Characterizations of smooth projective horospherical varieties of Picard number one I

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Joint work with T. Morimoto, S.-Y. Kim

Geometric structures

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It can be

- a G-reduction $\mathscr{P}$ of the frame bundle $\mathscr{F}(M)$,
  where $\mathscr{F}(M) = \cup_{x \in M}\{z : \mathbb{C}^n \to T_xM,$ linear isomorphism $\}$ and G is a subgroup of $GL(n, \mathbb{C})$;
- a subbundle $\mathcal{S}$ of $\mathbb{P}(TM)$ with typical fiber $\mathbf{S} \subset \mathbb{P}(V)$ where V is a vector space with $\dim V = \dim M$;

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- a G_0-subbundle $\mathscr{P}$ of the frame bundle $\mathscr{R}(M, F)$ of a filtered manifold (M, F) of type $\mathfrak{g}_-$, where G_0 is a subgroup of $G_0(\mathfrak{g}_-)$;
- a subbundle $\mathcal{S}$ of $\mathbb{P}(F^{-1})$ with typical fiber $\mathbf{S} \subset \mathbb{P}(\mathfrak{g}_{-1})$ such that $\mathbf{S} \subset \mathbb{P}(\mathfrak{g}_{-1})$ is isomorphic to $\mathcal{S}_x \subset \mathbb{P}(F^{-1})$ under a graded Lie algebra isomorphism $\mathfrak{g}_- \to \operatorname{gr} F_x$, etc.

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Local equivalence problem

- Two geometric structures $\mathscr{P} \subset \mathscr{F}(M)$ and $\mathscr{Q} \subset \mathscr{F}(N)$ are locally equivalent if there is a local map $\varphi : U \subset M \to V \subset N$ such that the induced map $\mathscr{F}(\varphi) : \mathscr{F}(M)|_U \to \mathscr{F}(N)|_V$ sends $\mathscr{P}|_U$ to $\mathscr{Q}|_V$.
- Two geometric structures $\mathcal{S} \subset \mathbb{P}(TM)$ and $\mathcal{T} \subset \mathbb{P}(TN)$ are locally equivalent if there is a local map $\varphi : U \subset M \to V \subset N$ such that the induced map $d\varphi : \mathbb{P}(TM)|_U \to \mathbb{P}(TN)|_V$ sends $\mathcal{S}|_U$ to $\mathcal{T}|_V$.
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We will consider a geometric structure naturally defined on a projective manifold covered with rational curves and its local equivalence.

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In this talk, every vector space is a complex vector space and every manifold is a complex manifold, etc.

Varieties of minimal rational tangents

(X, L), a projective manifold with an ample ($=$ positive) line bundle L

A nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ is called a *rational curve*.
A rational curve $f : \mathbb{P}^1 \to X$ is said to be *free* if f^*TX is semipositive, i.e., $f^*TX = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$, all a_i are nonnegative.
(Then deformations of f cover an open dense subset of X)

A free rational curve $f : \mathbb{P}^1 \to X$ such that the degree of f^*L is minimal among all free rational curve is called a minimal rational curve.

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Example

$X = Gr(r, V)$, the Grassmannian of r-subspaces of a vector space V

Fix $W_1 \subset W_2 \subset V$ with $\dim W_1 = r - 1$ and $\dim W_2 = r + 1$.

Then
$C_{W_1,W_2} := \{[W] \in Gr(r, V) : W_1 \subset W \subset W_2\} \simeq \{W/W_1 \subset W_2/W_1\}$
is a line $\mathbb{P}^1$ contained in $Gr(r, V) \subset \mathbb{P}(\wedge^r V)$ in Plücker embedding.

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$\widehat{\mathcal{K}} \subset Hom(\mathbb{P}^1, X)$ a connected component containing a m.r.c.

$\widehat{\mathcal{K}}^0 = \{$ free and generically injective $\} \subset \widehat{\mathcal{K}}$

$\mathcal{K} := \widehat{\mathcal{K}}^0 / Aut(\mathbb{P}^1)$, called a *minimal rational component*

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is a line $\mathbb{P}^1$ contained in $Gr(r, V) \subset \mathbb{P}(\wedge^r V)$ in Plücker embedding, and any such line $\mathbb{P}^1$ is of this form.

Thus $\mathcal{K} = \{(W_1, W_2) : W_1 \subset W_2 \subset V, \dim W_1 = r - 1, \dim W_2 = r + 1\}$ is a minimal rational component of $Gr(r, V)$.

