Constructions with Parabolic Geometries

Katja Sagerschnig

Center for Theoretical Physics, Warsaw

October 15, 2021

Lecture 2

- Short rewiev
- BGG equations and prolongations
- Fefferman-type constructions
- Holonomy and curved orbit decomposition

Review: Conformal standard tractors

$(M,[g])$ conformal manifold. $G = SO(p+1, q+1)$, $P \subset G$ stabilizer of null-line $\ell \in \mathbb{R}^{n+2}$, $n = p+q$.

1) Standard tractor bundle and connection

The normal conformal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ induces a linear connection $\nabla^{\mathcal{T}}$ on $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+2}$. In the splitting given by choice of g,

$$\mathcal{T} \cong \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]$$

and

$$\nabla_i^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_j \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + \mathrm{g}_{ij} \rho + \mathrm{P}_{ij} \sigma \\ \nabla_i \rho + \mathrm{P}_{ij} \mu^j \end{pmatrix} \tag{1}$$

Natural projection $\Pi : \mathcal{T} \to \mathcal{T}/\mathcal{T}^0 \cong \mathcal{E}[1], \quad (\sigma, \mu_i, \rho) \to \sigma$

2) Conformal-to-Einstein operator

Now suppose $S \in \Gamma(\mathcal{T})$ is a parallel tractor,

$$\nabla^{\mathcal{T}} S = \nabla_i^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_j \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + \mathrm{g}_{ij}\rho + \mathrm{P}_{ij}\sigma \\ \nabla_i \rho + \mathrm{P}_{ij}\mu^j \end{pmatrix} = 0.$$

2) Conformal-to-Einstein operator

Now suppose $S \in \Gamma(\mathcal{T})$ is a parallel tractor,

$$\nabla^{\mathcal{T}} S = \nabla_i^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_j \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + \mathsf{g}_{ij} \rho + \mathsf{P}_{ij} \sigma \\ \nabla_i \rho + \mathsf{P}_{ij} \mu^j \end{pmatrix} = 0.$$

From the first equ. and the trace of the second equ. we see that

$$S = (\sigma, \nabla_i \sigma, -\tfrac{1}{n}(\nabla^j \nabla_j \sigma - \mathsf{P}_j^j \sigma)) =; D(\sigma)$$

2) Conformal-to-Einstein operator

Now suppose $S \in \Gamma(\mathcal{T})$ is a parallel tractor,

$$\nabla^{\mathcal{T}} S = \nabla_i^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_j \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + \mathrm{g}_{ij} \rho + \mathrm{P}_{ij} \sigma \\ \nabla_i \rho + \mathrm{P}_{ij} \mu^j \end{pmatrix} = 0.$$

From the first equ. and the trace of the second equ. we see that

$$S = (\sigma, \nabla_i \sigma, -\tfrac{1}{n}(\nabla^j \nabla_j \sigma - \mathrm{P}_j^j \sigma)) =: D(\sigma)$$

Then

$$\nabla^{\mathcal{T}} D(\sigma) = \begin{pmatrix} 0 \\ \nabla_i \nabla_j \sigma + \mathrm{g}_{ij}(-\tfrac{1}{n}(\nabla^k \nabla_k \sigma - \mathrm{P}_k^k \sigma)) + \mathrm{P}_{ij} \sigma \\ * \end{pmatrix} = 0.$$

and $\sigma = \Pi(S)$ is a solution to the (conformally invariant) equation

$$\text{trace-free part}(\nabla_a \nabla_b \sigma + P_{ab} \sigma) = 0.$$

3) Prolonging the Conformal-to-Einstein equation

Conversely, suppose σ is a solution to the linear conformally invariant operator $\Theta : \Gamma(\mathcal{E}[1]) \to \Gamma(S^2_o T^* M[1])$,

$$\Theta(\sigma) = \text{trace-free part}(\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma)$$

3) Prolonging the Conformal-to-Einstein equation

Conversely, suppose σ is a solution to the linear conformally invariant operator $\Theta : \Gamma(\mathcal{E}[1]) \to \Gamma(S^2_o T^* M[1])$,

$$\Theta(\sigma) = \text{trace-free part}(\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma)$$

