Characterizations of smooth projective horospherical varieties of Picard number one II

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Main Theorem

Main Theorem (Hwang-Li, H.-Kim)

Let X be a smooth horospherical variety of Picard number one, and o be a general point in X.

Let M be a uniruled projective manifold of Picard number one with a family $\mathcal{K}$ of minimal rational curves. Assume that $(\mathcal{C}_x(M) \subset \mathbb{P}(T_xM))$ is projectively equivalent to $(\mathcal{C}_o(X) \subset \mathbb{P}(T_oX))$ for general $x \in M$.

Then M is biholomorphic to X.

(Hwang-Li)

$$(C_n, \varpi_k, \varpi_{k-1}),\ (G_2, \varpi_2, \varpi_1)$$

(H.-Kim)

$$(B_n, \varpi_{n-1}, \varpi_n)(n \geq 3),\ (B_3, \varpi_1, \varpi_3),\ (F_4, \varpi_2, \varpi_3)$$

Smooth horospherical varieties of Picard number one

(Kim)

Let X be a smooth non-homogeneous horospherical varieties of Picard number one. Let $\mathfrak{g} := Lie\,\mathrm{Aut}(X) = (\mathfrak{l} \oplus \mathbb{C}) \ltimes U$.

Then there is a grading on $\mathfrak{l}$ and U,

$$\mathfrak{l} = \bigoplus_{k=-\mu}^{\mu} \mathfrak{l}_k \quad \text{and} \quad U = \bigoplus_{k=-1}^{\nu} U_k,$$

such that, with the grading on $\mathfrak{g} = (\mathfrak{l} \oplus \mathbb{C}) \ltimes U$ defined by

$$\begin{aligned} \mathfrak{g}_0 &:= (\mathfrak{l}_0 \oplus \mathbb{C}) \rhd U_0 \\ \mathfrak{g}_p &:= \mathfrak{l}_p \oplus U_p \text{ for } p \neq 0, \end{aligned}$$

the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$ is identified with the tangent space of X at the base point o of X^0.

(Kim)

Furthermore,

- $\mathfrak{g}$ is the prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$;
- the variety of minimal rational tangents $\mathcal{C}_o(X)$ of X at the base point o is given by

$$\mathcal{C}_o(X) = \begin{cases} \mathsf{H}_{L_0}(U_{-1} \oplus \mathfrak{l}_{-1} \oplus \mathfrak{l}_{-2}) \text{ if } X \text{ is } (C_m, \alpha_{i+1}, \alpha_i) \text{ for } 1 \leq i < m \\ \mathsf{H}_{L_0}(U_{-1} \oplus \mathfrak{l}_{-1}), \text{ otherwise.} \end{cases}$$

cf. When $X = G/P$

$$\mathcal{C}_o(X) = \begin{cases} \mathsf{H}_{G_0}(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}) & \text{if } P \text{ is associated to a short root} \\ \mathsf{H}_{G_0}(\mathfrak{g}_{-1}) & \text{if } P \text{ is associated to a long root} \end{cases}$$

Example

$\mathcal{C}_0(X) \subset \mathbb{P}(D_o) \subset \mathbb{P}(T_oX)$ and $[\,,\,] : \wedge^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$

- $(G_2, \varpi_2, \varpi_1)$

$$\mathbb{P}\{cv + v^3 : c \in \mathbb{C}, v \in V\} \subset \mathbb{P}(V \oplus \mathrm{Sym}^3 V) = \mathbb{P}(\mathfrak{g}_{-1})$$

where $\dim V = 2$ and $L_0 = A_1$.

$$\begin{aligned} \wedge^2\mathfrak{g}_{-1} &= \wedge^2 V \oplus (V \wedge \mathrm{Sym}^3 V) \oplus (\wedge^2(\mathrm{Sym}^3 V)) \\ &= \wedge^2 V \oplus (V \wedge \mathrm{Sym}^3 V) \oplus (\mathrm{Sym}^4 V \oplus \mathbb{C}) \to \mathfrak{g}_{-2} = \mathbb{C} \end{aligned}$$

Geometric structures

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- $(B_n, \varpi_{n-1}, \varpi_n)$

  $\mathbb{P}\{cv + v^2 \otimes w : c \in \mathbb{C}, v \in V, w \in W\} \subset \mathbb{P}(V \oplus (\mathrm{Sym}^2 V \otimes W)) = \mathbb{P}(\mathfrak{g}$

  where $\dim V = 2$ and $\dim W = m - 1$ and $L_0 = A_1 \times A_{m-2}$.