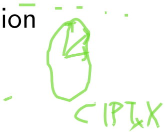
Varieties of minimal rational tangents

$x \in X$ a general point

$\mathcal{K}_x := \{$ minimal rational curves in $\mathcal{K}$ passing through $x\}$

Define $\Phi_x : \mathcal{K}_x \to \mathbb{P}(T_x X)$ by

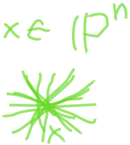
$$\Phi_x(C) = [T_x C] \in \mathbb{P}(T_x X) \quad \text{the tangents direction}$$



The image $\mathcal{C}_x := \overline{\Phi_x(\mathcal{K}_x)} \subset \mathbb{P}(T_x X)$ is called

the variety of minimal rational tangents of $(X, \mathcal{K})$ at x

$= \{$ tangent directions of minimal rational curves passing through $x\}$



Varieties of minimal rational tangents

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Any line $\mathbb{P}^1$ contained in $Gr(r, V) \subset \mathbb{P}(\wedge^r V)$ is of the form:
$C_{W_1,W_2} := \{[W] \in Gr(r, V) : W_1 \subset W \subset W_2\} \simeq \{W/W_1 \subset W_2/W_1\}$ for some $W_1 \subset W_2 \subset V$ with $\dim W_1 = r - 1$ and $\dim W_2 = r + 1$.
Thus $\mathcal{K} = \{(W_1, W_2) : W_1 \subset W_2 \subset V, \dim W_1 = r - 1, \dim W_2 = r + 1\}$ is a minimal rational component of $Gr(r, V)$.

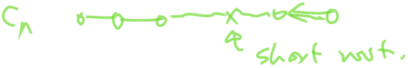
The tangent map is an embedding:

$$\Phi_{[W]} : \mathcal{K}_{[W]} \to \mathcal{C}_{[W]} \subset \mathbb{P}(TX) = \mathbb{P}(W^* \otimes (V/W)).$$

$\mathcal{K}_{[W]} = \{(W_1, W_2) : W_1 \subset W \subset W_2 \subset V\} \simeq \mathbb{P}(W^*) \times \mathbb{P}(V/W) = \mathcal{C}_{[W]}$.
the variety of minimal rational tangents of $(X, \mathcal{K})$ at $[W]$

Varieties of minimal rational tangents

For an irreducible L-representation space V let $\mathsf{H}_L(V)$ denote the highest weight orbit $\subset \mathbb{P}(V)$. More generally, for a finitely many irreducible L-representation spaces V_i $(i = 1, \ldots, r)$, let $\mathsf{H}_L(\oplus_{i=1}^r V_i)$ denote the closure of the L-orbit of the sum $\oplus_{i=1}^r v_i$ of highest weight vectors v_i of V_i in $\mathbb{P}(\oplus_{i=1}^r V_i)$.



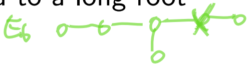
Example

When $X = G/P$ is a homogeneous variety associated with a maximal parabolic subgroup P, there is a grading $\mathfrak{g} = \bigoplus_{i=-\mu}^{\mu} \mathfrak{g}_i$ such that the tangent space T_oX at the base point $o \in X$ is the negative part $\bigoplus_{p<0} \mathfrak{g}_p$.

Then

$\overline{G_0(v_1 + v_2)}$

$$\mathcal{C}_o(X) = \begin{cases} \mathsf{H}_{G_0}(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}) & \text{if } P \text{ is associated to a short root} \\ \mathsf{H}_{G_0}(\mathfrak{g}_{-1}) & \text{if } P \text{ is associated to a long root} \end{cases}$$



where G_0 is the subgroup of G with Lie algebra $\mathfrak{g}_0$.

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$G/P \subset \mathbb{P}(V)$, G-equivariant embedding as the highest weight orbit
$\mathcal{K}(X) = \{\mathbb{P}^1\text{'s in } G/P \subset \mathbb{P}(V)\}$, a minimal rational component. Then

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Varieties of minimal rational tangents

(Hwang-Mok) K_X^{-1} is positive.

Let X be a Fano manifold of Picard number one. Suppose that there is a minimal rational component $\mathcal{H}$ with $p(\mathcal{H}), q(\mathcal{H}) > 0$ such that for a general point $x \in X$, the Gauss map for each irreducible component of $\mathcal{C}_x(X)$ at x as a subvariety of $\mathbb{P}(T_xX)$ is generically finite.

