Applications of Tractor Calculus in General Relativity

Jarosław Kopiński

Centre for Theoretical Physics PAS

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Overview

- Introduction to tractor calculus
- Asymptotically de Sitter spacetimes
- Spacetimes with initial isotropic singularities
- Conformal Cyclic Cosmology

Based on

- S. Curry and R. Gover, *An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity*, arXiv:1412.7559;
- R. Gover, J. Kopiński, *Higher fundamental forms of the conformal boundary of asymptotically de Sitter spacetimes*, Class. Quantum Grav. 40 015001 2023;
- R. Gover, J. Kopiński and A. Waldron, *The geometry of an isotropic Big Bang*, In prep.;

Notation and conventions

Let (M, g_{ab}) be a Lorentzian $n-$dimensional manifold with a metric g_{ab} of signature $(n-1, 1)$.
Abstract index notation:

- tangent bundle of M: $\mathcal{E}^a$
- cotangent bundle of M: $\mathcal{E}_a$

e.g. $g_{ab} \in \Gamma\left(\mathcal{E}_{(ab)}\right)$, where $\mathcal{E}_{(ab)}$ is a subbundle of symmetric 2-tensors.
The (anti)symmetrization brackets Let $T_{abc...} \in \Gamma\left(\mathcal{E}_{abc...}\right)$.

$$\begin{aligned} T_{(ab)c...} &= \frac{1}{2}\left(T_{abc...} + T_{bac...}\right), \\ T_{[ab]c...} &= \frac{1}{2}\left(T_{abc...} - T_{bac...}\right) \end{aligned} \tag{1}$$

Lowercase letters a,b,c, ... – tensor labels
Uppercase letters A,B,C, ... – tractor labels

Notation and conventions

Levi-Civita connection ∇_a

$$\nabla_a g_{bc} = 0, \quad \text{metric compatible} \tag{2}$$

$$\Gamma^a_{bc} - \Gamma^a_{cb} = 0 \quad \text{torsion} - \text{free} \tag{3}$$

where Γ^a_{bc} - Christoffel symbols.

Riemann tensor

$$2\nabla_{[a}\nabla_{b]} v_c =: R_{abc}{}^{d} v_d, \quad \text{for any } v_a \in \Gamma(\mathcal{E}_a) \tag{4}$$

Weyl and Schouten tensors Decomposition of Riemann tensor

$$R_{abcd} = W_{abcd} + 2g_{c[a}P_{b]d} + 2g_{d[b}P_{a]c}, \tag{5}$$

where W_{abcd} – Weyl tensor and P_{ab} – Schouten tensor,

$$P_{ab} := \frac{1}{n-2}\left(R_{ab} - \frac{R}{2(n-1)} g_{ab}\right). \tag{6}$$

Conformal transformations

Let $\widehat{g}_{ab} = \Omega^2 g_{ab}$ and $\psi_a := \partial_a \log \Omega$. Then for $v_a \in \Gamma(\mathcal{E}_a)$

$$\widehat{\nabla}_a v_b = \nabla_a v_b - \psi_a v_b - v_a \psi_b + g_{ab} v_c \psi^c, \tag{7}$$

e.g. $(dv)_{ab} = \nabla_{[a} v_{b]}$ is conformally invariant,

$$\widehat{\nabla}_{[a} v_{b]} = \nabla_{[a} v_{b]}. \tag{8}$$

Let $F_{ab} \in \Gamma\left(\mathcal{E}_{[ab]}\right)$. Then

$$\widehat{\nabla}^a F_{ab} = \Omega^{-2} \left[\nabla^a + (n-4)\, \psi^a\right] F_{ab} \tag{9}$$

i.e. $\mathrm{Div} : \Gamma\left(\mathcal{E}_{[ab]}\right) \to \Gamma(\mathcal{E}_a)$ given by

$$\mathrm{Div} : F_{ab} \to \nabla^b F_{ab} \tag{10}$$

is conformally covariant in dimension 4.

