Classifying homogeneous geometric structures (Lecture 1)

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Recall that the title of our GRIEG project is:

SCREAM

= Symmetry, Curvature Reduction, and EquivAlence Methods

My lecture series will focus on Cartan-geometric approaches to classifying (locally) homogeneous geometric structures (with an emphasis on "parabolic" geometries in low dimensions).

Today:

- Examples of homogeneous geometric structures.
- Motivate the notion of a (normalized) Cartan geometry. ("Cartan equivalence method" details will be done elsewhere.)
- Tanaka prolongation and its relevance.

Examples of (homogeneous) geometric structures

Some examples & basic questions

Structure	Symmetry condition
Riemannian (M, g)	$\mathcal{L}_X g = 0$
Conformal $(M, [g])$	$\mathcal{L}_X g = \lambda(X) g$
$(2,3,5)$-distribution (M^5, D)	$\mathcal{L}_X D \subset D$
2nd order ODE $(M^3, D = E \oplus V)$	$\mathcal{L}_X E \subset E$, $\mathcal{L}_X V \subset V$

Symmetries form a Lie algebra $\mathfrak{f} \subset \mathfrak{X}(M)$. For a given structure:

1. What is the maximum $\mathfrak{M}$ of $\dim(\mathfrak{f})$? (Assume $\dim(M)$ fixed.)
2. What is the "submaximal" (next realizable) sym dim $\mathfrak{S}$?
3. How can one classify *(locally)* *homogeneous* structures?

(Want: $\forall p \in M$, $ev_p : \mathfrak{f} \to T_p M$ is surjective. We'll encode data on $\mathfrak{f}/\mathfrak{f}^0$, where $\mathfrak{f}^0 =$ isotropy subalg at a chosen $p \in M$.)

Example: Riemannian geometry (M, g)

- A symmetry ("Killing v.f.") $X \in \mathfrak{X}(M)$ satisfies $\mathcal{L}_X g = 0$.
  Locally, $g = g_{ab} dx^a \otimes dx^b$, this is a linear PDE in $X = X^a \partial_{x^a}$:
  $X^c \partial_{x^c} g_{ab} + (\partial_{x^a} X^c) g_{cb} + (\partial_{x^b} X^c) g_{ac} = 0$.
- sym. dim. $\leq \binom{n+1}{2}$. Sharp on constant curvature spaces:
  - $\mathbb{R}^n \cong \mathbb{E}(n)/\mathrm{O}(n)$.
  - $S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n)$.
  - $H^n \cong \mathrm{O}(n,1)/\mathrm{O}(n)$.
-

n	$\mathfrak{M}$	$\mathfrak{S}$	Citation
2	3	1	Darboux / Koenigs (∼1890)
3	6	4	Wang (1947)
4	10	8	Egorov (1955)
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2}+1$	Wang (1947), Egorov (1949)

Example: 2nd order ODE

$y'' = f(x, y, y')$. Point transformations (PT): $\begin{cases} \tilde{x} = \tilde{x}(x,y) \\ \tilde{y} = \tilde{y}(x,y) \end{cases}$.

Letting $p = y'$ and $q = y''$, we can prolong:

$$\tilde{p} = \frac{\tilde{y}_x + p\tilde{y}_y}{\tilde{x}_x + p\tilde{x}_y}, \quad \tilde{q} = \frac{\tilde{p}_x + p\tilde{p}_y + q\tilde{p}_p}{\tilde{x}_x + p\tilde{x}_y}.$$

Symmetries are v.f. on (x, y)-space whose prolongation to (x, y, p, q)-space are tangent to $q = f(x, y, p)$.

Structure	$\mathfrak{M}$	Example	$\mathfrak{S}$	Example
2nd order ODE	8	$y'' = 0$	3	$y'' = \exp(y')$

$y'' = 0$ has symmetry alg $\mathfrak{sl}_3$. $\mathfrak{S} = 3$ due to Tresse (1896).