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Then for any choice of $(M, \mathcal{K})$ Fano manifold of Picard number one and a minimal rational component with $p(\mathcal{H}) = p(\mathcal{K})$,
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Then for any choice of $(M, \mathcal{K})$ Fano manifold of Picard number one and a minimal rational component with $p(\mathcal{H}) = p(\mathcal{K})$, any local biholomorphism $\varphi : U \to V$ where $U \subset X$ and $V \subset M$ are connected open subset, extends to a biholomorphic map $X \to M$

if the differential $\varphi_* : \mathbb{P}(T_xX) \to \mathbb{P}(T_{\varphi(x)}M)$ sends each irreducible component of $\mathcal{C}_x(X)$ to an irreducible component of $\mathcal{C}_{\varphi(x)}(M)$ for all generic $x \in U$.

Characterization

Question.

Let M be a uniruled projective manifold with a minimal rational component $\mathcal{K}$.
To what extent does the projective geometry of $\mathcal{C}_x(M) \subset \mathbb{P}(TM)$ for a general point $x \in M$ determine the biholomorphic geometry of M?

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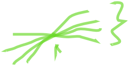

Let $X = G/P$ be a rational homogeneous variety of Picard number one associated to a long root and let o be a base point.
Let M be a uniruled projective manifold of Picard number one with a minimal rational component $\mathcal{K}$. Assume that $(\mathcal{C}_x(M) \subset \mathbb{P}(T_xM))$ is projectively equivalent to $(\mathcal{C}_o(X) \subset \mathbb{P}(T_oX))$ for general $x \in M$. Then M is biholomorphic to X.

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$X \supset X^{\circ} = G/H$ open dense

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Is M biholomorphic to X?

(Hwang-Li)

The answer is yes for

- (C_n, ϖ_k) Symplectic Grassmannian (to a short root)
- $(C_n, \varpi_k, \varpi_{k-1})$ Odd symplectic Grassmannian
- $(G_2, \varpi_2, \varpi_1)$

Characterization

Main Theorem (H.-Kim)

Let X be a smooth horospherical variety of Picard number one, one of the following types:

- $(B_n, \varpi_{n-1}, \varpi_n)$ $(n \geq 3)$
- $(B_3, \varpi_1, \varpi_3)$
- $(F_4, \varpi_2, \varpi_3)$

and o be a general point in X.

Let M be a uniruled projective manifold of Picard number one with a family $\mathcal{K}$ of minimal rational curves. Assume that $(\mathcal{C}_x(M) \subset \mathbb{P}(T_xM))$ is projectively equivalent to $(\mathcal{C}_o(X) \subset \mathbb{P}(T_oX))$ for general $x \in M$.

Then M is biholomorphic to X.

Smooth horospherical varieties of Picard number one

Let L be a simple algebraic group

$\{\varpi_1, \ldots, \varpi_n\}$ be a system of fundamental weights.

$v_{\varpi_i} \in V_{\varpi_i}$ highest weight vector in the irreducible representation,

P_{ϖ_i}, the isotropy subgroup of $[v_{\varpi_i}] \in \mathbb{P}(V_{\varpi_i})$.

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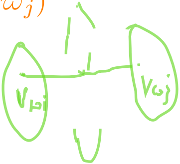
$X := \overline{L[v_{\varpi_i} \oplus v_{\varpi_j}]} \subset \mathbb{P}(V_{\varpi_i} \oplus V_{\varpi_j})$, denoted by (L, ϖ_i, ϖ_j)



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$Y := L.[v_{\varpi_i}]$, $Z := L.[v_{\varpi_j}]$, two closed orbits in X

$X^0 := X \backslash (Y \cup Z) \subset X$, a unique open orbit in X

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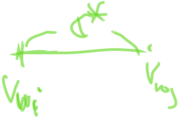

$$\begin{array}{ccc} X^0 = X \backslash (Y \cup Z) & \hookrightarrow & X \\ \downarrow \mathbb{C}^* & & \\ L/(P_{\varpi_i} \cap P_{\varpi_j}) & & \end{array}$$

Smooth horospherical varieties of Picard number one

For a reductive group L, a normal L-variety is said to be horospherical if it has an open L-orbit L/H whose isotropy group H contains the unipotent part of a Borel subgroup of L.