Conformal wave operator

Let $f \in \Gamma(\mathcal{E})$. We have

$$\widehat{\Delta} f = \Omega^{-2}\left[\Delta + (n-2)\,\psi^c \nabla_c\right] f \tag{11}$$

Assume that f changes with the metric, i.e.

$$f \to \widehat{f} = \Omega^{1-\frac{n}{2}} f \quad \text{when} \quad g_{ab} \to \widehat{g}_{ab} = \Omega^2 g_{ab} \tag{12}$$

Then

$$\overbrace{\left(\widehat{\Delta} - \frac{n-2}{4(n-1)}\widehat{R}\right)}^{\widehat{Y}} \widehat{f} = \Omega^{-1-\frac{n}{2}} \overbrace{\left(\Delta - \frac{n-2}{4(n-1)} R\right)}^{Y} f \tag{13}$$

is conformally covariant in the sense that

$$\widehat{Y}\left(\Omega^{1-\frac{n}{2}} f\right) = \Omega^{-1-\frac{n}{2}}\, Y f \tag{14}$$

for every $f \in \Gamma(\mathcal{E})$.

Conformal invariance and covariance

Transformation rules for parts of Riemann tensor,

$$\widehat{W}^a{}_{bcd} = W^a{}_{bcd}, \quad \widehat{P}_{ab} = P_{ab} - \nabla_a \psi_b + \psi_a \psi_b - \frac{1}{2} g_{ab} \psi_c \psi^c, \tag{15}$$

e.g.

$$\widehat{W}_{abcd} \widehat{W}^{abcd} = \Omega^{-6} W_{abcd} W^{abcd} \tag{16}$$

is conformally covariant.

Construction of conformally covariant tensors

Take derivatives of curvature etc. add lower order terms with undetermined coefficients to find conformal covariants and invariants, e.g. the Bach tensor

$$B_{ab} := \Delta P_{ab} - \nabla^c \nabla_a P_{bc} + P^{cd} W_{acbd} \tag{17}$$

transforms in the following way,

$$\widehat{B}_{ab} = \Omega^{-2} \left(B_{ab} + (n-4)\, \psi^c \psi^d W_{dabc} \right). \tag{18}$$

Conformal manifold

Let $(M, \mathbf{c})$ be a manifold with a conformal class $\mathbf{c}$:

$$g_{ab},\ \widehat{g}_{ab} \in \mathbf{c} \iff \widehat{g}_{ab}\big|_{x\in M} = s^2 g_{ab}\big|_{x\in M} \tag{19}$$

for every $x \in M$ with $s(x) > 0$. Thus, the conformal class $\mathbf{c}$ is a ray subbundle $\mathcal{Q} \subset \mathcal{E}_{(ab)}$.
Alternative view: $\mathcal{Q}$ is a principal $\mathbb{R}_+$-bundle with projection

$$\pi : \mathcal{Q} \to M \tag{20}$$

and principal action

$$\rho_s\,(x, g_x) = (x, s^2 g_x), \quad x \in M, s \in \mathbb{R}_+. \tag{21}$$

Conformal geometry

Bundle of conformal densities of weight w

Define $\mathcal{E}[w]$ as an associated bundle to $\mathcal{Q}$ with respect to the action of $\mathbb{R}_+$ on $\mathbb{R}$. A section $\Gamma(\mathcal{E}[w])$ can be identified with a function F:

$$\Gamma(\mathcal{E}[w]) \longleftrightarrow F : \mathcal{Q} \to \mathbb{R} \tag{22}$$

such that

$$F(\rho_s(x, g_x)) = F\left(x, s^2 g_x\right) = s^w F(x, g_x). \tag{23}$$

Let $\widehat{g}_{ab}$ and g_{ab} be two sections of $\mathcal{Q}$ such that $\widehat{g}_{ab} = \Omega^2 g_{ab}$. We can pullback F via this sections to get functions on M such that

$$\widehat{f} = \Omega^w f, \tag{24}$$

e.g. the conformal wave operator is an operator on $\mathcal{E}\left[1 - \frac{n}{2}\right]$.