Prolonging the equation $\rightsquigarrow$

$$\begin{aligned}
&\nabla_a \sigma - \mu_a = 0 \\
&\nabla_a \mu_b + \mathsf{P}_{ab}\sigma + g_{ab}\rho = 0 \\
&\nabla_a \rho
\end{aligned}$$

3) Prolonging the Conformal-to-Einstein equation

Conversely, suppose σ is a solution to the linear conformally invariant operator $\Theta : \Gamma(\mathcal{E}[1]) \to \Gamma(S^2_o T^* M[1])$,

$$\Theta(\sigma) = \text{trace-free part}(\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma)$$

Prolonging the equation $\rightsquigarrow$

$$\begin{aligned} &\nabla_a \sigma - \mu_a = 0 \\ &\nabla_a \mu_b + \mathsf{P}_{ab}\sigma + g_{ab}\rho = 0 \\ &\nabla_a \rho - \mathrm{P}_a{}^b \mu_b = 0 \end{aligned}$$

3) Prolonging the Conformal-to-Einstein equation

Conversely, suppose σ is a solution to the linear conformally invariant operator $\Theta : \Gamma(\mathcal{E}[1]) \to \Gamma(S^2_o T^* M[1])$,

$$\Theta(\sigma) = \text{trace-free part}(\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma)$$

Prolonging the equation $\rightsquigarrow$

$$\begin{aligned} \nabla_a \sigma - \mu_a &= 0 \\ \nabla_a \mu_b + \mathsf{P}_{ab}\sigma + g_{ab}\rho &= 0 \\ \nabla_a \rho - \mathrm{P}_a{}^b \mu_b &= 0 \end{aligned}$$

Theorem (Le Brun, Gover)

$$\{\text{ solutions } \sigma \text{ to } \Theta(\sigma) = 0 \} \overset{1-1}{\longleftrightarrow} \{\text{ parallel standard tractors } \nabla^{\mathcal{T}} S = 0 \}$$

Conformal Killing fields

If we start with the conformal adjoint tractor bundle

$$\mathcal{A} = \mathcal{G} \times_P \mathfrak{so}(p+1, q+1) = TM \oplus (\mathbb{R} \oplus \Lambda^2 TM[-2]) \oplus T^*M,$$

we we are lead to the conformal Killing quation

$$\text{tracefree part of } \nabla_{(a} k_{b)} = 0$$

Prolongation $\rightsquigarrow$

$$\begin{aligned}
&\nabla_a k_b - \nu g_{ab} - \mu_{ab} = 0 \\
&\nabla_a \nu - \rho_a + \mathrm{P}_{ab} k^b = 0 \\
&\nabla_a \mu_{bc} + 2\mathrm{P}_{a[b} k_{c]} + 2 g_{a[b} \rho_{c]} - W_{dabc} k^d = 0 \\
&\nabla_a \rho_b + \mathrm{P}_a^c \mu_{bc} + \mathrm{P}_{ab} \nu + C_{cab} k^c = 0
\end{aligned}$$

The tractor connection and differs from the prolongation connection by the Cotton and Weyl terms in blue.

Consider semisimple graded Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$ parabolic, and a G-representation on $\mathbb{V}$. Recall that $\mathfrak{g}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$.

Let $\partial^* : \Lambda^k \mathfrak{g}_+ \otimes \mathbb{V} \to \Lambda^{k-1}\mathfrak{g}_+ \otimes \mathbb{V}$, $\partial^* \circ \partial^* = 0$, be the Lie algebra homology differential (Kostant Codifferential)

$$\begin{aligned}\partial^*(Z_1 \wedge \cdots \wedge Z_k \otimes v) := &\sum_i (-1)^i Z_1 \wedge \ldots \hat{Z}_i \cdots \wedge Z_k \otimes Z_i \cdot v \\ &+ \sum_{i,j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \ldots \hat{Z}_i \ldots \hat{Z}_j \cdots \wedge Z_k \otimes v\end{aligned}$$

The homologies $\mathbb{H}_k(\mathfrak{g}_+, \mathbb{V}) = \ker(\partial^*)/\mathrm{im}(\partial^*)$ are completely reducible P-representations (P_+ acting trivially). Their structure can be determined using Kostant's theorem.

