of a spacetime $(\widetilde{M}, \widetilde{g}_{ab})$:

- M is a manifold with a boundary Σ
- $\widetilde{M} \equiv M \setminus \Sigma$
- Ω is a smooth function such that:
  - $\Omega > 0$ on $M \setminus \Sigma$
  - $\Omega = 0$, $d\Omega \neq 0$ on Σ
- $\widetilde{g}_{ab} = \Omega^{-2} g_{ab}$ and $\underline{\widetilde{T}_{ab} = \Omega^q T_{ab},\ \ q \geqslant 0}$
- Unphysical tensors g_{ab}, T_{ab} regular on M

Σ – end state of the evolution of the spacetime (conf. ∞)

4-dimensional de Sitter spacetime with $\Lambda = 3$

Hypersurface described by the equation

$$-t^2 + x^2 + y^2 + z^2 + w^2 = 1 \tag{26}$$

in 5-dimensional Minkowski space. Induced metric has the following form

$$g_{dS} = -dt^2 + (\cosh t)^2 \, g_{S^3}. \tag{27}$$

Focus on $t \in [0, \infty)$. Introduce new coordinate τ such that

$$\cosh t = \frac{1}{\cos \tau}. \tag{28}$$

Then $\tau \in [0, \pi/2]$ and the metric reads

$$g_{dS} = \frac{1}{\cos^2 \tau} \left(-d\tau^2 + g_{S^3} \right). \tag{29}$$

so $\Omega = \cos \tau$ and $\Omega = 0 \iff \tau = \pi/2 \iff t = \infty$.

Conformal Einstein field equations

Singular equation for the unphysical metric g_{ab}

We have

$$\widetilde{g}_{ab} = \Omega^{-2} g_{ab}, \tag{30}$$

so

$$\widetilde{R}_{ab} = R_{ab} + \frac{n-2}{\Omega}\nabla_a \nabla_b \Omega + \frac{1}{\Omega} g_{ab}\left(\Delta\Omega - \frac{n-1}{\Omega}\nabla_c \Omega \nabla^c \Omega\right) \tag{31}$$

Consequences:

- Einstein field equations (EFEs) are singular at Σ
- multiplying by Ω^2 does not help as the principal part vanishes at Σ

Regularization – Conformal EFEs ($n = 4, q \geqslant 1$):
H. Friedrich, *On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations*, Proc. Roy. Soc. Lond. A, 375, 169, 1981.

Conformal Einstein field equations in 4 dimensions

Conformal transformation rule for $\widetilde{P}_{ab}$ and EFEs imply

$$\nabla_a \nabla_b \Omega = -\Omega P_{ab} + s g_{ab} + \frac{1}{2}\Omega \mathring{\tilde{T}}_{ab}, \tag{32}$$

where

$$s := \frac{1}{4}\Delta\Omega + \frac{1}{24} R\Omega \tag{33}$$

is the Friedrich scalar. Promote the field s and P_{ab} to the level of unknowns and close the system.

Equation for s

Apply derivative to (32), commute, use Bianchi etc.

$$\nabla_a s = -P_{ab}\nabla^b \Omega + \frac{1}{6}\nabla^c \left(\Omega \mathring{\tilde{T}}_{ab}\right) \tag{34}$$

Conformal Einstein field equations in 4 dimensions

Equation for P_{ab} – Bianchi identities lead to

$$2\nabla_{[b}P_{c]a} = w^{d}{}_{abc}\nabla_{d}\Omega + \Omega Q_{abc} \tag{35}$$

and

$$\nabla_{d} w^{d}{}_{abc} = Q_{abc} \tag{36}$$

where

$$w_{abcd} := \Omega^{-1} W_{abcd}, \quad Q_{abc} := \Omega^{-1}\widetilde{A}_{abc} \tag{37}$$

are the rescaled Weyl and physical Cotton tensor.