Bundle of conformally weighted tensors

Let $\mathcal{E}^{b\cdots}_{a\ldots}[w] := \mathcal{E}^{b\cdots}_{a\ldots} \otimes \mathcal{E}[w]$ for any tensor bundle $\mathcal{E}^{b\cdots}_{a\ldots}$.

Conformal metric

Tautological inclusion $\tilde{g} : \mathcal{Q} \rightarrow \pi^* \mathcal{E}_{(ab)}$,

$$\tilde{g}\left(x, s^2 g_x\right) = \left(x, s^2 g_x\right) \tag{25}$$

may be identified with a canonical section $\mathbf{g}_{ab} \in \Gamma\left(\mathcal{E}_{(ab)}[2]\right)$.
The choice of specific $g_{ab} \in \mathbf{c}$ is equivalent to the choice of scale $\sigma_g \in \Gamma\left(\mathcal{E}[1]\right)$, i.e.

$$g_{ab} = \sigma_g^{-2} \mathbf{g}_{ab} \in \Gamma\left(\mathcal{E}_{(ab)}[0]\right), \tag{26}$$

with the property that the corresponding function $\tilde{\sigma}_g$ on $\mathcal{Q}$ takes value 1 along the section g_{ab}.

Conformal geometry

Calculus with conformal densities

Choice of scale σ_g (choice of metric $g_{ab} \in \mathbf{c}$) determines a connection on $\mathcal{E}[w]$,

$$\nabla^g_a \tau = \sigma_g^w \partial_a \left(\sigma_g^{-w} \tau \right), \quad \tau \in \mathcal{E}[w] \tag{27}$$

with the immediate generalizations to weighted tensor bundles.
Consequences:

$$\nabla^g_a \sigma_g = 0, \tag{28}$$

$$\nabla^g_a \mathbf{g}_{bc} = \nabla^g_a \left(\sigma_g^2 g_{bc} \right) = 0, \tag{29}$$

so we can use $\mathbf{g}_{ab}$ instead of g_{ab} to raise and lower indices.

Conformal transformation rule

If $\widehat{g}_{ab} = \Omega^2 g_{ab}$, $\tau \in \mathcal{E}[w]$ then

$$\nabla^{\hat{g}} \tau = \nabla^g \tau + w \psi_a \tau. \tag{30}$$

Almost Einstein equation and tractors

Let $g_{ab},\ \widehat{g}_{ab} \in \mathbf{c}$ with $\widehat{g}_{ab} = \Omega^2 g_{ab}$ and $\widehat{g}_{ab}$ be the Einstein metric, i.e.

$$\widehat{R}_{ab} = \lambda \widehat{g}_{ab}, \quad \lambda = \mathrm{const}, \tag{31}$$

or

$$\widehat{P}_{ab} - \frac{1}{n}\widehat{g}_{ab}\widehat{P}_c{}^c = 0. \tag{32}$$

In terms of g_{ab} this reads

$$P_{ab} - \nabla_a \psi_b + \psi_a \psi_b - g_{ab}\left(\ldots\right) = 0. \tag{33}$$

Alternative look - g_{ab} and $\widehat{g}_{ab}$ are determined by scales. Let

$$\widehat{g}_{ab} = \sigma_{\hat{g}}^{-2}\mathbf{g}_{ab}. \tag{34}$$

Then $\Omega = \frac{\sigma_g}{\sigma_{\hat{g}}}$ and

$$\nabla^g_a \nabla^g_b \sigma_{\hat{g}} = \sigma_{\hat{g}}\left(-\nabla_a \psi_b + \psi_a \psi_b\right). \tag{35}$$