Reformulation: Consider (x, y, p, q)-space equipped with $\langle dy - p dx,\ dp - q dx \rangle$. On $q = f(x, y, p)$, get a line field:

$$E = \langle \partial_x + p\partial_y + f\partial_p \rangle.$$

Wrt (prolonged) PT, $V = \langle \partial_p \rangle$ is also distinguished.

Geometric structure: Let M be (x, y, p)-space with a contact distribution $D = \ker\{dy - p dx\} = \langle \partial_x + p\partial_y, \partial_p \rangle \subset TM$. A 2nd order ODE $\leftrightarrow$ splitting $D = E \oplus V$. Note $[D, D] = TM$.

2nd order ODE and $\mathfrak{f} = \mathfrak{sl}_3$? Consider $\mathfrak{p} = \mathfrak{f}^0 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \subset \mathfrak{f}$.

Then $\mathfrak{f}/\mathfrak{f}^0$ admits an $\mathfrak{f}^0$-invariant filtration:

$$\mathfrak{f}^{-1}/\mathfrak{f}^0 = \begin{pmatrix} & & \\ * & & \\ 0 & * & \end{pmatrix} \quad \subset \quad \mathfrak{f}/\mathfrak{f}^0.$$

This corresponds to $\mathfrak{sl}_3$-invariant data $D = E \oplus V$ on $M = SL_3/P$.

Example: $(2,3,5)$-distributions

Consider $(M^5, D \subset TM)$ with rank growth $(2,3,5)$:

$$D \subset [D,D] \subset [D,[D,D]] = TM.$$

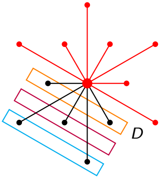
Monge form $z' = f(x,y,y',y'',z)$: In (x,y,p,q,z)-space, let

$$D = \ker\{dy - pdx,\ dp - qdx,\ dz - fdx\}$$
$$= span\{\partial_x + p\partial_y + q\partial_p + f\partial_z,\ \partial_q\}.$$

where $f = f(x,y,p,q,z)$ satisfies $f_{qq} \neq 0$.

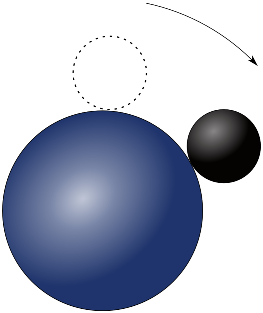
Structure	$\mathfrak{M}$	Example	$\mathfrak{S}$	Example
$(2,3,5)$-distribution	14	$z' = (y'')^2$	7	$z' = \exp(y'')$

$z' = (y'')^2$ has sym alg $Lie(G_2)$. $\mathfrak{S} = 7$ due to Cartan (1910).



Consider a 2-sphere rolling on another without twisting or slipping.

- Configuration space M is 5-dimensional.
- No twisting or slipping $\Rightarrow$ constraints on velocity space TM. Get rank 2 distribution $D \subset TM$ of allowable directions.



Let $\rho \geq 1$ be the ratio of the radii. If $\rho \neq 1$, get $(2,3,5)$-geometry.

- $\rho \neq 3$: $\mathfrak{so}(3) \times \mathfrak{so}(3)$ symmetry
- $\rho = 3$: (split) Lie(G_2) symmetry *(Bryant, Zelenko, Bor–Montgomery, Baez–Huerta)*

Riemannian metrics (on surfaces)

(Local) equivalence of Riemannian metrics

Q: Does $\exists \varphi : (M^n, g) \to (\tilde{M}^n, \tilde{g})$ s.t. $\varphi^* \tilde{g} = g$?

Locally diagonalize, e.g. $g = (\theta^1)^2 + ... + (\theta^n)^2$, get o.n. coframes $\{\theta^i\}$ and $\{\tilde{\theta}^i\}$. Reformulate Q as a Cartan equivalence problem:

Q: Does $\exists \varphi : M \to \tilde{M}$ s.t. $\varphi^* \tilde{\theta}^i = g^i_j(x)\theta^j$ for $g : M \to O(n)$?