$\mathfrak{g}_- = \mathfrak{g}_{-1} + \mathfrak{g}_{-2}$

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$$\begin{aligned} & \wedge^2(\mathrm{Sym}^2 V \otimes W) \\ = \ & (\wedge^2(\mathrm{Sym}^2 V) \otimes \mathrm{Sym}^2 W) \oplus (\mathrm{Sym}^2(\mathrm{Sym}^2 V) \otimes \wedge^2 W) \\ = \ & (\wedge^2(\mathrm{Sym}^2 V) \otimes \mathrm{Sym}^2 W) \oplus ((\mathrm{Sym}^4 V \oplus (\mathrm{Sym}^4 V)^\perp) \otimes \wedge^2 W) \\ & \to (\mathrm{Sym}^4 V)^\perp \otimes \wedge^2 W = \mathfrak{g}_{-2} \end{aligned}$$

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($\dim V = 3$ and $\dim W = 2$ and $L_0 = A_2 \times A_1$, respectively)
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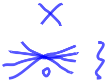
Geometric structures

Let X be a smooth horospherical variety of Picard number one, one of the following types **(model)**

- $(B_n, \varpi_{n-1}, \varpi_n)$ $(n \geq 3)$
- $(B_3, \alpha_1, \alpha_3)$
- $(F_4, \varpi_2, \varpi_3)$



and o be a general point.

Let M be a uniruled projective manifold with a minimal rational component $\mathcal{K}$. Assume that $(\mathcal{C}_x(M) \subset \mathbb{P}(T_x M))$ is projectively equivalent to $(\mathcal{C}_o(X) \subset \mathbb{P}(T_o X))$ for general $x \in M$.

Geometric structures

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$$\begin{array}{ccc} \mathcal{C}(X) \subset \mathbb{P}(TX) & & \mathcal{C}(M) \subset \mathbb{P}(TM) \\ \downarrow & & \downarrow \\ X & & M \end{array}$$

$$(\mathcal{C}_x(M) \subset \mathbb{P}(T_xM)) \overset{\text{proj.equiv.}}{\simeq} (\mathcal{C}_o(X) \subset \mathbb{P}(T_oX)) \text{ for general } x \in M$$

Geometric structures

What we have:

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Want to show: There exists connected open subsets $U \subset X$ and $V \subset M$ and a biholomorphism

$$\varphi : U \to V$$

whose differential $\varphi_* : \mathbb{P}(T_x X) \to \mathbb{P}(T_{\varphi(x)} M)$ sends $\mathcal{C}_x(X)$ to $\mathcal{C}_x(M)$ for all generic $x \in U$.

Geometric structures

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(Local equivalence problem)

Proof when X is G/P associated to a long root

Step. 1 There is a one-to-one correspondence

$$\{\mathbf{S}\text{-structures }\} \leftrightarrow \{G_0\text{-structures }\}$$

such that $\mathcal{S}_1 \overset{\text{equiv.}}{\simeq} \mathcal{S}_2$ if and only if $\mathscr{P}_1^{(0)} \overset{\text{equiv.}}{\simeq} \mathscr{P}_2^{(0)}$.

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Step. 2 (Tanaka) There is a one-to-one correspondence

$$\{G_0\text{-structures }\} \leftrightarrow \{\text{ Cartan connections of type } G/P\}$$

such that $\mathscr{P}_1^{(0)} \overset{\text{equiv.}}{\simeq} \mathscr{P}_2^{(0)}$ if and only if $\mathscr{P}_1 \overset{\text{equiv.}}{\simeq} \mathscr{P}_2$.