$\mathbb{H}_0(\mathfrak{g}_+, \mathbb{V}) = \mathbb{V}/\mathbb{V}^0$, where $\mathbb{V}^0 = \mathfrak{g}_+ \cdot \mathbb{V}$ is largest P-inv. filtrant.

One can turn the algebraic constructions to geometry:

- Via Cartan connection $\Lambda^k T^*M \otimes \mathcal{V} \cong \mathcal{G} \times_P \Lambda^k \mathfrak{p}_+ \otimes \mathbb{V}$,
- $\partial^* : \Lambda^k T^*M \otimes \mathcal{V} \to \Lambda^{k-1} T^*M \otimes \mathcal{V}$ induced bundle maps,
- bundle projections $\Pi : \ker(\partial^*) \to \ker(\partial^*)/\mathrm{im}(\partial^*) := \mathcal{H}_k$, where

$$\mathcal{H}_k \cong \mathcal{G} \times_P \mathbb{H}_k(\mathfrak{g}_+, \mathbb{V})$$

One can turn the algebraic constructions to geometry:

- Via Cartan connection $\Lambda^k T^* M \otimes \mathcal{V} \cong \mathcal{G} \times_P \Lambda^k \mathfrak{p}_+ \otimes \mathbb{V}$,
- $\partial^* : \Lambda^k T^* M \otimes \mathcal{V} \to \Lambda^{k-1} T^* M \otimes \mathcal{V}$ induced bundle maps,
- bundle projections $\Pi : \ker(\partial^*) \to \ker(\partial^*)/\mathrm{im}(\partial^*) := \mathcal{H}_k$, where

$$\mathcal{H}_k \cong \mathcal{G} \times_P \mathbb{H}_k(\mathfrak{g}_+, \mathbb{V})$$

BGG machinery yields sequence of inv. differential operators, called (curved) BGG-sequences (Čap-Slovák-Souček, Calderbank-Diemer)

$$\Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1) \to \Gamma(\mathcal{H}_2) \to \dots$$

1. Construct natural differential splitting operator characterized by

$$D : \Gamma(\mathcal{H}_0) = \Gamma(\mathcal{V}/\mathcal{V}^0) \to \Gamma(\mathcal{V}), \quad \Pi(D(\sigma)) = \sigma \quad \text{and} \quad \partial^* \circ \nabla^{\mathcal{V}} \circ D = 0$$

2. Then define 1st BGG as as

$$\Theta_0 = \Pi \circ \nabla^{\mathcal{V}} \circ D : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1)$$

Let $\mathfrak{g}$ be complex simple Lie alg. and $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra corresponding to a set of simple roots $\Sigma \subset \Delta^0$.

Associate to $\mathfrak{p}$ is subset $W^{\mathfrak{p}} \subset W$ of the Weyl group with induced graph structure (Hasse diagram): $\quad W^{\mathfrak{p}} \longleftrightarrow W \cdot \delta^{\mathfrak{p}} \quad$ via $\quad w \mapsto w^{-1}(\delta^{p})$, where $\delta^{\mathfrak{p}}$ is the sum of all fundamental weights corr. to Σ

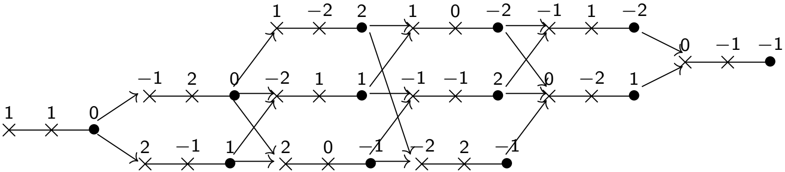
Elements of length 1 in $W^{\mathfrak{p}}$: $s_{\alpha_i}(\lambda) = \lambda - \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\alpha_i$ for $\alpha_i \in \Sigma$.