KEY IDEA: Build a bundle $\mathcal{G}$ that incorporates ambiguity, e.g. for metrics, $\mathcal{G} = F_{on}(M)$ suffices. Find a canonical coframing (aka. "connection", "absolute parallelism", etc.) there. This is the "solution in the sense of Élie Cartan".

The orthonormal frame bundle

- Frame at $x \in M^n$ is $u : \mathbb{R}^n \xrightarrow{\cong} T_x M$. Coframe: u^{-1}.
- Frame bundle $\pi : F(M) \to M$. ($\pi^{-1}(x) =$ all frames at x.) This is a principal $\mathrm{GL}(n;\mathbb{R})$-bundle; right action $R_a(u) = u \circ a$.

Given (M^n, g), fix std metric $(\mathbb{R}^n, g_0)$, restrict $F(M)$ to $F_{on}(M)$ (isometric frames), a principal $\mathrm{O}(n)$-bundle. O.n. coframing $\{\bar{\theta}^i\}$, i.e. with $g = (\bar{\theta}^1)^2 + \ldots + (\bar{\theta}^n)^2$, is a section of $\pi : F_{on}(M) \to M$.

- Soldering form: $\Theta \in \Omega^1(F(M); \mathbb{R}^n)$, $\Theta_u(\xi) = u^{-1}\pi_*(\xi)$.
- Principal connection: $\Upsilon \in \Omega^1(F_{on}(M); \mathfrak{so}(n))$ s.t.:
  1. $R_a^*\Upsilon = \mathrm{Ad}_{a^{-1}}\Upsilon$, $\forall a \in \mathrm{O}(n)$.
  2. $\Upsilon(\zeta_X) = X$, $\forall X \in \mathfrak{so}(n)$.

  ($\zeta_X =$ fundamental vertical v.f.: $\zeta_X|_u = \frac{d}{dt}\Big|_{t=0} R_{\exp(tX)} u$.)

Note: $\mathrm{rank}\ker(\Upsilon) = n$, i.e. Υ is not a coframing of $F_{on}(M)$.

Local coframing on $\mathcal{G} = F_{on}(M)$ for (M^2, g)

Let $g = (\bar\theta^1)^2 + (\bar\theta^2)^2$. Lift to $\mathcal{G}$: Let $\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} \bar\theta^1 \\ \bar\theta^2 \end{pmatrix}$.

Exercise

Show $\begin{cases} d\theta^1 = dt \wedge \theta^2 + A\theta^1 \wedge \theta^2 = \gamma \wedge \theta^2 \\ d\theta^2 = -dt \wedge \theta^1 + B\theta^1 \wedge \theta^2 = -\gamma \wedge \theta^1 \end{cases}$, *where* $\gamma = dt + A\theta^1 + B\theta^2$.

$$\Rightarrow \quad \begin{cases} 0 = d^2\theta^1 = -d\gamma \wedge \theta^2 \\ 0 = d^2\theta^2 = +d\gamma \wedge \theta^1 \\ d\gamma = c\,\theta^1 \wedge \theta^2 \end{cases}, \text{ so } \quad \boxed{\begin{cases} d\theta^1 = \gamma \wedge \theta^2 \\ d\theta^2 = -\gamma \wedge \theta^1 \\ d\gamma = c\,\theta^1 \wedge \theta^2 \end{cases}}$$

These "structure equations" uniquely determine $\omega = (\theta^1, \theta^2, \gamma)$, i.e. a "canonical coframing". Dual basis: $(\partial_{\theta^1}, \partial_{\theta^2}, \partial_\gamma)$.

Exercise

Show that $\mathcal{L}_{\partial_\gamma} \theta^1 = \theta^2, \mathcal{L}_{\partial_\gamma} \theta^2 = -\theta^1$, $\mathcal{L}_{\partial_\gamma} \gamma = 0$, *and* $\mathcal{L}_{\partial_\gamma} c = 0$.