Regularized conformal transformation rule for R

$$6\Omega s - 3\nabla_{a}\Omega\nabla^{a}\Omega + \frac{1}{4}\widetilde{T} = \Lambda \tag{38}$$

Conformal Einstein field equations in 4 dimensions

Summary

$$\widetilde{R}_{ab} - \frac{1}{2}\widetilde{R}\widetilde{g}_{ab} + \Lambda\widetilde{g}_{ab} = \widetilde{T}_{ab}, \tag{39}$$

with

$$\widetilde{g}_{ab} = \Omega^{-2} g_{ab}, \quad \widetilde{T}_{ab} = \Omega^{q} T_{ab} \tag{40}$$

in 4 dimensions is equivalent to

$$\nabla_a \nabla_b \Omega = -\Omega P_{ab} + s g_{ab} + \frac{1}{2}\Omega \mathring{\widetilde{T}}_{ab}, \tag{41}$$

$$\nabla_a s = -P_{ab}\nabla^b \Omega + \frac{1}{6}\nabla^c \left(\Omega \mathring{\widetilde{T}}_{ab}\right) \tag{42}$$

$$2\nabla_{[b} P_{c]a} = w^d{}_{abc}\nabla_d \Omega + \Omega Q_{abc} \tag{43}$$

$$\nabla_d w^d{}_{abc} = Q_{abc} \tag{44}$$

$$\Lambda = 6\Omega s - 3\nabla_a \Omega \nabla^a \Omega + \frac{1}{4}\widetilde{T} \tag{45}$$

with solution in the form of

$$(\Omega, g_{ab}, s, P_{ab}, w_{abcd}, T_{ab}). \tag{46}$$

Conformal Einstein field equations in 4 dimensions

Constraints on Σ

Conformal EFEs induce 10 constraints on Σ. Analysis in vacuum leads to

Theorem

Given a 3-dimensional metric $\overline{g}_{ab}$ and a divergence-free and trace-free (with respect to $\overline{g}_{ab}$) tensor d_{ab} there is a solution to the vacuum conformal constraint equations on Σ.

Conformal covariance: $\theta^2\overline{g}_{ab}$, $\theta^{-1}d_{ab}$ for some $\theta \in C^\infty(\Sigma)$ also give rise to a solution of constraints.

Goal: use tractor calculus to study the geometry of Σ in a more efficient way

Tractor calculus with stress-energy tensor

EFEs again (n dimensions)

$$\mathring{\tilde{R}}_{ab} = \mathring{\tilde{T}}_{ab}, \quad \tilde{R} = \frac{2}{n-2}\left(n\Lambda - \tilde{T}_c{}^c\right) \tag{47}$$

Almost Einstein equation with stress-energy tensor

We have

$$\mathring{\tilde{P}}_{ab} = \frac{1}{n-2}\mathring{\tilde{T}}_{ab} \tag{48}$$

so

$$\nabla_a \nabla_b \tilde{\sigma} + \tilde{\sigma} P_{ab} - \frac{1}{n}\mathbf{g}_{ab}\left(\Delta\tilde{\sigma} + \tilde{\sigma} P_c{}^c\right) = \frac{\tilde{\sigma}}{n-2}\mathring{\tilde{T}}_{ab} \tag{49}$$

where $\tilde{\sigma}$ is the scale corresponding to the physical metric $\tilde{g}_{ab}$.

Stress-energy tensor density Let

$$\tau_{ab} := \tilde{\sigma}^{-q}\tilde{T}_{ab}, \qquad \tau_{ab} \in \Gamma\left(\mathcal{E}_{(ab)}[-q]\right). \tag{50}$$

Weight $-q$ and the decay rate of the stress-energy tensor
Previously

$$\widetilde{g}_{ab} = \Omega^{-2} g_{ab} \tag{51}$$

but

$$\widetilde{g}_{ab} = \widetilde{\sigma}^{-2} \mathbf{g}_{ab}, \quad g_{ab} = \sigma_g^{-2} \mathbf{g}_{ab}, \tag{52}$$

so

$$\Omega = \frac{\widetilde{\sigma}}{\sigma_g} \tag{53}$$

$$\tau_{ab} = \widetilde{\sigma}^{-q} \widetilde{T}_{ab} = \sigma_g^{-q} T_{ab} \tag{54}$$

where T_{ab} is the unphysical stress-energy tensor, so

$$\widetilde{T}_{ab} = \left(\frac{\widetilde{\sigma}}{\sigma_g} \right)^q T_{ab} = \Omega^q T_{ab}. \tag{55}$$

Tractor calculus with stress-energy tensor

Ultimately,

$$\nabla_a \nabla_b \widetilde{\sigma} + \widetilde{\sigma} P_{ab} - \frac{1}{n} \mathbf{g}_{ab} \left(\Delta \widetilde{\sigma} + \widetilde{\sigma} P_c{}^c \right) = \frac{\widetilde{\sigma}^{q+1}}{n-2} \mathring{\tau}_{ab} \tag{56}$$