Let $\mathfrak{g}$ be complex simple Lie alg. and $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra corresponding to a set of simple roots $\Sigma \subset \Delta^0$.

Associate to $\mathfrak{p}$ is subset $W^{\mathfrak{p}} \subset W$ of the Weyl group with induced graph structure (Hasse diagram): $W^{\mathfrak{p}} \longleftrightarrow W \cdot \delta^{\mathfrak{p}}$ via $w \mapsto w^{-1}(\delta^p)$, where $\delta^{\mathfrak{p}}$ is the sum of all fundamental weights corr. to Σ

Elements of length 1 in $W^{\mathfrak{p}}$: $s_{\alpha_i}(\lambda) = \lambda - \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\alpha_i$ for $\alpha_i \in \Sigma$.

Example: $\mathfrak{g} = \mathfrak{sl}(4,\mathbb{R})$, $\Sigma = \{\alpha_1, \alpha_2\}$

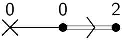


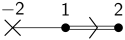
Kostant's Theorem

Let $\mathbb{V}$ be an irreducible $\mathfrak{g}$-representation (λ highest weight of $\mathbb{V}^*$).

- The irreducible components of the $\mathfrak{g}_0$-representation $\mathbb{H}_k(\mathfrak{g}_+, \mathbb{V})$ are in bijective correspondence with elements in $W^{\mathfrak{p}}$ of length k.
- Their highest weights can be determined using the affine Weyl group action $w \bullet \lambda = w(\lambda + \delta^{\mathfrak{g}}) - \delta^{\mathfrak{g}}$ (and dualizing).

Example: $\mathfrak{g} = \mathfrak{so}(3,4)$, $\Sigma = \{\alpha_1\}$, $\mathbb{V} = \Lambda^3\mathbb{R}^{7*}$

- $\mathbb{H}_0 = \Lambda^2\mathbb{R}^{5^*}[3]$, 

- $\mathbb{H}_1 = \mathbb{R}^{5^*} \odot \Lambda^2\mathbb{R}^{5^*}[3]$, 

,
$\odot$ denotes the highest weight component in

$$\mathbb{R}^5 \otimes \Lambda^k\mathbb{R}^5 \cong \Lambda^{k-1}\mathbb{R}^5 \oplus \Lambda^{k+1}\mathbb{R}^5 \oplus \mathbb{R}^5 \odot \Lambda^k\mathbb{R}^5$$

- $\rightsquigarrow$ conformal Killing operator $\phi_{ab} \mapsto pr_{\odot}(\nabla_a\phi_{ab})$

The first operators in BGG sequences define linear overdetermined systems of PDE.

Examples

- Conformal: conformal-to-Einstein, conformal Killing operators twistor operator,...
- Projective: Killing type equations, metrizability equation for projective structures,...

The PDEs are of finite type and can be prolonged to linear connections on tractor bundles such that solutions to the equation are in bijective correspondence with parallel sections of the connection.

There are general prolongation procedures (Branson-Čap-Eastwood -Gover) and in particular one that produces invariant connections on the associated tractor bundles (Hammerl-Silhan-Somberg-Souceck).

Prolongation Connection

We call the connection $\nabla^{pr} = \nabla^{\mathcal{V}} + \Phi$ on the tractor bundle $\mathcal{V}$ such that

$$\left\{ \text{solutions to 1st BGG} \right\} \xleftrightarrow{1-1} \left\{ \text{tractors } S \in \Gamma(\mathcal{V}) \text{ s.t. } \nabla^{prol} S = 0 \right\}$$

prolongation connection.

Prolongation Connection

We call the connection $\nabla^{pr} = \nabla^{\mathcal{V}} + \Phi$ on the tractor bundle $\mathcal{V}$ such that

$$\{\,\text{solutions to 1st BGG}\} \overset{1-1}{\longleftrightarrow} \{\,\text{tractors } S \in \Gamma(\mathcal{V}) \text{ s.t. } \nabla^{prol} S = 0\,\}$$

prolongation connection.