Coframe rank and symmetry

We saw $g \rightsquigarrow \exists! \omega$. Symmetries of g correspond to symmetries Φ of the coframing ω, i.e. $\Phi^* \omega = \omega$. Since $d \circ \Phi^* = \Phi^* \circ d$, then Φ preserves structure functions $\gamma_{ij}{}^k$, where $d\omega^k = \frac{1}{2}\gamma_{ij}{}^k \omega^i \wedge \omega^j$. For metrics, we get $\Phi^* c = c$. Rinse & repeat:

$$dc = c_1 \theta^1 + c_2 \theta^2 + c_3 \gamma.$$

Then Φ preserves c_1, c_2, c_3. Keep going... The rank r of ω is the number of indep. fcns obtained via this process. General thm:

Theorem (c.f. Olver, "Equivalence, Invariants, Symmetry", Thm 8.22)

A coframe ω of rank $r \geq 0$ on an m-mfld has $\dim(sym) = m - r$.

Note: If $r = 0$, we get str. eqns for a Lie alg/grp:

$$d\omega^k = -\frac{1}{2} C_{ij}{}^k \omega^i \wedge \omega^j \quad \Longleftrightarrow \quad [e_i, e_j] = C_{ij}{}^k e_k,$$

where $\{e_i\}$ is the dual basis to $\{\omega^i\}$.

Symmetry gap for surface metrics

Thm: Any (M^2, g) cannot have precisely 2 Killing vectors.

Proof: On $\mathcal{G} = F_{on}(M)$, we saw $\exists!$ coframing $\omega = (\theta^1, \theta^2, \gamma)$ with:

$$\begin{cases} d\theta^1 = \gamma \wedge \theta^2 \\ d\theta^2 = -\gamma \wedge \theta^1 \\ d\gamma = c\,\theta^1 \wedge \theta^2 \end{cases} \quad \Rightarrow \quad \begin{cases} 0 = d^2\theta^1 = d^2\theta^2 \\ 0 = d^2\gamma = dc \wedge \theta^1 \wedge \theta^2 \\ dc = f\theta^1 + g\theta^2 \end{cases}$$

Assuming $\dim(sym) = 2$, then $rank(\omega) = 1$, so c is nonconstant, and f, g are fcns of c. Then

$$\begin{cases} 0 = d^2c \wedge \theta^1 = f\gamma \wedge \theta^2 \wedge \theta^1 \\ 0 = d^2c \wedge \theta^2 = g\gamma \wedge \theta^2 \wedge \theta^1 \end{cases} \quad \Rightarrow \; f = g = 0 \Rightarrow \; c \text{ constant} \quad ⋇$$

Cartan geometry

Towards Cartan geometry

Klein geometry $(G \to G/P, \omega_G)$

$\overset{\text{(curvature)}}{\rightsquigarrow}$

Cartan geometry $(\mathcal{G} \to M, \omega)$

$\rotatebox{90}{$\rightsquigarrow$}$ $\rotatebox{90}{$\rightsquigarrow$}$

Euclidean geometry $(\mathbb{R}^n, g_0)$

$\overset{\text{(curvature)}}{\rightsquigarrow}$

Riemannian geometry (M^n, g)

Let G be a Lie group, $P \leq G$ a closed subgroup.

Definition

A Cartan geometry $(\mathcal{G} \to M, \omega)$ *of type* (G, P) *consists of a (right) principal* P*-bundle* $\mathcal{G} \to M$ *with a Cartan connection* $\omega \in \Omega^1(\mathcal{G}; \mathfrak{g})$*:*

1. ω *is a coframing:* $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ *linear iso* $\forall u \in \mathcal{G}$*;*
2. ω *is* P*-equivariant:* $R_p^*\omega = \mathrm{Ad}_{p^{-1}} \circ \omega$, $\forall p \in P$*;*
3. ω *reproduces fund. vertical v.f.:* $\omega(\zeta_A) = A$, $\forall A \in \mathfrak{p}$.