Prolongation

$$\nabla_a^{\mathcal{T}} I_{\tilde{\sigma}} = \frac{\tilde{\sigma}^q}{(n-1)(n-2)} \begin{pmatrix} 0 \\ (n-1)\, \tilde{\sigma} \mathring{\tau}_{ab} \\ -(q+1)\, \mathring{\tau}_{ab} \nabla^b \tilde{\sigma} - \tilde{\sigma} \nabla^b \mathring{\tau}_{ab} \end{pmatrix}, \tag{57}$$

The $I_{\tilde{\sigma}}^2$ equation and the scalar curvature

$$\widetilde{R} = \frac{2}{n-2} \left(n\Lambda - \widetilde{T}_c{}^c \right) \tag{58}$$

leads to

$$I_{\tilde{\sigma}}^2 = -\frac{2\Lambda}{(n-1)(n-2)} + \frac{2\widetilde{\sigma}^{q+2} \tau}{n(n-1)(n-2)}. \tag{59}$$

Tractor calculus with stress-energy tensor

Almost Einstein vs. Almost Einstein with stress-energy tensor

$$\nabla_a^{\mathcal{T}} I_{\tilde{\sigma}} = 0, \quad I_{\tilde{\sigma}}^2 = -1 + \mathcal{O}\left(\tilde{\sigma}^n\right) \tag{60}$$

and

$$\nabla_a^{\mathcal{T}} I_{\tilde{\sigma}} = T_{aA}, \quad I_{\tilde{\sigma}}^2 = -\frac{2\Lambda}{(n-1)(n-2)} + \frac{2\tilde{\sigma}^{q+2}\tau}{n(n-1)(n-2)}. \tag{61}$$

Remark Conformally covariant equation

$$D_A \left(I_{\tilde{\sigma}}\right)_B = T_{AB} \tag{62}$$

Use construction of conformal fundamental forms to derive constraints relating geometry of Σ with the stress-energy tensor (in 4 dimensions)

First constraint

$$\nabla I_{\tilde{\sigma}}^2 = 2h\left(I_{\tilde{\sigma}}, \nabla I_{\tilde{\sigma}}\right) \implies (q-2)\,\tau_{a\hat{n}} + \hat{n}_a \tau = -\sqrt{\frac{3}{\Lambda}}\tilde{\sigma}\nabla_b \tau_a{}^b. \tag{63}$$

Conformal fundamental forms and matter fields

Extension of the second fundamental form

$$E_{ab} = \nabla^g_a \widehat{n}_b + \widetilde{\sigma} P_{ab} + \rho \mathbf{g}_{ab} = \frac{1}{2}\widetilde{\sigma}^{q+1}\mathring{\tau}_{ab} \tag{64}$$

so if $q \geqslant 0$ (asymptotically de Sitter spacetime)

$$\mathring{K}_{ab} = 0 \quad \text{on} \quad \Sigma \tag{65}$$

i.e. ∂M is an umbilic hypersurface.
Third fundamental form We have

$$\mathring{K}^{(3)}_{ab} = \mathring{\top}\delta E_{ab} = W^{\top}_{\hat{n}ab\hat{n}} \tag{66}$$

on the other hand

$$\mathring{\top}\delta\left(\frac{1}{2}\widetilde{\sigma}^{q+1}\mathring{\tau}_{ab}\right) = -\frac{1}{2}\left(q+1\right)\widetilde{\sigma}^{q}\mathring{\top}\left(\tau_{ab}\right)\big|_{\Sigma} \tag{67}$$

Hence for $q = 0$

$$W^{\top}_{\hat{n}a\hat{n}b} = \frac{1}{2}\mathring{\top}\left(\tau_{ab}\right) \quad \text{on} \quad \Sigma. \tag{68}$$

Fourth fundamental form We have

$$\mathring{K}^{(4)}_{ab} = \mathring{\top}\delta^2 E_{ab} = 0 \quad \text{on} \quad \Sigma \tag{69}$$

if this hypersurface is umbilic. Hence

$$0 = \mathring{\top}\delta^2\left(\frac{1}{2}\widetilde{\sigma}^{q+1}\mathring{\tau}_{ab}\right) = \begin{cases} \nabla^{g}_{\hat{n}}\tau^{\top}_{ab} + \frac{2}{3}H\tau^{\top}_{ab} \quad \text{for} \quad q = 0, \\ \mathring{\top}\left(\tau_{ab}\right) \quad \text{for} \quad q = 1, \\ 0 \quad \text{for} \quad q \geqslant 2 \end{cases} \tag{70}$$

on ∂M.