Normal BGG solutions

If $S \in \Gamma(\mathcal{V})$ is parallel for the (normal) tractor connection, $\nabla^{\mathcal{V}} S = 0$,, then it projects to a solution $\sigma = \Pi(S)$ of the 1st BGG equation.

BGG solutions corresponding to parallel tractors are called normal solutions.

Prolongation Connection

We call the connection $\nabla^{pr} = \nabla^{\mathcal{V}} + \Phi$ on the tractor bundle $\mathcal{V}$ such that

$$\{\,\text{solutions to 1st BGG}\,\} \overset{1-1}{\longleftrightarrow} \{\,\text{tractors } S \in \Gamma(\mathcal{V}) \text{ s.t. } \nabla^{prol} S = 0\,\}$$

prolongation connection.

Normal BGG solutions

If $S \in \Gamma(\mathcal{V})$ is parallel for the (normal) tractor connection, $\nabla^{\mathcal{V}} S = 0$,, then it projects to a solution $\sigma = \Pi(S)$ of the 1st BGG equation.

BGG solutions corresponding to parallel tractors are called normal solutions.

Example

Normal conformal Killing fields are those that in addition satisfy $W_{dabc} k^c = 0$ and $C_{dab} k^d = 0$.

Consider hom. spaces G/Q and $\tilde{G}/\tilde{P}$, a homomorphism $i : G \to \tilde{G}$ s.t. G-orbit of $e\tilde{P} \subset \tilde{G}/\tilde{P}$ is open and $Q = i^{-1}(\tilde{P})$.
Given Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, Q),

- extend the structure group $\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$
- let $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ be unique Cartan connection extending ω.

$\rightsquigarrow$ Cartan geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ of type $(\tilde{G}, \tilde{P})$

Consider hom. spaces G/Q and $\tilde{G}/\tilde{P}$, a homomorphism $i : G \to \tilde{G}$ s.t. G-orbit of $e\tilde{P} \subset \tilde{G}/\tilde{P}$ is open and $Q = i^{-1}(\tilde{P})$.
Given Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, Q),

- extend the structure group $\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$
- let $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ be unique Cartan connection extending ω.

$\rightsquigarrow$ Cartan geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ of type $(\tilde{G}, \tilde{P})$

Nurowski's conformal structures

- Recall that $G_2 \subset SO(3,4)$ and $G_2/P \cong SO(3,4)/\tilde{P} \cong \mathbb{P}(\mathcal{N})$.
- Apply construction to reg., normal parabolic geometry ass. with $(2,3,5)$ distribution $\rightsquigarrow$ $(2,3,5)$ distribution $\mathcal{D} \subset TM$ determines a natural conformal structure $[g_{\mathcal{D}}]$ of signature $(2,3)$ on M.
- Moreover, the so obtained conformal parabolic geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ turns out to be *normal*.
- Hence the tractor connections induced by ω resp. $\tilde{\omega}$ coincide.

- Recall: $G_2 \subset SO(3,4)$ is stabilizer of a generic 3-form Φ
- It follows that conformal structures associated with $(2,3,5)$ distributions admit tractor 3-form

  $$\Phi \in \Gamma(\Lambda^3 \mathcal{T}), \quad \nabla^{\mathcal{T}} \Phi = 0,$$

  and thus (decomposable) normal conf. Killing 2-form

  $$\phi = \Pi(\Phi) \in \Gamma(\Lambda^2 T^* M[3])$$
- Nurowski's conformal structures admit twistor spinors. These objects can be used to characterize the conformal structures

- Recall: $G_2 \subset SO(3,4)$ is stabilizer of a generic 3-form Φ
- It follows that conformal structures associated with $(2,3,5)$ distributions admit tractor 3-form

$$\Phi \in \Gamma(\Lambda^3 \mathcal{T}), \quad \nabla^{\mathcal{T}} \Phi = 0,$$

  and thus (decomposable) normal conf. Killing 2-form

$$\phi = \Pi(\Phi) \in \Gamma(\Lambda^2 T^* M[3])$$

- Nurowski's conformal structures admit twistor spinors. These objects can be used to characterize the conformal structures

Fefferman spaces

- related to the inclusion $i : SU(n+1,1) \to SO(2n+2,2)$.
- produces a Lorentzian conformal structure on a circle bundle over a partially integrable CR structure of hypersurface type.
- induced conformal geometry is normal iff the original structure was integrable, i.e., CR.

