Constructions with Parabolic Geometries

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This will be an introductory lecture

Plan for lecture 1

- Rewiew of parabolic geometries
- Tractor bundles and connections

(Some) literature

- Parabolic Geometries I (Čap-Slovák)
- Thomas's Structure Bundle for Conformal, Projective and Related Structures (Bailey, Eastwood, Gover)
- An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity (Curry, Gover)
- Two constructions with parabolic geometries (Čap)

Example: conformal structures

Conformal structure of signature (p,q) $(p+q>2)$: equivalence class of (pseudo-)Riemannian metrics, where

$$\hat{g} \sim g \iff \hat{g} = \Omega^2 g, \quad 0 < \Omega \in C^\infty(M).$$

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Homogeneous model: Consider $\mathbb{R}^{n+2}$ with Lorentzian inner product and the null-cone

$$\mathcal{N} = \{X \in \mathbb{R}^{n+2} \setminus \{0\} : \sum_{i=1}^{n+1} (x^i)^2 - (x^0)^2 = 0\}.$$

The space of lines in $\mathcal{N}$ can be identified with the sphere $\mathbb{S}^n$,

$$\pi : \mathcal{N} \to \mathbb{P}(\mathcal{N}) \cong \mathbb{S}^n. \tag{1}$$

Then $\mathbb{S}^n$ inherits a well-defined conformal structure $[g]$: any section of (1) determines a metric and different sections lead to conformally related metrics (the usual round metric arises from the section $x^0 = 1$).

Example: conformal structures

$SO(n+1,1)$ acts linearly on $\mathbb{R}^{n+2}$ and by isometries, and descends to an action on $\mathbb{P}(\mathcal{N})$ by conformal transformations, which is transitive. This leads to an identification

$$SO(n+1,1)/P \cong \mathbb{P}(\mathcal{N}) \cong \mathbb{S}^n,$$

where $P \subset SO(n+1,1)$ is the parabolic subgroup stabilizing a null-line in $\mathbb{R}^{n+2}$.

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For other signatures, the construction is analogous, one obtains

$$SO(p+1,q+1)/P \cong \mathbb{P}(\mathcal{N}) \cong \mathbb{S}^p \times \mathbb{S}^q/\mathbb{Z}_2$$

Example: (2,3,5) distributions

A <u>(2,3,5) distribution</u> is a rank 2-distribution $\mathcal{D} \subset TM$ on 5-manifold such that

$$[\mathcal{D}, \mathcal{D}] \quad \text{has constant rank 3 and} \quad [\mathcal{D}, [\mathcal{D}, \mathcal{D}]] = TM,$$

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The Lie algebra of infinitesimal symmetries of the rank 2 distribution associated with the Hilbert-Cartan equation $z' = (y'')^2$,

$$\mathcal{D} = \left\langle \partial_x + p\partial_y + q\partial_p + q^2\partial_z, \partial_q \right\rangle,$$

is the exceptional Lie algebra $\mathfrak{g}_2$

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Any $\Phi \in \Lambda^3\mathbb{R}^{7*}$ determines bilinear form

$$H_\Phi(X,Y)\mathrm{vol} = (X \lrcorner \Phi) \wedge (Y \lrcorner \Phi) \wedge \Phi,$$

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Homogeneous model: G_2 acts transitively on null-lines and

$$G_2/P \cong \mathbb{P}(\mathcal{N}) \cong S^2 \times S^3/\mathbb{Z}_2,$$

where $P \subset G_2$ is stabilizer of null-line $\ell \subset \mathbb{R}^{3,4}$.

Consider a semisimple Lie algebra with a grading

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \underbrace{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k}_{\mathfrak{g}_+}$$

compatible with the Lie bracket, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and such that $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$. Then $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is a parabolic subalgebra.

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Example: conformal grading

$$\mathfrak{so}(n+1,1) = \left\{ \begin{pmatrix} \mu & Z^t & 0 \\ Y & M & -Z \\ 0 & -Y^t & -\mu \end{pmatrix} \right\}$$

$$= \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}_{\mathfrak{p}} \qquad\qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

Let G be a Lie group and P a closed subgroup. A Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) is given by

- principal bundle $\mathcal{G} \to M$ with structure group P and
- Cartan connection, i.e., a P-equivariant 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ that maps fundamental vector fields to its generators and defines an isomorphism $\omega : T_u\mathcal{G} \to \mathfrak{g} \ \forall u \in \mathcal{G}$

Homogeneous model: $G \to G/P$ equipped with Maurer Cartan form ω.