Almost Einstein equation and tractors

The conformal-to-Einstein equation is then

$$A^g_{ab}\sigma_{\hat{g}} = 0, \tag{36}$$

where

$$A^g_{ab} := \nabla^g_a \nabla^g_b + P_{ab} - \frac{1}{n} g_{ab} \left(\Delta^g + P_c{}^c \right) \tag{37}$$

Operator A_{ab} is conformally covariant,

$$A_{ab} : \mathcal{E}[1] \to \mathcal{E}_{(ab)_0}[1]. \tag{38}$$

Einstein metric - evaluate (36) in the scale $\sigma_{\hat{g}}$.
Since $\nabla^{\hat{g}}\sigma_{\hat{g}} = 0$,

$$\widehat{P}_{ab} - \frac{1}{n}\widehat{g}_{ab}\widehat{P}_c{}^c = 0. \tag{39}$$

Almost Einstein equation

$$\nabla^g_a \nabla^g_b \sigma + P^g_{ab}\sigma + \mathbf{g}_{ab}\rho = 0, \tag{40}$$

where the trace terms are absorbed by $\rho \in \Gamma(\mathcal{E}[-1])$.

Prolongation

Let $\mu_a \in \Gamma(\mathcal{E}_a[1])$ and

$$\mu_a := \nabla^g_a \sigma \tag{41}$$

Equation (40) reads

$$\nabla^g_a \mu_b + P^g_{ab}\sigma + \mathbf{g}_{ab}\rho = 0 \tag{42}$$

Apply derivative, contract with $\mathbf{g}_{ab}$, use the Bianchi identity to get

$$\nabla^g_a \rho - P^g_{ab}\mu^b = 0. \tag{43}$$

Tractor bundle

On a conformal manifold $(M, \mathbf{c})$ define pre-tractor bundle $[\mathcal{T}]_g$ as a pair consisting of a direct sum bundle and $g \in \mathbf{c}$,

$$[\mathcal{T}]_g := (\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1], g)\,, \tag{44}$$

where $V^A = (\sigma, \mu_a, \rho) \in \Gamma(\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1])$.

Connection on $[\mathcal{T}]_g$

$$\nabla^{\mathcal{T}}_a V^A = \nabla^{\mathcal{T}}_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla^g_a \sigma - \mu_a \\ \nabla^g_a \mu_b + P^g_{ab}\sigma + \mathbf{g}_{ab}\rho \\ \nabla^g_a \rho - P^g_{ab}\mu^b \end{pmatrix}. \tag{45}$$

Parallel sections of $\nabla^{\mathcal{T}}_a \iff$ solutions of the almost Einstein equation.

Equivalence relations among the direct sum bundles $[\mathcal{T}]_g$

In the prolongation procedure

$$\mu_a = \nabla_a^g \sigma, \quad \rho = -\frac{1}{n}\left(\Delta^g \sigma + \left(\mathrm{tr}\ P^g\right)\sigma\right) \tag{46}$$

Under the conformal change with $\widehat{g}_{ab} = \Omega^2 g_{ab}$,

$$\nabla_a^{\hat{g}} \sigma = \nabla_a^g \sigma + \psi_a \sigma \tag{47}$$

and

$$\Delta^{\hat{g}} \sigma + \left(\widehat{\mathrm{tr}} P^{\hat{g}}\right)\sigma = \Delta^g \sigma + \left(\mathrm{tr}\ P^g\right)\sigma + n\psi^b \nabla_b^g \sigma + \frac{n}{2}\sigma\psi^a\psi_a. \tag{48}$$

Thus, we decree that

$$\widehat{\sigma} = \sigma, \quad \widehat{\mu}_a = \mu_a + \psi_a \sigma, \quad \widehat{\rho} = \rho - \psi^b \mu_b - \frac{1}{2}\sigma\psi^a\psi_a. \tag{49}$$

