Constructions with Parabolic Geometries

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Lecture 3

- Some examples from projective geometry
  (References at the end)

Projective structure on n-dimensional manifold M is an equivalence class $[\nabla]$ of torsion-free connections sharing the same geodesics up to reparametrization:

$$\hat{\nabla} \sim \nabla \iff \hat{\nabla}_X Y = \nabla_X Y + \Upsilon(X) Y + \Upsilon(Y) X$$

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Homogeneous model: projective sphere

$$G/P \cong \mathbb{S}^n = \mathbb{P}_+(\mathbb{R}^{n+1}),$$

where $G = SL(n+1, \mathbb{R})$ and

$$P = \{ \begin{pmatrix} \det(A)^{-1} & Z^t \\ 0 & A \end{pmatrix} : A \in GL_+(n, \mathbb{R}), Z^t \in \mathbb{R}^{n*} \}.$$

Geodesics are great circles (mapped to straight lines by central projection).

The Lie algebra $\mathfrak{p}$ of P is the non-neg. part in the grading

$$\mathfrak{sl}(n+1,\mathbb{R}) = \left\{ \begin{pmatrix} -tr(A) & \phi \\ X & A \end{pmatrix}, A \in \mathfrak{gl}(n,\mathbb{R}), X \in \mathbb{R}^n, \phi \in \mathbb{R}^{n*} \right\}$$

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Associated with a projective structure $(M, [\nabla])$ we have the normal parabolic geometry

$$(\mathcal{G} \to M, \omega) \quad \omega \in \Omega^1(\mathcal{G}, \mathfrak{sl}(n+1,\mathbb{R}))$$

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modeled on $SL(n+1,\mathbb{R})/P$.

For any representation $\rho : SL(n+1,\mathbb{R}) \to \mathrm{End}(\mathbb{V})$ we have an associated tractor bundle

$$\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$$

and induced tractor connection $\nabla^{\mathcal{V}}$.

A connection ∇ from the projective class corresponds to a G_0-equivariant section $\iota : \mathcal{G}_0 \to \mathcal{G}$, where $\mathcal{G}_0$ denotes the (oriented) frame bundle. Then

$$\iota^*\omega_- \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_-), \quad \iota^*\omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0), \quad \iota^*\omega_+ \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_+)$$

correspond to the soldering form θ, the connection ∇, and the projective Schouten tensor $\mathsf{P}_{ab} = \frac{1}{(n-1)(n+1)}(n\mathrm{Ric}_{ab} + \mathrm{Ric}_{ba})$.

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Moreover, a choice of ∇ determines an identification of the tractor bundle

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with a sum of (weighted) tensor bundles and a formula for the tractor connection via

$$\nabla^{\mathcal{V}}_X S = \nabla_X S + X \bullet S - \mathsf{P}(X) \bullet S,$$

where $\bullet$ is induced map from actions of $\mathfrak{g}_-$ resp. $\mathfrak{g}_+$ on $\mathbb{V}$.

- We have

$$\mathcal{E}(w) := (\Lambda^n TM)^{\frac{w}{n+1}}$$

(bundle of projective densities of weight w). Then for connections $\hat{\nabla}$ and ∇ projectively related by Υ,

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$$\mathsf{P}_{ab} = \tfrac{1}{n-1} \mathrm{Ric}_{ab}$$

and their curvature decomposes as

$$R_{bc}{}^d{}_a = W_{bc}{}^d{}_a + 2\delta_{[b}{}^d \mathsf{P}_{c]a},$$

where $W_{bc}{}^d{}_a$ is the projective Weyl tensor.

Cotractor bundle and connection

Consider the dual of the standard representation of $\mathfrak{sl}(n+1,\mathbb{R})$. Then $\mathfrak{p}$ preserves an n-dimensional subspace in $\mathbb{R}^{n+1*}$. Under $\mathfrak{g}_0 \cong \mathfrak{gl}(n,\mathbb{R})$ the representation decomposes

$$\mathbb{R}^{n+1*} \cong_{\mathfrak{g}_0} \mathbb{V}_0 \oplus \mathbb{V}_1 \cong_{\mathfrak{g}_0} \mathbb{V}_0 \oplus (\mathbb{V}_0 \otimes \mathfrak{g}_+)$$

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Any choice of ∇ gives identification

$$\mathcal{T}^* \cong_{\nabla} \mathcal{G}_0 \times_{G_0} \mathbb{R}^{n+1*} \cong \mathcal{E}(1) \oplus T^*M(1)$$

and explicit representation of tractor connection

$$\nabla_i^{\mathcal{T}^*}(\sigma, \mu_j) = (\nabla_i\sigma - \mu_i, \nabla_i\mu_j + \sigma \mathsf{P}_{ij}),$$