Furthermore, if any section of $\mathcal{H}^2_k(\mathscr{P}^{(0)}) := \mathscr{P}^{(0)} \times_{G_0} H^2(\mathfrak{g}_-, \mathfrak{g})_k$ vanishes for all $k \geq 1$, then $\mathscr{P}^{(0)} \to M$ is flat.

Proof when X is G/P associated to a long root

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Step. 3 Any section of $\mathcal{H}^2_k(\mathscr{P}^{(0)}) \to M$ vanishes for all $k \geq 1$.

Geometric structures

$o \in X = G/P$

$\mathfrak{g} = \oplus_i \mathfrak{g}_i$, the Lie algebra of $G = \mathrm{Aut}^0(X)$ with a grading

$\mathfrak{g}_- = \oplus_{i<0} \mathfrak{g}_i$ identified with $T_o X$

Geometric structures

$o \in X = G/P$ ($o \in X^0 = G/G^0 \subset X$ when X is horospherical)

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$\mathfrak{g}_- = \oplus_{i<0} \mathfrak{g}_i$ identified with $T_o X$

Denote by $H^2(\mathfrak{g}_-, \mathfrak{g})$ the Lie algebra cohomology of the $\mathfrak{g}_-$-module $\mathfrak{g}$, the cohomology of the following complex:

$$0 \to \mathfrak{g} \xrightarrow{\partial} C^1(\mathfrak{g}_-, \mathfrak{g}) := \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}) \xrightarrow{\partial} C^2(\mathfrak{g}_-, \mathfrak{g}) := \mathrm{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g}) \xrightarrow{\partial} \dots .$$

Denote by $H^2(\mathfrak{g}_-, \mathfrak{g})_k$ the cohomology of the following restriction:

$$0 \to \mathfrak{g}_k \xrightarrow{\partial} C^1(\mathfrak{g}_-, \mathfrak{g})_k \xrightarrow{\partial} C^2(\mathfrak{g}_-, \mathfrak{g})_k \xrightarrow{\partial} C^3(\mathfrak{g}_-, \mathfrak{g})_k \xrightarrow{\partial} \dots ,$$

where $C^1(\mathfrak{g}_-, \mathfrak{g})_k := \{\phi \in C^1(\mathfrak{g}_-, \mathfrak{g}) : \phi(\mathfrak{g}_p) \subset \mathfrak{g}_{p+k}\}$,
$C^2(\mathfrak{g}_-, \mathfrak{g})_k := \{\phi \in C^2(\mathfrak{g}_-, \mathfrak{g}) : \phi(\mathfrak{g}_{p_1} \wedge \mathfrak{g}_{p_2}) \subset \mathfrak{g}_{p_1+p_2+k}\}$, etc.

Geometric structures

$\mathbf{S} := \mathcal{C}_o(X) \subset \mathbb{P}(\mathfrak{g}_{-1})$ the variety of mrt of X at the base point o

$\mathfrak{g}_0 \subset \mathfrak{g}_0(\mathfrak{g}_-) \rightsquigarrow G_0 \subset G_0(\mathfrak{g}_-)$ where $G_0(\mathfrak{g}_-)$ is the automorphism group of the graded Lie algebra $\mathfrak{g}_-$

A *distribution* D on a complex manifold M is a subbundle of the tangent bundle TM of M. A distribution D is called *of type* $\mathfrak{g}_-$ if for each $x \in M$ the symbol algebra $\mathrm{Symb}_x(D)$ is isomorphic to $\mathfrak{g}_-$ as a graded Lie algebra. In this case, the pair (M, D) is called a *filtered manifold of type* $\mathfrak{g}_-$.

For each $x \in M$, let $\mathscr{R}_x$ be the set of all isomorphisms $r\colon \mathfrak{g}_- \to \mathrm{Symb}_x(D)$ of graded Lie algebras. Then $\mathscr{R}(M) := \cup_{x\in M}\mathscr{R}_x$ is a principal $G_0(\mathfrak{g}_-)$-bundle on M. We call $\mathscr{R}(M)$ the *frame bundle* of (M, D).