Tractor bundle

In the matrix form

$$[\mathcal{T}]_{\hat{g}} \ni \begin{pmatrix} \widehat{\sigma} \\ \widehat{\mu}_b \\ \widehat{\rho} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \psi_b & \delta_b^c & 0 \\ -\frac{1}{2}\psi^a\psi_a & -\psi^c & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu_c \\ \rho \end{pmatrix}. \qquad (50)$$

The transformation is by a group action. This defines the equivalence relation $(\widehat{\sigma}, \widehat{\mu}_a, \widehat{\rho}) \sim (\sigma, \mu_a, \rho)$ and

$$\mathcal{T} = \bigsqcup_{g \in \mathbf{c}} [\mathcal{T}]_g / \sim \qquad (51)$$

is the **tractor bundle** on $(M, \mathbf{c})$ with and since $\nabla_a^{\mathcal{T}}(\sigma, \mu_a, \rho)$ transforms in accordance with (50) it determines a connection on $\mathcal{T}$.

Tractor metric and curvature

Tractor metric The formula

$$V^A = (\sigma, \mu_a, \rho) \to 2\sigma\rho + \mathbf{g}^{ab}\mu_a\mu_b =: h(V, V) \tag{52}$$

defines a tractor metric h_{AB} which is preserved by $\nabla^{\mathcal{T}}_a$. We have

$$V_A = h_{AB}V^B, \quad V^A = h^{AB}V_B \tag{53}$$

Tractor curvature Since the tractor connection is coupled with the Levi-Civita connection

$$\left(\nabla^{\mathcal{T}}_a\nabla^{\mathcal{T}}_b - \nabla^{\mathcal{T}}_b\nabla^{\mathcal{T}}_a\right)V^C = \kappa_{ab}{}^C{}_D V^D \tag{54}$$

Since $\nabla^{\mathcal{T}}_a h_{BC} = 0$

$$\kappa_{ab}{}^{CD} = -\kappa_{ab}{}^{DC}. \tag{55}$$

Invariant D operator

In the scale g_{ab},

$$\overset{g}{\kappa}_{ab}{}^{C}{}_{D} = \begin{pmatrix} 0 & 0 & 0 \\ -A_{ab}{}^{c} & W_{ab}{}^{c}{}_{d} & 0 \\ 0 & A_{abd} & 0 \end{pmatrix}, \tag{56}$$

where $A_{abc} := 2\nabla_{[a}P_{b]c}$ is the Cotton tensor. In particular,

$$\overset{g}{\kappa}_{ab}{}^{C}{}_{D} \equiv 0 \iff W_{ab}{}^{c}{}_{d} = 0.$$

Thomas-D operator Let $V_A \in \Gamma\left(\mathcal{T}\left[w\right]\right)$. The expression

$$D_A V_B \overset{g}{:=} \begin{pmatrix} \left(n + 2w - 2\right) wV \\ \left(n + 2w - 2\right) \nabla^{\mathcal{T}}_{a} V \\ -\left[\Delta^{g} + w\left(\mathrm{tr}\ P^{g}\right)\right] V \end{pmatrix} \tag{57}$$

transforms as an element of $\Gamma\left(\mathcal{E}_{AB}[w]\right)$.

Scale tractor and almost Einstein equation

Let

$$I_\sigma := \frac{1}{n} D\sigma \tag{58}$$

be the scale tractor associated to the scale σ. The AE eq. reads

$$\nabla_a^{\mathcal{T}} I_\sigma = 0, \tag{59}$$

i.e. conformal to Einstein – there exists a parallel tractor I_σ. How to recover scale σ from I_σ?