- $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ and $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$.
- Symmetry algebra: $\mathfrak{inf}(\mathcal{G}, \omega) = \{\xi \in \mathfrak{X}(\mathcal{G})^P : \mathcal{L}_\xi \omega = 0\}$.

Example

Flat model: $(G \to G/P, \omega_G)$, where ω_G is the Maurer–Cartan form on G, i.e. $\omega_G(g) = (L_{g^{-1}})_*$. MC eqn: $d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$.

Curvature: $K = d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(\mathcal{G};\mathfrak{g})$, i.e.

$$K(\xi,\eta) = d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)].$$

- $K = 0$ ("flat") $\leftrightarrow$ locally equiv. to $(G \to G/P, \omega_G)$.
- K is P-equivariant: $R_p^* K = \mathrm{Ad}_{p^{-1}} \circ K$, $\forall p \in P$.
- K is horizontal, i.e. $K(\zeta_A, \cdot) = 0$, $\forall A \in \mathfrak{p}$. (Axiom 2 for ω implies $-\mathrm{ad}_A \circ \omega = \mathcal{L}_{\zeta_A}\omega = \iota_{\zeta_A} d\omega$.)

Definition (Curvature function)

$\kappa : \mathcal{G} \to \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ *via* $\kappa(x,y) = K(\omega^{-1}(x), \omega^{-1}(y))$.

κ is P-equivariant, and codomain is a P-module. Ideally, impose P-inv. normalization conditions on κ to pin down ω uniquely.

Example

$(\mathcal{G} \to M, \omega)$ is torsion-free if κ is valued in $\bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$.

Fundamental theorem of Riemannian geometry

Let $(G, P) = (\mathbb{E}(n), O(n))$ with Lie algebras $(\mathfrak{g}, \mathfrak{p}) = (\mathfrak{e}(n), \mathfrak{so}(n))$.
We have $\mathfrak{g} = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} : A \in \mathfrak{so}(n), b \in \mathbb{R}^n \right\}$.

Theorem

There is an equivalence of categories btw Riemannian metrics and torsion-free Cartan geometries of type $(\mathbb{E}(n), O(n))$.

Write $\omega = \Upsilon + \Theta \in \Omega^1(\mathcal{G}, \mathfrak{so}(n) \oplus \mathbb{R}^n)$. Then Υ is the Levi-Civita (principal) connection and Θ is the soldering form.

Example ($n = 2$ case)

$$\omega = \begin{pmatrix} 0 & -\gamma & \theta^1 \\ \gamma & 0 & \theta^2 \\ 0 & 0 & 0 \end{pmatrix}, \; K = \begin{pmatrix} 0 & -d\gamma & d\theta^1 - \gamma\wedge\theta^2 \\ d\gamma & 0 & d\theta^2 + \gamma\wedge\theta^1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c\theta^1\wedge\theta^2 & 0 \\ c\theta^1\wedge\theta^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

by torsion-freeness and horizontality. Thus, $\begin{cases} d\theta^1 = \gamma \wedge \theta^2 \\ d\theta^2 = -\gamma \wedge \theta^1 \\ d\gamma = c\, \theta^1 \wedge \theta^2 \end{cases}$.

Cartan's equivalence method and refinements

Beyond metrics, i.e. "$\mathrm{O}(n)$-structures", one can consider other G_0-structures $\mathcal{G}_0 \subset F(M)$, where $G_0 \leq \mathrm{GL}(n, \mathbb{R})$. In general, $\nexists$ distinguished coframing on $\mathcal{G}_0$, e.g. $G_0 = \mathrm{CO}(n)$.

Strategy: Build a new bundle... In general, get a tower of bundles...

This process is known as Cartan's equivalence method, e.g. for $(2, 3, 5)$, see Cartan's 1910 "5-variables" paper for a tour-de-force application.