A parabolic geometry is a Cartan geometry of type (G, P), where G is semisimple and $P \subset G$ is parabolic.

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Remarks

- Parabolic geometries may have non-trivial automorphisms that equal the identity to first order in a point.
- They do not determine canonical linear connections on the tangent bundle of the manifold.

A Cartan connection induces an identification

$$TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}.$$

The Curvature of the Cartan connection is the 2-form

$$\kappa = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}, \mathfrak{g}).$$

P-equivariant and horizontal and can thus be equivalently viewed as

$$\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$$

The curvature vanishes identically iff the geometry is locally equivalent to the homogeneous model.

For $\mathfrak{p} \subset \mathfrak{g}$ parabolic, $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{g}_+$ as P-modules. Define $\mathfrak{g}^i = \mathfrak{g}_i \oplus \mathfrak{g}_{i+1} \cdots \oplus \mathfrak{g}_k$. Then $(\mathcal{G}, \omega)$ is called

- regular if $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ (homogeneity ≥ 1)
- normal if $\partial^* \kappa = 0$, where $\partial^* : \Lambda^2 \mathfrak{g}_+ \otimes \mathfrak{g} \to \mathfrak{g}_+ \otimes \mathfrak{g}$ denotes the Lie alg. homology differential (*Kostant codifferential*).

Theorem (Tanaka,...)

Equivalence of categories between

$$\{\text{"underlying structures"}\} \longleftrightarrow \{\text{regular, normal parabolic geometries}\}$$

- In most cases, the "underlying structures" admit description as filtered G_0-structures and these can be prolonged to Cartan geometries (see Dennis GRIEG lecture).

- To pass from $(\mathcal{G}, \omega)$ to "underlying structures" form

$$\mathcal{G}_0 = \mathcal{G}/P_+,$$

which has structure group $G_0 \cong P/P_+$, and descend the Cartan connection to a family of partially defined 1-forms. In the conformal case $G_0 = CO(p, q)$ and one recovers the conformal frame bundle.

- A G_0-equivariant section ι of $\mathcal{G} \to \mathcal{G}_0$ is called Weyl structure. Then

$$\iota^*\omega = \iota^*\omega_- + \iota^*\omega_0 + \sigma^*\omega_1 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+).$$

Then $\iota^*\omega_0$ is a principal connection, called Weyl connection. In the conformal case it corresponds to a torsion-free connection ∇ compatible with $[g]$. In this case $\iota^*\omega_-$ and $\iota^*\omega_+$ correspond to the soldering form θ and Schouten tensor

$$\mathsf{P}_{ij} = \tfrac{1}{(n-2)}(Ric_{ij} - \tfrac{1}{2(n-1)} S\, g_{ij}).$$

Given any P-representation $\rho : P \to \mathrm{End}(\mathbb{V})$, one can form associated vector bundles

$$\mathcal{V} = \mathcal{G} \times_P \mathbb{V}.$$

In general, these do not come with induced linear connections (e.g. $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, and other tensor bundles).

However, if one starts with a G-representation, then there are such connections.

Examples

- ρ standard representation of matrix group G $\rightsquigarrow$ standard tractors $\mathcal{T}$
- ρ adjoint representation on $\mathfrak{g}$ $\rightsquigarrow$ adjoint tractors $\mathcal{A}$

Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). Consider the extended principal G-bundle

$$\tilde{\mathcal{G}} = \mathcal{G} \times_P G \to M.$$

$\exists$ unique extension of $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ to principal connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$.

Let $\rho : G \to \mathrm{End}(\mathbb{V})$ be a representation of G. Then

$$\mathcal{V} = \mathcal{G} \times_P \mathbb{V} = \tilde{\mathcal{G}} \times_G \mathbb{V} \to M$$

is called a tractor bundle. The principal connection $\tilde{\omega}$ induces a linear connection $\nabla^{\mathcal{V}}$, called a tractor connection.