L/H
$\downarrow (\mathbb{C}^*)^r$
$L/N_L(H) = L/P$

(Pasquier)

Classification of non-homogeneous smooth horospherical varieties of Picard number one.

1. $(B_n, \varpi_{n-1}, \varpi_n)$ $(n \geq 3)$
2. $(B_3, \varpi_1, \varpi_3)$
3. $(C_n, \varpi_k, \varpi_{k-1})$ $(n \geq 2,\ 2 \leq k \leq n)$
4. $(F_4, \varpi_2, \varpi_3)$
5. $(G_2, \varpi_2, \varpi_1)$

Smooth horospherical varieties of Picard number one

- (C_n, ϖ_k) is a symplectic Grassmannian

$$Gr_\omega(k, \mathbb{C}^{2n}) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^{2n}).$$

- $(C_n, \varpi_k, \varpi_{k-1})$ is an odd symplectic Grassmannian

$$Gr_{\omega'}(k, \mathbb{C}^{2n+1}) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^{2n+1}).$$

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$$Gr_{\omega'}(k, \mathbb{C}^{2n+1}) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^{2n+1}).$$

$$\begin{array}{ccc} \mathbb{P}(\wedge^{k-1}\mathbb{C}^{2n} \oplus \wedge^k \mathbb{C}^{2n}) & \to & \mathbb{P}(\wedge^k \mathbb{C}^{2n+1}) \\ (e_1 \wedge \cdots \wedge e_{k-1},\ e_1 \wedge \ldots e_{k-1} \wedge e_k) & \mapsto & e_1 \wedge \cdots \wedge e_{k-1} \wedge e_0 \\ & & + e_1 \wedge \cdots \wedge e_{k-1} \wedge e_k \\ & & = e_1 \wedge \cdots \wedge e_{k-1} \wedge (e_0 + e_k) \end{array}$$

where $\{e_1, \ldots, e_{2n}\}$ is a basis of $(\mathbb{C}^{2n}, \omega)$ with a symplectic form ω and $\{e_0, e_1, \ldots, e_{2n}\}$ is a basis of $(\mathbb{C}^{2n+1}, \omega')$ with a skew-sym form of max rk.

Smooth horospherical varieties of Picard number one

- (F_4, ϖ_4) is a hyperplane section of $\mathbb{OP}^2 = (E_6, \varpi_6)$

- $(B_3, \varpi_1, \varpi_3)$ is a hyperplane section of the spin variety $\mathbf{S}_5 = (D_5, \varpi_5)$

Smooth horospherical varieties of Picard number one

X	$\mathrm{Aut}^0(X)$
1. $(B_n, \varpi_{n-1}, \varpi_n)$	$(SO(2n+1) \times \mathbb{C}^*) \ltimes V_{\varpi_n}$
2. $(B_3, \varpi_1, \varpi_3)$	$(SO(7) \times \mathbb{C}^*) \ltimes V_{\varpi_3}$
3. $(C_n, \varpi_k, \varpi_{k-1})$	$((Sp(2n) \times \mathbb{C}^*)/\{\pm 1\}) \ltimes V_{\varpi_1}$
4. $(F_4, \varpi_2, \varpi_3)$	$(F_4 \times \mathbb{C}^*) \ltimes V_{\varpi_4}$
5. $(G_2, \varpi_2, \varpi_1)$	$(G_2 \times \mathbb{C}^*) \ltimes V_{\varpi_1}$

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Remark. $G := \mathrm{Aut}^0(X)$ is not reductive.

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Remark. $G := \mathrm{Aut}^0(X)$ is not reductive.

Put $L :=$ the semisimple part of G
and $U :=$ the unipotent part of G.

so that

$$G = (L \times \mathbb{C}^*) \ltimes U$$

and the Lie algebra $\mathfrak{g}$ of G is $(\mathfrak{l} \oplus \mathbb{C}) \ltimes U$.

Smooth horospherical varieties of Picard number one

(Kim)

Let X be a smooth non-homogeneous horospherical varieties of Picard number one.

Then there is a grading on $\mathfrak{l}$ and U,

$$\mathfrak{l} = \bigoplus_{k=-\mu}^{\mu} \mathfrak{l}_k \quad \text{and} \quad U = \bigoplus_{k=-1}^{\nu} U_k,$$

such that the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$, where the grading is defined by

$$\begin{aligned} \mathfrak{g}_0 &:= (\mathfrak{l}_0 \oplus \mathbb{C}) \rhd U_0 \\ \mathfrak{g}_p &:= \mathfrak{l}_p \oplus U_p \text{ for } p \neq 0, \end{aligned}$$

is identified with the tangent space of X at the base point o of X^0.