$\geq$ 1960's: Tanaka, Morimoto, Yamaguchi $\rightsquigarrow$ further refinements:

- Study of filtered manifolds / filtered G_0-structures.
- Tanaka prolongation (upper bound on sym dim)
- Harmonic theory & fundamental (harmonic) curvature.

Suggested reading:

- Olver, "Equivalence, invariants, symmetry" (2009).
- Zelenko, "On Tanaka's prolongation procedure for filtered structures of constant type" (2009).
- Čap, "On canonical Cartan connections associated to filtered G-structures" (2017).

Tanaka theory

Distributions & symbol algebras

Let $D^{-1} := D \subsetneq TM$ a distribution, form the weak-derived flag, i.e. $D^i = [D, D^{i+1}]$ for $i < 0$ (assume constant rank). Suppose $D^{-\nu} = TM$, $\exists \nu > 0$, i.e. "bracket-generating". Get a filtration

$$D =: D^{-1} \subset D^{-2} \subset \ldots \subset D^{-\nu} = TM.$$

Fix $x \in M$, take associated-graded: Let $\mathfrak{g}_i(x) := D^i(x)/D^{i+1}(x)$,

$$\mathfrak{m}(x) := \mathfrak{g}_{-1}(x) \oplus \mathfrak{g}_{-2}(x) \oplus \ldots \oplus \mathfrak{g}_{-\nu}(x).$$

The Lie bracket of v.f. induces a tensorial ("Levi") bracket on each $\mathfrak{m}(x)$, turning it into a nilpotent graded Lie algebra (NGLA) called the symbol algebra. We'll assume $\mathfrak{m}(x) \cong \mathfrak{m}$, $\forall x \in M$ as NGLA.

Example

Let $X, Y \in D^{-1}$. Then $[fX, gY] = fX(g)Y - gY(f)X + fg[X, Y]$, so $[fX, gY] \equiv fg[X, Y] \mod D^{-1}$.

Filtered G_0-structures

Given (M, D) as before, one has a natural (graded) frame bundle:

$$F_{gr}(M) = \bigcup_{x \in M} \left\{ u : \mathfrak{m} \xrightarrow{\cong} \mathfrak{m}(x) \text{ NGLA iso.} \right\}$$

Structure group: $\text{Aut}_{gr}(\mathfrak{m})$. Structure algebra: $\mathfrak{der}_{gr}(\mathfrak{m})$.
Note: $\mathfrak{der}_{gr}(\mathfrak{m}) \hookrightarrow \mathfrak{gl}(\mathfrak{g}_{-1})$ since $\mathfrak{g}_{-1}$ generates $\mathfrak{m} = \mathfrak{g}_-$.

Can specify reduction: $G_0 \leq \text{Aut}_{gr}(\mathfrak{m})$, so $\mathfrak{g}_0 \leq \mathfrak{der}_{gr}(\mathfrak{m})$.

Analogous to $O(n)$-structure (metrics) being an $\text{O}(n)$-reduction of $F(M)$, a filtered G_0-structure is a G_0-reduction $\mathcal{G}_0 \subset F_{gr}(M)$.

We have a vertical distribution to $\mathcal{G}_0 \to M$. Try to choose a horizontal complement canonically. (We phrased this dually earlier.) If not, build a new bundle: i.e. "geometrically prolong" to $\mathcal{G}_1$, then $\mathcal{G}_2$ if necessary, etc.

Symbol algebra examples

Example (Riemannian geometry)

$\mathfrak{m} = \mathfrak{g}_{-1} \cong \mathbb{R}^n$ (abelian) and $\mathfrak{g}_0 = O(n) \leq \text{Aut}_{gr}(\mathfrak{m}) \cong \text{GL}(n;\mathbb{R})$.