The Cotton tensor

Let $q \geqslant 1$. Then Gauss-Codazzi and $\mathring{K}^{(3)}_{ab}$ constraint imply

$$W_{abcd} = 0 \quad \text{on} \quad \Sigma \tag{71}$$

Let

$$W_{abcd} = \widetilde{\sigma} w_{abcd} + \mathcal{O}\left(\widetilde{\sigma}^2\right) \tag{72}$$

Then

$$q^* \left([\nabla^{\mathcal{T}}_a, \nabla^{\mathcal{T}}_b] I_{\tilde{\sigma}} \right) = \widetilde{\sigma} \left(A_{cab} + w_{abc\hat{n}} \right) + \mathcal{O}\left(\widetilde{\sigma}^2 \right) \\ = \widetilde{\sigma}^{q+1} \left(\nabla_{[a} \tau_{b]c} + \dots \right) \tag{73}$$

Ultimately

$$A^{\top}_{a\hat{n}b} = w^{\top}_{a\hat{n}b\hat{n}} \quad \text{on} \quad \Sigma \tag{74}$$

Divergence of the Cotton tensor

Use the relation between the Cotton tensor and w_{abcd} to get

$$\overline{\nabla}^g_b A^{\top}_{a\hat{n}}{}^b = \begin{cases} j_a - \frac{1}{3} \overline{\nabla}^g_a \tau & \text{for} \quad q = 1, \\ \tau^{\top}_{a\hat{n}} & \text{for} \quad q = 2, \\ 0 \quad \text{on} & \text{for} \quad q > 2 \end{cases} \tag{75}$$

on Σ, where $\overline{\nabla}^g$ is the hypersurface connection and

$$j_a = \lim_{\tilde{\sigma} \to 0} \left(\frac{\tau^{\top}_{a\hat{n}}}{\widetilde{\sigma}} \right).$$

Conformal fundamental forms and matter fields

The fifth fundamental form Let $q \geqslant 1$. Then

$$\mathring{K}^{(5)}_{ab} := \mathring{\top}\delta^3 E_{ab} = 6\mathring{\top}\left(B_{ab}\right) \quad \text{on} \quad \Sigma. \tag{76}$$

Ultimately

$$\mathring{\top}\left(B_{ab}\right) = \begin{cases} \frac{1}{3}\left(9\nabla^{g}_{\hat{n}}\tau^{\top}_{ab} - 3\overline{g}_{ab}\nabla^{g}_{\hat{n}}\tau_{\hat{n}\hat{n}} + 8H\tau^{\top}_{ab} - H\overline{g}_{ab}\tau_{\hat{n}\hat{n}}\right) & \text{for} \quad q = 1, \\ -3\mathring{\top}\left(\tau_{ab}\right) \quad \text{for} \quad q = 2, \\ 0 \quad \text{for} \quad q > 2 \end{cases} \tag{77}$$

on Σ.

Commutator of D operators and the Bach tensor

$$D_{[A}D_{B]}\left(I_{\tilde{\sigma}}\right)_C = 2\sqrt{\frac{\Lambda}{3}}X_C X_{[A}Z_{B]}{}^c B_{c\hat{n}} \tag{78}$$

on Σ. On the other hand,

$$D_{[A}D_{B]}I_{\tilde{\sigma}} = -X_{[A}\left(\Delta - P_c{}^c\right)\left(2Z_{B]}{}^a\nabla_a I_{\tilde{\sigma}} - X_{B]}\nabla^b\nabla_b I_{\tilde{\sigma}}\right). \tag{79}$$

Ultimately

$$\begin{aligned} B_{\hat{n}\hat{n}} = 0, \quad & B^{\top}_{a\hat{n}} = \nabla^g_{\hat{n}}\tau^{\top}_{a\hat{n}} + \frac{1}{3}\overline{\nabla}^g_a\tau \quad \text{for} \quad q = 1, \\ B_{\hat{n}\hat{n}} = 0, \quad & B^{\top}_{a\hat{n}} = -\tau^{\top}_{a\hat{n}} \quad \text{for} \quad q = 2. \end{aligned} \tag{80}$$

on Σ.

Conclusion

Summary:

- we applied the construction of conformal fundamental forms to derive a constraint on the matter fields on asymptotically de Sitter background
- projected parts of Weyl and Bach tensors and the divergence of Cotton are related to the stress-energy tensor on the conformal boundary Σ

Thank you!