Let $S \in \Gamma(\mathcal{T})$, $\nabla^{\mathcal{T}} S = 0$, be a conformal parallel standard tractor. Then

$$S =_g (\sigma, \nabla_a \sigma, -\tfrac{1}{n}(\Delta\sigma + \mathsf{P}_a^a \sigma)).$$

Let h denote the tractor metric,

$$h(S,S) =_g -\tfrac{2}{n}\sigma(\Delta\sigma + \mathsf{P}_a^a\sigma) + g^{ab}\nabla_a\sigma\nabla_b\sigma$$

Then (Gover):

- $M_0 = \{x|\sigma(x) \neq 0\}$ open and dense and carries an Einstein metric $\hat{g} = \frac{1}{\sigma^2}\mathrm{g}$ whose Einstein constant is neg. multiple of $h(S,S)$.
- $M_1 = \{x|\sigma(x) = 0\}$, $h(S,S) \neq 0$: σ, is defining density and M_1 an embedded hypersurface with induced conformal structure.

Recall: We extended the Cartan connection ω to a principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$. For G-representation $\mathbb{V}$ we defined the tractor connection $\nabla^{\mathcal{V}}$ as induced connection on $\mathcal{V} = \tilde{\mathcal{G}} \times_G \mathbb{V}$.

Let $\mathcal{H} = \ker(\tilde{\omega}) \subset T\tilde{\mathcal{G}}$ be the horizontal subbundle. A curve $\tilde{c} : [0,1] \to \tilde{\mathcal{G}}$ is called horizontal if $\tilde{c}'(t) \in \mathcal{H}_{\tilde{c}(t)}$ for all t.
For any loop $c : [0,1] \to M$ in $x \in M$ and point $u \in \tilde{\mathcal{G}}_x$ in the fibre, the horizontal lift $\tilde{c}_u : [0,1] \to \tilde{\mathcal{G}}$ is the unique horizontal curve such that $\pi \circ \tilde{c}_u = c$ and $\tilde{c}_u(0) = u$.

The holonomy group at $u \in \tilde{\mathcal{G}}$ is defined as

$$Hol_u(\omega) := Hol_u(\tilde{\omega}) = \{g \in G \mid \exists \text{ loop in x such that } \tilde{c}_u(1) = u \cdot g\}$$

The conjugacy class of Hol_u in G is independent of the chosen point u.

Conjugation does not preserve the subgroup P and the relative position matters.

Suppose you have a parallel tractor $S \in \Gamma(\mathcal{V})$, $\nabla^{\mathcal{V}} S = 0$. It corresponds to a G-equivariant function $f : \tilde{\mathcal{G}} \to \mathbb{V}$, which is constant on horizontal curves.

The image $f(\tilde{\mathcal{G}}) = \mathcal{O} \cong G/H \subset \mathbb{V}$ is a G-orbit, called the G-type of S. ($Hol(\omega)$ contained in H.)

For each $x \in M$, the image $f(\mathcal{G}_x) \subset \mathcal{O}$ of the fibre of $\mathcal{G}$ is a P-orbit, called the P-type of S at the point x.

Suppose you have a parallel tractor $S \in \Gamma(\mathcal{V})$, $\nabla^{\mathcal{V}} S = 0$. It corresponds to a G-equivariant function $f : \tilde{\mathcal{G}} \to \mathbb{V}$, which is constant on horizontal curves.

The image $f(\tilde{\mathcal{G}}) = \mathcal{O} \cong G/H \subset \mathbb{V}$ is a G-orbit, called the G-type of S. ($Hol(\omega)$ contained in H.)

For each $x \in M$, the image $f(\mathcal{G}_x) \subset \mathcal{O}$ of the fibre of $\mathcal{G}$ is a P-orbit, called the P-type of S at the point x.