(Kim)

Furthermore, the variety of minimal rational tangents $\mathcal{C}_o(X)$ of X at the base point o is given by

$$\mathcal{C}_o(X) = \begin{cases} \mathsf{H}_{L_0}(U_{-1} \oplus \mathfrak{l}_{-1} \oplus \mathfrak{l}_{-2}) \text{ if } X \text{ is } (C_m, \alpha_{i+1}, \alpha_i) \text{ for } 1 \leq i < m \\ \mathsf{H}_{L_0}(U_{-1} \oplus \mathfrak{l}_{-1}), \text{ otherwise.} \end{cases}$$

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cf. When $X = G/P$

$$\mathcal{C}_o(X) = \begin{cases} \mathsf{H}_{G_0}(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}) & \text{if } P \text{ is associated to a short root} \\ \mathsf{H}_{G_0}(\mathfrak{g}_{-1}) & \text{if } P \text{ is associated to a long root} \end{cases}$$

Main Theorem

Main Theorem (Hwang-Li, H.-Kim)

Let X be a smooth horospherical variety of Picard number one, and o be a general point in X.

Let M be a uniruled projective manifold of Picard number one with a family $\mathcal{K}$ of minimal rational curves. Assume that $(\mathcal{C}_x(M) \subset \mathbb{P}(T_x M))$ is projectively equivalent to $(\mathcal{C}_o(X) \subset \mathbb{P}(T_o X))$ for general $x \in M$.

Then M is biholomorphic to X.

(Hwang-Li)

$$(C_n, \varpi_k, \varpi_{k-1}),\ (G_2, \varpi_2, \varpi_1)$$

(H.-Kim)

$$(B_n, \varpi_{n-1}, \varpi_n)(n \geq 3),\ (B_3, \varpi_1, \varpi_3),\ (F_4, \varpi_2, \varpi_3)$$

Let X be a smooth horospherical variety of Picard number one, one of the following types **(model)**

- $(B_n, \varpi_{n-1}, \varpi_n)$ $(n \geq 3)$
- $(B_3, \alpha_1, \alpha_3)$
- $(F_4, \varpi_2, \varpi_3)$

and o be a general point.

Let M be a uniruled projective manifold with a family $\mathcal{K}$ of minimal rational curves. Assume that $(\mathcal{C}_x(M) \subset \mathbb{P}(T_xM))$ is projectively equivalent to $(\mathcal{C}_o(X) \subset \mathbb{P}(T_oX))$ for general $x \in M$.

Geometric structures

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$$\begin{array}{ccc} \mathcal{C}(X) \subset \mathbb{P}(TX) & \qquad & \mathcal{C}(M) \subset \mathbb{P}(TM) \\ \downarrow & & \downarrow \\ X & & M \end{array}$$

$$(\mathcal{C}_x(M) \subset \mathbb{P}(T_xM)) \overset{\text{proj.equiv.}}{\simeq} (\mathcal{C}_o(X) \subset \mathbb{P}(T_oX)) \text{ for general } x \in M$$

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What we have:

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Want to show: There exists connected open subsets $U \subset X$ and $V \subset M$ and a biholomorphism

$$\varphi : U \to V$$

whose differential $\varphi_* : \mathbb{P}(T_x X) \to \mathbb{P}(T_{\varphi(x)} M)$ sends $\mathcal{C}_x(X)$ to $\mathcal{C}_x(M)$ for all generic $x \in U$.

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(Local equivalence problem)

Local to Global

(Hwang-Mok) Cantan-Fubini type extension theorem

Let X be a Fano manifold of Picard number one. ......
Then for any choice of $(M, \mathcal{K})$ Fano manifold of Picard number one and a minimal rational component with $p(\mathcal{H}) = p(\mathcal{K})$,

if the differential $\varphi_* : \mathbb{P}(T_x X) \to \mathbb{P}(T_{\varphi(x)} M)$ sends each irreducible component of $\mathcal{C}_x(X)$ to an irreducible component of $\mathcal{C}_{\varphi(x)}(M)$ for all generic $x \in U$,

then any local biholomorphism $\varphi : U \to V$ where $U \subset X$ and $V \subset M$ are connected open subset, extends to a biholomorphic map $X \to M$.

This will complete the proof of Main Theorem. □

Thank you!