Geometric structures

$$G_0\text{-structures } \mathscr{P}^{(0)} \subset \mathscr{R}(M) \quad \leftrightarrow \quad \mathbf{S}\text{-structures } \mathcal{S} \subset \mathbb{P}(D)$$

Given a closed subgroup $G_0 \subset G_0(\mathfrak{g}_-)$, a G_0-structure on (M, D) is a G_0-subbundle $\mathscr{P}^{(0)}$ of the frame bundle $\mathscr{R}(M)$ of (M, D).
Two G_0-structures $\mathscr{P}_1^{(0)}$ on (M_1, D_1) and $\mathscr{P}_2^{(0)}$ on (M_2, D_2) are *equivalent* if there is a biholomorphism $\varphi : M_1 \to M_2$ such that $d\varphi : TM_1 \to TM_2$ induces an isomorphism from $\mathscr{P}_1^{(0)}$ onto $\mathscr{P}_2^{(0)}$.

A fiber subbundle $\mathcal{S} \subset \mathbb{P}D$ is called an $\mathbf{S}$-structure on (M, D) if for each $x \in M$, the fiber $\mathcal{S}_x \subset \mathbb{P}D_x$ is isomorphic to $\mathbf{S} \subset \mathbb{P}\mathfrak{g}_{-1}$ under a graded Lie algebra isomorphism $\mathfrak{g}_- \to \mathrm{Symb}_x(D)$.
Two $\mathbf{S}$-structures $\mathcal{S}_1$ on (M_1, D_1) and $\mathcal{S}_2$ on (M_2, D_2) are said to be *equivalent* if there exists a biholomorphism $\phi\colon M_1 \to M_2$ such that $d\phi\colon \mathbb{P}TM_1 \to \mathbb{P}TM_2$ sends $\mathcal{S}_1 \subset \mathbb{P}TM_1$ to $\mathcal{S}_2 \subset \mathbb{P}TM_2$.

Proof when X is $(G_2, \varpi_2, \varpi_1)$

Step. 1 There is a one-to-one correspondence

$$\{\mathbf{S}\text{-structures }\} \leftrightarrow \{G_0\text{-structures }\}$$

such that $\mathcal{S}_1 \overset{\text{equiv.}}{\simeq} \mathcal{S}_2$ if and only if $\mathscr{P}_1^{(0)} \overset{\text{equiv.}}{\simeq} \mathscr{P}_2^{(0)}$.

Step. 2 (Hwang-Li) If any section of

$$C^2(\mathscr{P}^{(0)})_k/\partial C^1(\mathscr{P}^{(0)})_k := \mathscr{P}^{(0)} \times_{G_0} (C^2(\mathfrak{g}_-, \mathfrak{g})_k/\partial C^1(\mathfrak{g}_-, \mathfrak{g})_k)$$

vanishes for all $k \geq 1$, then there exists a Cartan connection of type G/G^0 which is flat so that $\mathscr{P}^{(0)} \to M$ is flat.

Step. 3 Any section of $C^2(\mathscr{P}^{(0)})_k/\partial C^1(\mathscr{P}^{(0)})_k \to M$ vanishes for all $k \geq 1$.

Proof of Main Theorem

Step. 1 There is a one-to-one correspondence

$$\{\mathbf{S}\text{-structures }\} \leftrightarrow \{G_0\text{-structures }\}$$

such that $\mathcal{S}_1 \overset{\text{equiv.}}{\simeq} \mathcal{S}_2$ if and only if $\mathscr{P}_1^{(0)} \overset{\text{equiv.}}{\simeq} \mathscr{P}_2^{(0)}$.

Step. 2 (H.-Morimoto) There is a one-to-one correspondence

$$\{G_0\text{-structures }\} \leftrightarrow \{\text{ step prolongations of type } (G_0, G_1, \ldots, G_\nu)\}$$

such that $\mathscr{P}_1^{(0)} \overset{\text{equiv.}}{\simeq} \mathscr{P}_2^{(0)}$ if and only if $\mathscr{P}_1 \overset{\text{equiv.}}{\simeq} \mathscr{P}_2$.