Splitting operator

We need the tractor connection

$$\nabla^{\mathcal{T}^*} : \Gamma(\mathcal{T}^*) \to \Omega^1(M, \mathcal{T}^*), \ \nabla_i^{\mathcal{T}^*}(\sigma, \mu_j) = (\nabla_i \sigma - \mu_i, \nabla_i \mu_j + \sigma \mathsf{P}_{ij})$$

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and the Kostant codifferential $\partial^* : \Omega^1(M, \mathcal{T}^*) \to \Gamma(\mathcal{T}^*)$ induced by

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The BGG splitting operator is then characterized by

$$D : \Gamma(\mathcal{E}(1)) \to \Gamma(\mathcal{T}^*), \quad \Pi(D(\sigma)) = \sigma, \quad \partial^* \nabla^{\mathcal{T}^*} D(\sigma) = 0$$

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To determine it explicitly we compute

$$\partial^*(\nabla^{\mathcal{T}^*}(\sigma, \mu_i)) = (0, \nabla_i \sigma - \mu_i)$$

and thus

$$D(\sigma) = (\sigma, \nabla_i \sigma).$$

First BGG operator

First we compute the homologies

$$\mathcal{H}_0 = \mathcal{E}(1) \quad (\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet}) \quad \text{and} \quad \mathcal{H}_1 = S^2T^*M(1) \quad (\overset{-3}{\times}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet})$$

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From the splitting operator and the tractor connection

$$D(\sigma) = (\sigma, \nabla_i\sigma), \quad \nabla_i^{\mathcal{T}^*}(\sigma, \mu_j) = (\nabla_i\sigma - \mu_i, \nabla_i\mu_j + \sigma \mathsf{P}_{ij}),$$

we get for the first BGG operator

$$\Theta^{\mathcal{T}^*} : \Gamma(\mathcal{E}(1)) \to \Gamma(S^2T^*M(1))$$

$$\begin{aligned} \Theta^{\mathcal{T}^*}(\sigma) &= \Pi(\nabla^{\mathcal{T}^*} D(\sigma)) = \Pi\left((0, \nabla_i\nabla_j\sigma + \mathsf{P}_{ij}\sigma)\right) \\ &= \nabla_i\nabla_j\sigma + \mathsf{P}_{ij}\sigma \end{aligned}$$

The tractor connection $\nabla^{\mathcal{T}^*}$ is the prolongation connection for the BGG equation $\nabla_i \nabla_j \sigma + \mathsf{P}_{ij}\sigma = 0$. Hence, every solution σ is normal and corresponds to $S \in \Gamma(\mathcal{T}^*), \nabla^{\mathcal{T}^*} S = 0$.

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Curved orbit decomposition (Č-G-H)

- Homogeneous model: hyperplane through origin in $\mathbb{R}^{n+1}$ decomposes $\mathbb{S}^{n+1}$ into $\mathbb{S}^n$ and two open hemispheres
- In general

$$M = M_+ \cup M_0 \cup M_-,$$

zero set M_0 of σ: is a separating hypersurface with an induced projective structure
open $M_\pm$ curved orbits: σ is non-vanishing and determines Ricci-flat connection characterized by $\hat{\nabla}\sigma = 0$ (then $\hat{P}_{ij} = 0$ and thus $\hat{Ric}_{ij} = 0$) and $\hat{\nabla}_i \sigma = \nabla_i \sigma + \Upsilon_i \sigma$, so $\hat{\nabla}$ is related to ∇ via $\Upsilon = -\frac{1}{\sigma}\nabla\sigma$.

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Remark: related to projective compactifications (Čap-Gover).

Standard Tractors

Standard tractor bundle has distinguished line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ corresp. to line $\ell \subset \mathbb{R}^{n+1}$ preserved by P, $\Pi : \mathcal{T} \to \mathcal{T}/\mathcal{T}^1 \cong TM(-1)$.

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$$\nabla_a X^b - \frac{1}{n}\delta_a{}^b \nabla_c X^c = 0, \quad X^a \in \Gamma(TM(-1))$$

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Adjoint Tractors

The first BBG operator for the adjoint tractor bundle $\mathcal{A} = \mathcal{G} \times_P \mathfrak{sl}(n+1, \mathbb{R})$ is related to the infinitesimal automorphism operator for projective structures

$$X^d \mapsto \underbrace{(\nabla_{(a}\nabla_{b)} X^c + \mathrm{P}_{ab} X^c)_o}_{\text{first BGG}} + \underbrace{W_{d(a}{}^c{}_{b)}}_{\text{proj Weyl}} X^d$$