Curved Orbit Decomposition (Čap-Gover-Hammerl)

The manifold M decomposes according to P-types $M = \bigcup_{i \in P \backslash \mathcal{O}} M_i$

- The decomposition $M = \cup M_i$ is locally diffeomorphic to the H-orbit decomposition of G/P.
- The Cartan geometry $(\mathcal{G} \to M, \omega)$ reduces to Cartan geometries $(\mathcal{G}_i \to M_i, \omega_i)$ of the same type $(H, H \cap P_i)$ as in the homogeneous model (P_i a conjugate of P)

$\mathbb{V} \supset \mathbb{V}^0 \supset \ldots \supset \mathbb{V}^N$ be the P-inv. filtration, proj. $\Pi : \mathbb{V} \to \mathbb{V}/\mathbb{V}^0 = \mathbb{H}_0$, $\sigma = \Pi(S)$. Then $\sigma(x) = 0$ iff the P-type of S at x is contained in $\mathbb{V}^0 \cap \mathcal{O}$.
$\rightsquigarrow$ zero set of $\sigma = \Pi(S)$ is union of curved orbits.

$\mathbb{V} \supset \mathbb{V}^0 \supset \ldots \supset \mathbb{V}^N$ be the P-inv. filtration, proj. $\Pi : \mathbb{V} \to \mathbb{V}/\mathbb{V}^0 = \mathbb{H}_0$, $\sigma = \Pi(S)$. Then $\sigma(x) = 0$ iff the P-type of S at x is contained in $\mathbb{V}^0 \cap \mathcal{O}$.
$\rightsquigarrow$ zero set of $\sigma = \Pi(S)$ is union of curved orbits.

Conformal Standard Tractors

$G = SO(p+1, q+1)$, P stabilizer of null line ℓ; filtration

$$\mathcal{T} \supset \mathcal{T}^0 = (\mathcal{T}^1)^\perp \supset \mathcal{T}^1.$$

Three essentially different G-types:

- $h(S,S) = 1$, then $H = \mathrm{Stab}_G(s) = SO(p, q+1)$, two P-types: $\{x : S_x \in \mathcal{T}^0\}$ and $\{x : S_x \notin \mathcal{T}^0\}$
- $h(S,S) = -1$, then $H = \mathrm{Stab}_G(s) = SO(p+1, q)$, two P-types: $\{x : S_x \in \mathcal{T}^0\}$ and $\{x : S_x \notin \mathcal{T}^0\}$
- $h(S,S) = 0$ then $H = \mathrm{Stab}_G(s) = SO(p,q) \ltimes \mathbb{R}^{p,q}$, three P-types: $\{x : S_x \in \mathcal{T}^1\}$, $\{x : S_x \notin \mathcal{T}^1$ and $S_x \in \mathcal{T}^0\}$ and $\{x : S_x \notin \mathcal{T}^0\}$

Suppose $h(S, S) > 0$. Geometric structure:

- On M_0: Cartan geometry of type $(SO(p, q+1), SO(p, q))$; from normality of the initial Cartan connection one deduces that one obtains Einstein metric.
- On M_1: normal conformal Cartan geometry of type $(SO(p, q+1), P)$

Suppose $h(S, S) > 0$. Geometric structure:

- On M_0: Cartan geometry of type $(SO(p, q+1), SO(p, q))$; from normality of the initial Cartan connection one deduces that one obtains Einstein metric.
- On M_1: normal conformal Cartan geometry of type $(SO(p, q+1), P)$

Conformal Tractor 3-form of type G_2

Conformal manifold of signature $(2, 3)$ with parallel $\Phi \in \Lambda^3(\mathcal{T}^*)$ s.t.

$$H = \mathrm{Stab}_{SO(3,4)}(\Phi) = G_2.$$

Since $G_2/P \cong SO(3,4)/\tilde{P}$, there is only one P-type and so no non-trivial curved orbit decomposition. One obtains a reg. normal parabolic geometry of type (G_2, P) on all of M.

Almost Einstein (2,3,5) distributions

Suppose the conformal manifold is equipped with

- a parallel tractor 3-form $\Phi \in \Lambda^3(\mathcal{T}^*)$ of type G_2
- and a parallel standard tractor $S \in \Gamma(\mathcal{T})$.