Furthermore, if any section of $\mathcal{H}^2_k(\mathscr{P}^{(0)}) := \mathscr{P}^{(0)} \times_{G_0} H^2(\mathfrak{g}_-, \mathfrak{g})_k$ vanishes for all $k \geq 1$, then the corresponding $\mathscr{P} \to M$ is a Cartan connection of type G/G^0 which is flat so that $\mathscr{P}^{(0)} \to M$ is flat.

Step. 3 Any section of $\mathcal{H}^2_k(\mathscr{P}^{(0)}) \to M$ vanishes for all $k \geq 1$.

Geometric structures

Fix a set of subspaces $W = \{W^1_\ell, W^2_{\ell+1}\}_{\ell \geq 0}$ such that

$$\begin{aligned} \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g})_\ell &= W^1_\ell \oplus \partial \mathfrak{g}_\ell \\ \mathrm{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g})_{\ell+1} &= W^2_{\ell+1} \oplus \partial\, \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g})_{\ell+1}. \end{aligned}$$

(H.-Morimoto)

Let $\mathscr{P}^{(0)}$ be a G_0-structure on a filtered manifold M of type $\mathfrak{g}_-$. Then for each $\ell \geq 1$, there is a step prolongation

$$\mathscr{S}^{(\ell)}_W \mathscr{P}^{(0)} \xrightarrow{G_\ell} \mathscr{S}^{(\ell-1)}_W \mathscr{P}^{(0)} \longrightarrow \cdots \longrightarrow \mathscr{S}^{(1)}_W \mathscr{P}^{(0)} \xrightarrow{G_1} \mathscr{P}^{(0)} \xrightarrow{G_0} M$$

of type $(\mathfrak{g}_-, G_0, \cdots, G_\ell)$.

(We call the limit $\mathscr{S}_W \mathscr{P} = \lim_\ell \mathscr{S}^{(\ell)}_W \mathscr{P}$ the W-*normal complete step prolongation of* $\mathscr{P}$.)

Geometric structures

If, furthermore, $H^0(M, \mathcal{H}^0_k(\mathscr{P}^{(0)}))$ is zero for all $k \geq 1$, then the W-normal complete step prolongation $\mathscr{S}_W \mathscr{P}^{(0)}$ of $\mathscr{P}^{(0)}$ is a Cartan connection of type G/G^0 which is flat, and $\mathscr{P}^{(0)}$ is locally isomorphic to the standard G_0-structure on G/G^0.

Geometric structures

Proof of Main Theorem (Step.3)

Recall

Let X be a smooth horospherical variety of Picard number one, one of the following type

- $(B_n, \varpi_{n-1}, \varpi_n)$ $(n \geq 3)$
- $(B_3, \alpha_1, \alpha_3)$
- $(F_4, \varpi_2, \varpi_3)$

and o be a general point in X.
Let M be a uniruled projective manifold of Picard number one with a family $\mathcal{K}$ of minimal rational curves. Assume that

$$(\mathcal{C}_x(M) \subset \mathbb{P}(T_xM)) \overset{\text{proj.equiv.}}{\simeq} (\mathcal{C}_o(X) \subset \mathbb{P}(T_oX)) \text{ for general } x \in M.$$

Want to show: M is biholomorphic to X.

Proof of Main Theorem (Step.3)

Enough to show: There is a connected open subset $M^0 \subset M$ such that $\mathcal{C}(M)|_{M^0}$ defines an $\mathbf{S}$-structure on M^0, and for the corresponding G_0-structure $\mathscr{P}^{(0)}$ on M^0,

$$H^0(M^0, \mathcal{H}^2_k(\mathscr{P}^{(0)})) = 0 \text{ for all } k \geq 1.$$

$0 \to \mathfrak{g} \xrightarrow{\partial_0} Hom(\mathfrak{l}_-, \mathfrak{g}) \xrightarrow{\partial_0} Hom(\wedge^2 \mathfrak{l}_-, \mathfrak{g}) \xrightarrow{\partial_0} \cdots$

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Step 3-1 Lie algebra cohomology.