Tractor projectors Let

$$X^A : \mathcal{E}[-1] \to \mathcal{E}^A, \quad Z^{Aa} : \mathcal{E}_a[1] \to \mathcal{E}^A, \quad Y^A : \mathcal{E}[-1] \to \mathcal{E}^A \tag{60}$$

such that any tractor U^A can be decomposed as

$$U^A = Y^A \sigma + Z^{Aa} \mu_a + X^a \rho, \tag{61}$$

e.g. the tractor metric $h_{AB} = 2X_{(A} Y_{B)} + Z_A{}^c Z_{Bc}$ and

$$\sigma = X_A I_\sigma^A. \tag{62}$$

The I_σ^2 and scalar curvature

The I_σ^2 It can be checked that

$$h\left(I_\sigma, I_\sigma\right) \stackrel{g}{=} \mathbf{g}^{ab}\nabla_a^g\sigma\nabla_b^g\sigma - \frac{2}{n}\sigma\left(\Delta^g + \mathrm{tr}P^g\right)\sigma \tag{63}$$

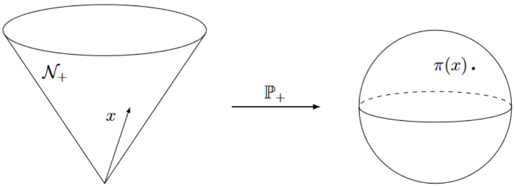
In particular for $g_\sigma = \sigma^{-2}\mathbf{g}$

$$h\left(I_\sigma, I_\sigma\right) \stackrel{g_\sigma}{=} -\frac{1}{n\left(n-1\right)}R^{g_\sigma} \tag{64}$$

i.e. I^2 is a conformally covariant notion of the scalar curvature.

Simplest model – conformal geometry of S^2

Consider Minkowski spacetime with a metric η_{ab} and the set of its null vectors. They form a *null cone* $\mathcal{N}$. Let $\mathcal{N}_+$ be a part of $\mathcal{N} \setminus \{0\}$ generated by the future-pointing null vectors:



Let $\mathbb{P}_+$ be a map which takes $x \in \mathcal{N}_+$ to its equivalence class with the following equivalence relation,

$$x \sim x' \quad \text{iff} \quad x' = \alpha x \quad \text{for} \quad \alpha > 0. \tag{65}$$

Then

$$\mathbb{P}_+(\mathcal{N}_+) \equiv S^2. \tag{66}$$

Example – conformal sphere

Let π be the submersion,

$$\pi : \mathcal{N}_+ \to S^2. \tag{67}$$

Each $x \in \mathcal{N}_+$ determines a positive metric g_x on S^2 via

$$g_x(y, z) = \eta(y', z'), \quad y := \pi(y'), \quad z := \pi(z'), \tag{68}$$

which is independent of the choices of lifts y' and z'.

The conformal metric $\mathbf{g}_{ab}$ of S^2 can be defined as restriction of η_{ab} to the vector fields in $T\mathcal{N}_+$ which are lifts of the vector fields from S^2.

Alternative 'definition' of $\mathbf{g}_{ab}$ – Minkowski metric in spherical co-ordinates

$$\eta = -dt^2 + dr^2 + r^2 g_{S^2} \tag{69}$$

Example – conformal sphere

Introduce null coordinates u, v,

$$u := t + r, \quad v := t - r \tag{70}$$

Then

$$\eta = -dudv + \left(\frac{u-v}{2}\right)^2 g_{S^2}. \tag{71}$$

Metric on a null cone ($v = 0$) is degenerate with signature $(0, +, +)$. Restriction to the vectors tangent to S^2 reads

$$\mathbf{g} = \frac{u^2}{4} g_{S^2}, \tag{72}$$

e.g. $u = 2$ corresponds to the standard round sphere metric.

Tractor bundle

Equivalence relation on $T\mathcal{N}_+$ coming from equivalence relation on $\mathcal{N}_+$:

$$U_x \sim V_y \quad \text{iff} \quad x, y \in \pi^{-1}(x'), x' \in S^2 \quad \text{and} \quad U_x \text{ and } V_y \text{ parallel}$$

Then $T\mathcal{N}_+/\sim$ is the *standard tractor bundle* of (S^2, c).