Almost Einstein (2,3,5) distributions

Suppose the conformal manifold is equipped with

- a parallel tractor 3-form $\Phi \in \Lambda^3(\mathcal{T}^*)$ of type G_2
- and a parallel standard tractor $S \in \Gamma(\mathcal{T})$.

Inserting S into Φ gives parallel tractor 2-form $\mathbb{K} = \iota_S \Phi \in \Gamma(\Lambda^2 \mathcal{T}^*)$, which corresponds to normal conformal Killing field $\xi = \Pi(\mathbb{K}) \in \Gamma(TM)$.

Almost Einstein (2,3,5) distributions

Suppose the conformal manifold is equipped with

- a parallel tractor 3-form $\Phi \in \Lambda^3(\mathcal{T}^*)$ of type G_2
- and a parallel standard tractor $S \in \Gamma(\mathcal{T})$.

Inserting S into Φ gives parallel tractor 2-form $\mathbb{K} = \iota_S \Phi \in \Gamma(\Lambda^2 \mathcal{T}^*)$, which corresponds to normal conformal Killing field $\xi = \Pi(\mathbb{K}) \in \Gamma(TM)$.

There are again three different G-types, and

$$H = \mathrm{Stab}_{G_2}(s) = \begin{cases} \mathrm{SL}(3, \mathbb{R}) & \text{iff} \quad h(s,s) < 0 \\ \mathrm{SU}(1,2) & \text{iff} \quad h(s,s) > 0 \\ \mathrm{SL}(2) \ltimes P_+ & \text{iff} \quad h(s,s) = 0 \end{cases}$$

Almost Einstein (2,3,5) distributions

Suppose the conformal manifold is equipped with

- a parallel tractor 3-form $\Phi \in \Lambda^3(\mathcal{T}^*)$ of type G_2
- and a parallel standard tractor $S \in \Gamma(\mathcal{T})$.

Inserting S into Φ gives parallel tractor 2-form $\mathbb{K} = \iota_S \Phi \in \Gamma(\Lambda^2 \mathcal{T}^*)$, which corresponds to normal conformal Killing field $\xi = \Pi(\mathbb{K}) \in \Gamma(TM)$.

There are again three different G-types, and

$$H = \mathrm{Stab}_{G_2}(s) = \begin{cases} \mathrm{SL}(3, \mathbb{R}) & \text{iff} \quad h(s,s) < 0 \\ \mathrm{SU}(1,2) & \text{iff} \quad h(s,s) > 0 \\ \mathrm{SL}(2) \ltimes P_+ & \text{iff} \quad h(s,s) = 0 \end{cases}$$

Geometric structure for $h(S,S) \neq 0$:

- open curved orbit M_0: (g, ξ) defines a (para-) Sasaki-Einstein structure of signature $(2,3)$ (g Einstein metric, ξ is Killing field satisfying $g_{ab}\xi^a\xi^b = 1$ and $\nabla_a \nabla_b \xi^c = \epsilon(g_{ab}\xi^c - \delta^c{}_a \xi_b)$)
- hypersurface M_1: (locally) Fefferman spaces over 3-dim. (para-) CR manifolds

Some references

- Baston, Eastwood, *The Penrose Transform*
- Čap, Slovák, Souček, *Bernstein-Gelfand-Gelfand sequences*
- Calderbank, Diemer, *Differential invariants and curved Bernstein-Gelfand-Gelfand sequences*
- Branson, Čap, Eastwood, Gover, *Prolongations of geometric overdetermined systems*
- Hammerl, Silhan, Somberg, Souček, *On a new normalization for tractor covariant derivatives*
- Nurowski, *Differential equations and conformal structures*
- Hammerl, Sagerschnig,*Conformal structures associated to generic rank 2 distributions on 5-manifolds*
- Čap, Gover, Hammerl, *Holonomy Reductions of Cartan Geometries and Curved Orbit Decompositions*
- Sagerschnig, Willse, *The Geometry of Almost Einstein (2,3,5) Distributions*