Filtration

Any irred. G-representation admits a grading s.t. $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$, and in particular a P-invariant filtration $\rightsquigarrow$ filtration of the corr. tractor bundle:

$$\mathcal{V} \supset \mathcal{V}^0 \supset \cdots \supset \mathcal{V}^N \quad \text{projection } \Pi : \mathcal{V} \to \mathcal{V}/\mathcal{V}^0$$

Example: Conformal standard tractors

Consider standard representation of $G = \mathrm{SO}(n+1,1)$ on $\mathbb{R}^{n+2}$, and $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+2}$

- P-inv. filtration $\mathbb{R}^{n+2} \supset \ell^\perp \supset \ell \rightsquigarrow$ filtration of $\mathcal{T}$:

$$\mathcal{T} \supset \mathcal{T}^0 = (\mathcal{T}^1)^\perp \supset \mathcal{T}^1,$$

- G-inv. $\langle , \rangle \in S^2(\mathbb{R}^{n+2})^* \rightsquigarrow$ tractor metric $h \in S^2\mathcal{T}^*$
- Cartan connection induces tractor connection $\nabla^{\mathcal{T}}$, and $\nabla^{\mathcal{T}} h = 0$.

To write $(\mathcal{T}, \nabla^{\mathcal{T}})$ explicitly, choose a metric $g \in [g]$ with L.C. connection ∇ (or any Weyl connection). It determines a reduction $\sigma : \mathcal{G}_0 \to \mathcal{G}$ from P to G_0 and thus a splitting of the filtration into a direct sum of bundles associated to $\mathcal{G}_0$:

$$\mathcal{T} \cong \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]$$
$$S \mapsto (\sigma, \mu_j, \rho)$$

Changing the metric $\hat{g} = e^{2f} g$ put $\Upsilon = df$, then the identification changes explicitly as follows

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu_a} \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a^b & 0 \\ -\frac{1}{2}\Upsilon^b\Upsilon_b & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix}$$

$\mathcal{E}[w]$ denotes bundle of conformal densities of weight w (choice of metric $g \in$ c trivializes $\mathcal{E}[w]$ and allows to identify densities with functions; changing the metric, these transform as $\hat{f} = \Omega^w f$).

In the splitting given by g, the tractor connection is

$$\nabla_i^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_j \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + \mathrm{g}_{ij}\rho + \mathrm{P}_{ij}\sigma \\ \nabla_i \rho + \mathrm{P}_{ij}\mu^j \end{pmatrix} \tag{2}$$

∇ L.C. connection, P_{ij} Schouten tensor

Remark

- Can define $(\mathcal{T}, \nabla^{\mathcal{T}})$ via (2) and transformation rules. One can recover $(\mathcal{G}, \omega)$ from $(\mathcal{T}, \nabla^{\mathcal{T}})$.

Now suppose $S \in \Gamma(\mathcal{T})$ is a parallel tractor, $\nabla^{\mathcal{T}} S = 0$. Then from (2) we see that

$$S = (\sigma, \nabla_i \sigma, -\tfrac{1}{n}(\nabla^j \nabla_j \sigma - \mathrm{P}_j^j \sigma))$$

and σ is a solution to the equation

$$\text{trace-free part}(\nabla_a \nabla_b \sigma + P_{ab}\sigma) = 0.$$

Conformal-to-Einstein operator

linear confor. invariant operator $D : \Gamma(\mathcal{E}[1]) \to \Gamma(S^2 T^* M \otimes \mathcal{E}[1])$,

$$\Theta(\sigma) = \text{trace-free part}(\nabla_a \nabla_b \sigma + P_{ab}\sigma)$$

Nowhere vanishing solutions $\sigma \in \mathcal{E}_+[1]$ to $\Theta(\sigma) = 0$ correspond to Einstein metrics in the conformal class via $\sigma \mapsto \sigma^{-2}\mathrm{g}$, where g denotes the conformal metric.

Conversely, prolonging the equation $\rightsquigarrow$

$$\begin{aligned}
&\nabla_a \sigma - \mu_a = 0 \\
&\nabla_a \mu_b + \mathrm{P}_{ab}\sigma + g_{ab}\rho = 0 \\
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$$\{\text{solutions } \sigma \text{ to } \Theta(\sigma) = 0\} \xleftrightarrow{1-1} \{\text{parallel standard tractors } \nabla^{\mathcal{T}} S = 0\}$$

via a differential splitting operator $\sigma \mapsto (\sigma, \nabla_i \sigma, -\frac{1}{n}(\nabla^j \nabla_j \sigma - \mathsf{P}^j_j \sigma))$