1. If $(\mathfrak{g}_-, \mathfrak{g}_0)$ is of type $(B_m, \alpha_{m-1}, \alpha_m)$, where $m > 3$, then $H^2(\mathfrak{g}_-, \mathfrak{g})_k$ vanishes except for $k = 1, 2$, and

$\mathfrak{g}_- = \mathfrak{l}_- + \mathfrak{u}_-$

$$\begin{aligned} H^2(\mathfrak{g}_-, \mathfrak{g})_1 &\subset \wedge^2 \mathfrak{g}_{-1}^* \otimes U_{-1} \\ H^2(\mathfrak{g}_-, \mathfrak{g})_2 &\subset \wedge^2 \mathfrak{g}_{-1}^* \otimes U_0 \subset (\wedge^2 \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1})^* \otimes U_{-1}. \end{aligned}$$

$\wedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g} >$

2. If $(\mathfrak{g}_-, \mathfrak{g}_0)$ is of type $(F_4, \alpha_2, \alpha_3)$, then $H^2(\mathfrak{g}_-, \mathfrak{g})_k$ vanishes except for $k = 1$, and

$$H^2(\mathfrak{g}_-, \mathfrak{g})_1 \subset \wedge^2 \mathfrak{g}_{-1}^* \otimes U_{-1}$$

Remark. $\mathfrak{g}$ is not reductive, so that we cannot apply Kostant's theory directly.

$\mathfrak{g} = \mathfrak{u} + \mathfrak{c} + \mathfrak{l}$ $\phi = \phi_{\mathfrak{u}} + \phi_{\mathfrak{l}+\mathfrak{c}}$

Step 3-1 Lie algebra cohomology.

$\mathfrak{g}_- = \mathfrak{u}_- + \mathfrak{l}_- \ni X = X^1 + X^0$

$Y = Y^1 + Y^0$

$Z = Z^1 + Z^0$

$\phi \in \mathrm{Hom}(\wedge^2 S_-, \mathfrak{g})_K \quad \partial\phi = 0$

$0 = \partial\phi(X, Y, Z) = [X, \phi(Y, Z)] - [Y, \phi(X, Z)] + [Z, \phi(X, Y)]$
$- \phi([X, Y], Z) + \phi([X, Z], Y) - \phi([Y, Z], X)$

(I) $\mathfrak{l}$: $[X^0, \phi_{\mathfrak{l}+\mathfrak{c}}(Y^1 + Y^0, Z^1 + Z^0)] - [\cdots$
$- \phi_{\mathfrak{l}+\mathfrak{c}}([X^0, Y^0], Z^1 + Z^0] + \cdots$

(II) $\mathfrak{u}$

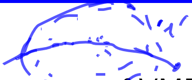
$X^1 = Y^1 = Z^1 = 0.$

$\partial_0(\phi|_{\mathfrak{l}_-}) = 0$

$H^1(\mathfrak{l}_-, \mathfrak{g})_K =$
$H^2(\mathfrak{l}_-, \mathfrak{g})_K = 0$

$\phi|_{\mathfrak{l}_-} = \partial\eta \ \cdots$

Proof of Main Theorem (Step.3)



Step 3-2 Parallel transport of VMRT.

Let $\mathbf{S} \subset \mathbb{P}(\mathfrak{g}_{-1})$ be the variety of minimal rational tangents of $(B_m, \alpha_{m-1}, \alpha_m)$, $m \geq 3$ or $(F_4, \alpha_2, \alpha_3)$ at the base point.
Let $\pi : \mathbb{P}E \to \mathbb{P}^1$ be the projectivization of a holomorphic vector bundle E over $\mathbb{P}^1$ and let $\mathcal{C} \subset \mathbb{P}E$ be an irreducible subvariety. Denote by ϖ the restriction of π to $\mathcal{C}$. Assume that

1. $\mathcal{C}_t := \varpi^{-1}(t) \subset \mathbb{P}E_t := \pi^{-1}(t)$ is projectively equivalent to $\mathbf{S} \subset \mathbb{P}(\mathfrak{g}_{-1})$ for all $t \in \mathbb{P}^1 - \{t_1, \ldots, t_k\}$;
2. for a general section $\sigma \subset \mathcal{C}$ of ϖ,
   the relative second fundamental form and
   the relative third fundamental form of $\mathcal{C}$ along σ are constants.