Example (2nd order ODE: $D = \langle \partial_x + p\partial_y + f\partial_p \rangle \oplus \langle \partial_p \rangle$)

$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = \langle e_1, e_2 \rangle \oplus \langle e_3 \rangle$ with $[e_1, e_2] = e_3$ (Heisenberg). Have splitting $\mathfrak{g}_{-1} = L_1 \oplus L_2$ and $\mathfrak{g}_0 =$ rescalings along L_1 and L_2 (2-dim). (Here, $\mathfrak{g}_0 \hookrightarrow \mathfrak{der}_{gr}(\mathfrak{m}) \cong \mathfrak{gl}(2,\mathbb{R})$.)

Example ((2, 3, 5)-dist. $D = \langle D_x := \partial_x + p\partial_y + q\partial_p + f\partial_z, \partial_q \rangle$)

$$T := [\partial_q, D_x] = \partial_p + f_q\partial_z \neq 0, \quad [\partial_q, T] = f_{qq}\partial_z,$$
$$[T, D_x] = \partial_y + S\partial_z, \quad S = f_p + f_q f_z - D_x(f_q).$$

Thus, $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} = \langle e_1, e_2 \rangle \oplus \langle e_3 \rangle \oplus \langle e_4, e_5 \rangle$, where:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = e_5.$$

Tanaka prolongation

Given NGLA $\mathfrak{m}$ and $\mathfrak{g}_0 \leq \mathfrak{der}_{gr}(\mathfrak{m})$, let $pr(\mathfrak{m}, \mathfrak{g}_0)$ be the GLA s.t.

1. $pr_{\leq 0}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}_0$.
2. If $X \in pr_+(\mathfrak{m}, \mathfrak{g}_0)$ s.t. $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$.
3. $pr(\mathfrak{m}, \mathfrak{g}_0)$ is maximal among all GLA satisfying (1) and (2).

Special case: When $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{m})$, we just write $pr(\mathfrak{m})$.

Theorem (Tanaka 1970)

- $pr(\mathfrak{m}, \mathfrak{g}_0)$ *is unique up to isomorphism.*
- $\dim(pr(\mathfrak{m}, \mathfrak{g}_0))$ *is an upper bound for the symmetry algebra of a filtered G_0-structure.*

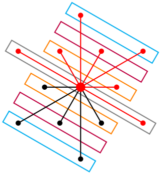
IDEA: Positive parts of this algebraic prolongation correspond to the geometric tower of bundles: $\ldots \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0 \to M$.

Also: Kruglikov, "Finite-dimensionality in Tanaka theory" (2011).

The height of $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ is the maximal $k \geq 0$ s.t. $\mathrm{pr}_k(\mathfrak{m}, \mathfrak{g}_0) \neq 0$.

Structure	$\mathfrak{m}$	$\mathfrak{g}_0$	$pr(\mathfrak{m}, \mathfrak{g}_0)$	Height
Metrics	$\mathfrak{g}_{-1} = \mathbb{R}^n$	$\mathfrak{so}(n)$	$\mathfrak{e}(n)$	0
Conformal	$\mathfrak{g}_{-1} = \mathbb{R}^n$	$\mathfrak{co}(n)$	$\mathfrak{so}(1, n+1)$	+1
2nd order ODE	$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	$\mathfrak{sl}_3$	+2
$(2, 3, 5)$	$\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$	$\mathfrak{gl}(\mathfrak{g}_{-1})$	$Lie(G_2)$	+3

$$\mathfrak{p} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \subset \mathfrak{sl}_3,$$

$\mathfrak{p}_1 =$  $\subset Lie(G_2)$

Definition

A parabolic geometry *is a Cartan geometry of type (G, P), where G is a semisimple Lie group and P is a parabolic subgroup.*

Upshot: Cartan geometry is:

- a "nice" soln of the Cartan equivalence problem.
- a unifying "upstairs" framework, despite a zoo of "downstairs" structures.

Next lecture:

- Normalization conditions for parabolic geometries (to get categorical equivalence to underlying structures).
- Kostant's theorem and harmonic curvature.
- Cartan reduction method for classifying (homogeneous) geometric structures.