Consider

$$\Lambda^2\mathcal{T}^* = \mathcal{G} \times_P \Lambda^2\mathbb{R}^{n+1*} \cong_\nabla T^*M(2) \oplus \Lambda^2 T^*M(2)$$

with tractor connection

$$\nabla_a^{\Lambda^2\mathcal{T}^*} \begin{pmatrix} k_b \\ \mu_{bc} \end{pmatrix} = \begin{pmatrix} \nabla_a k_b - \mu_{ab} \\ \nabla_a \mu_{bc} + 2\mathrm{P}_{a[b} k_{c]} \end{pmatrix}.$$

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Then $\partial^*\nabla^{\Lambda^2\mathcal{T}^*} \begin{pmatrix} k_b \\ \mu_{bc} \end{pmatrix} = 0$ iff $\mu_{[ab]} = \nabla_{[a}k_{b]}$, hence the splitting operator

$$D(k_b) = \begin{pmatrix} k_b \\ \nabla_{[a}k_{b]} \end{pmatrix},$$

Killing-type operator $\Lambda^2\mathcal{T}^*$

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$$D(k_b) = \begin{pmatrix} k_b \\ \nabla_{[a}k_{b]} \end{pmatrix},$$

and we get as first BGG operator $\Theta^{\Lambda^2\mathcal{T}^*} : T^*M(2) \to S^2T^*M(2)$

$$k_a \to \nabla_{(a}k_{b)}.$$

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Prolongation connection is

$$\nabla_a^{prol} \begin{pmatrix} k_b \\ \mu_{bc} \end{pmatrix} = \nabla_a^{\Lambda^2 \mathcal{T}^*} \begin{pmatrix} k_b \\ \mu_{bc} \end{pmatrix} - \begin{pmatrix} 0 \\ W_{bc}{}^d{}_a k_d \end{pmatrix}$$

Tractor Metrics $S^2\mathcal{T}^*$

First BGG operator

Consider $S^2\mathcal{T}^* = \mathcal{G} \times_P S^2\mathbb{R}^{n+1*}$, then

$$\mathcal{H}_0 = \overset{2}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} = \mathcal{E}(2) \quad \text{and} \quad \mathcal{H}_1 = \overset{-4}{\times}\!\!-\!\!\!-\!\!\overset{3}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} = S^3T^*M(2)$$

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and the first BGG turns out to be

$$\Theta^{S^2\mathcal{T}^*}(\sigma) = \nabla_{(a}\nabla_b\nabla_{c)}\sigma + 4\mathsf{P}_{(ab}\nabla_{c)}\sigma + 2(\nabla_{(a}\mathsf{P}_{bc)})\sigma$$

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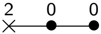
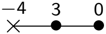
$\mathcal{H}_0 =$ 

$= \mathcal{E}(2)$ and $\mathcal{H}_1 =$ 

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Parallel tractor metric (Č-G-H)

Now assume we have a non-degenerate parallel tractor $h \in \Gamma(S^2\mathcal{T}^*)$ of signature (p, q) with corresponding normal solution $\sigma = \Pi(h) \in \Gamma(\mathcal{E}(2))$.

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- signature of h determines the G-type
- three P-types according to whether restriction of h to distinguished line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ is positive, zero, or negative $\rightsquigarrow$

$$M = M_+ \cup M_0 \cup M_-$$

Homogeneous Model

Consider $H = SO(n,1)$ preserving bilinear form of sig. (n,1) on $\mathbb{R}^{n+1}$ and H-orbit decomposition of the projective sphere $G/P \cong \mathbb{P}_+(\mathbb{R}^{n+1}) \cong \mathbb{S}^n$:

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- one open orbit of pos. rays isomorphic to $SO(n,1)/SO(n-1,1)$ with induced de Sitter metric
- two open orbits of negative rays isomorphic to $SO(n,1)/SO(n)$ with induced hyperbolic metric

Homogeneous Model

Consider $H = SO(n,1)$ preserving bilinear form of sig. (n,1) on $\mathbb{R}^{n+1}$ and H-orbit decomposition of the projective sphere $G/P \cong \mathbb{P}_+(\mathbb{R}^{n+1}) \cong \mathbb{S}^n$:

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- M_0, the zero set of σ, is embedded hypersurface with normal conformal Cartan geometry of signature $(p-1, q-1)$
- Einstein metrics of signature $(p-1, q)$ and $(p, q-1)$ on open curved orbits ($g_{ab} = \mathsf{P}_{ab}$ of connection ∇ determined by scale σ).

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- Normal non-deg. solutions ϕ^{jk} correspond to Einstein metrics (Armstrong, Čap-Gover-Macbeth)
- Bij. correspondence between *non-deg. sections* of $S^2\mathcal{T}$ and of $S^2\mathcal{T}^*$.

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