Then for any $t \in \mathbb{P}^1$, $\mathcal{C}_t \subset \mathbb{P}(E_t)$ is projectively equivalent to $\mathbf{S} \subset \mathbb{P}(\mathfrak{g}_{-1})$.

Proof of Main Theorem (Step.3)

Step 3-3 $H^0(M^0, \mathcal{H}^2_k(\mathscr{P}^{(0)})|_{M^0}) = 0$ for all $k \geq 1$

Proof. By **Step 3-1** it suffices to show

1. $H^0(M^0, \wedge^2 D^* \otimes \mathcal{U}_{-1}) = 0$ and $H^0(M^0, (\wedge^2 D \otimes D)^* \otimes \mathcal{U}_{-1}) = 0$ when $(\mathfrak{m}, \mathfrak{g}_0)$ is of type $(B_m, \alpha_{m-1}, \alpha_m)$;
2. $H^0(M^0, \wedge^2 D^* \otimes \mathcal{U}_{-1}) = 0$ when $(\mathfrak{m}, \mathfrak{g}_0)$ is of type $(F_4, \alpha_2, \alpha_3)$.

Claim: For $C \subset M^0 \subset M$,
$D|_C, \mathcal{U}_{-1}|_C$ have the same splitting type as those in X.
(Use the projective geometry of $(\mathcal{C}_x(M) \subset \mathbb{P}(T_x M))$.)

Then, the existence of a nontrivial bundle map $\varphi : \wedge^2 D \to \mathcal{V}$ gives rise to a contradiction to the irreducibility of $\mathcal{U}_{-1}$. □

Main Theorem (H.-Kim)

Let X be a smooth horospherical variety of Picard number one of the following type

- $(B_n, \varpi_{n-1}, \varpi_n)$ $(n \geq 3)$
- $(B_3, \alpha_1, \alpha_3)$
- $(F_4, \varpi_2, \varpi_3)$

and o be a general point in X.

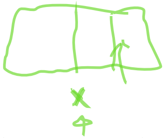
Let M be a uniruled projective manifold of Picard number one with a family $\mathcal{K}$ of minimal rational curves. Assume that

$$(\mathcal{C}_x(M) \subset \mathbb{P}(T_x M)) \overset{\text{proj.equiv.}}{\simeq} (\mathcal{C}_o(X) \subset \mathbb{P}(T_o X)) \text{ for general } x \in M.$$

Then M is biholomorphic to X.

Applications

X, smooth horospherical variety of Picard number one



Fact Pasquier - Perrin

$H^1(X, TX) = 0$, i.e., X is locally rigid except for $X = (G_2, \varpi_2, \varpi_1)$.

Question

Let $\pi : \mathfrak{X} \to \Delta$ be a smooth and projective morphism from a complex manifold $\mathfrak{X}$ to the unit disc Δ. Assume that for any $t \in \Delta - \{0\}$, $X_t := \pi^{-1}(t)$ is biholomorphic to X, then is $X_0 := \pi^{-1}(0)$ also biholomorphic to X?

Assume $X_t := \pi^{-1}(t)$ is biholomorphic to X for any $t \in \Delta - \{0\}$

Show: there is a minimal rational component $\mathcal{K}_0$ on X_0 such that $\mathcal{C}_x(X_0) \subset \mathbb{P}(T_x X_0)$ is projectively equivalent to $\mathcal{C}_o(X) \subset \mathbb{P}(T_o X)$

Then by our characterization theorem, X_0 is biholomorphic to X.

Thank you!