Frobenius integrability and Cartan geometries

Omid Makhmali

Center for Theoretical Physics
Polish Academy of Sciences
Warsaw

March 7, 2022

GRIEG seminar

Outline of lectures

Lecture one:

(1) Frobenius integrability in 4D conformal structures
(2) Frobenius integrability in 3D conformal structures

Lecture two:

(1) Frobenius integrability in (2,3,5)-geometries
(2) Integrable (2,3,5)-geometries from scalar 4th order ODEs

Lecture three:

(1) Parabolic quasi-contact cone structures and quasi-contactification
(2) Frobenius integrability in quasi-contactified structures

Pfaffian systems and Frobenius integrability

Given a subring $I \subset \bigwedge T^*M$, on a manifold M, I is called an (algebraic) *ideal* if it is closed under wedge product:

$$\alpha \in I \Rightarrow \alpha \wedge \beta \in I \quad \text{for all } \beta \in \bigwedge T^*M$$

I is called a differential ideal, or an *exterior differential system* (EDS), if it is closed under exterior derivative

$$\mathrm{d}I \subset I$$

Finding the differential ideal:

$$I = \langle S \rangle_{alg} \Rightarrow I_{diff} = \langle S, \mathrm{d}S \rangle_{alg}$$

An integral manifold of I is a submanifold $f\colon S \to M$ s.t. $f^* I = 0$.

Example : Solutions $y'' = F(x,y,y')$ in the space (x,y,y') are integ curves of

$I = \langle \mathrm{d}y - y'\mathrm{d}x, \mathrm{d}y' - F(x,y,y')\mathrm{d}x \rangle$ i.e. integral curves of $V = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + F\frac{\partial}{\partial y'}$

Pfaffian systems and Frobenius Integrability

A Pfaffian system is a diff ideal generated by finitely many 1-forms. The notation "mod I" or "modulo I" means *modulo the ideal* I, e.g.

$$\mathrm{d}I \equiv 0 \mod I \Longrightarrow I \text{ is a differential ideal.}$$

Theorem (The Frobenius theorem): Let I be a Pfaffian system on an n-dimensional manifold M, s.t I is Frobenius/completely integrable i.e.

$$I = \langle \theta^1, \cdots, \theta^{n-k} \rangle_{alg}$$

for some constant k in a neighborhood of a point $p \in M$. Then there is a local coordinate $(x^1, \cdots, x^n)$ around p such that

$$I = \langle \mathrm{d}x^{k+1}, \cdots, \mathrm{d}x^n \rangle_{alg}.$$

The maximal k-dimensional integral manifolds of I are given by

$$x^{k+1} = \mathrm{const}, \quad \cdots, \quad x^n = \mathrm{const}.$$

4D conformal geometry of signature (2,2)

Let $[g]$ be the conformal class of $g = \eta^0\eta^3 - \eta^1\eta^2$ with sign $(2,2)$.

The solution of the equivalence problem for $[g]$ can be given as

Cartan (parabolic) geometries $(\pi : \mathcal{G} \to M, \psi)$ of type $(\mathrm{SO}(3,3), P_1)$
$P_1 = \mathrm{CO}(2,2) \ltimes \mathrm{N}^4$: Stabilizer of a null line in $\mathbb{R}^6$ (parabolic subgroup).

$\mathcal{G}$ is the *prolongation* of the $\mathrm{CO}(2,2)$-bundle of frames on M.

The Cartan connection ψ is $\mathfrak{so}(3,3)$-valued 1-form expressed as follows.

4D conformal geometry: structure equations

$$\psi = \begin{pmatrix} -\phi_0 & -\xi_0 & -\xi_1 & -\xi_2 & -\xi_3 & 0 \\ \omega^0 & -\phi_1 & \gamma_1 & \gamma_2 & 0 & \xi_3 \\ \omega^1 & \theta^1 & -\phi_2 & 0 & \gamma_2 & -\xi_2 \\ \omega^2 & \theta^2 & 0 & \phi_2 & \gamma_1 & -\xi_1 \\ \omega^3 & 0 & \theta^2 & \theta^1 & \phi_1 & \xi_0 \\ 0 & -\omega^3 & \omega^2 & \omega^1 & -\omega^0 & \phi_0 \end{pmatrix}$$

which is $\mathfrak{so}(3,3)$-valued wrt to

$$\langle u, w\rangle = u_1 w_6 + w_1 u_6 + u_2 w_5 + w_2 u_5 - u_3 w_4 - w_3 u_4$$

The conformal class $[\mathbf{g}]$ of $\mathbf{g} = \omega^0\omega^3 - \omega^1\omega^2 \in S^2 T^*\mathscr{G}$ is well-defined.

For any section $s\colon M \to \mathscr{G}$, one has $s^*\mathbf{g} \in [g]$.

ω^i's are the *lifted coframe* i.e. at $u \in \mathscr{G}$ with $u_0 = h \in \mathrm{CO}(2,2)$

$$\omega^i_u = (h^{-1})^i_j \pi^* \eta^j.$$

4D conformal geometry: structure equations

The curvature 2-form of the Cartan connection is given by

$$\Psi := \mathrm{d}\psi + \psi \wedge \psi = \begin{pmatrix} 0 & -\Xi_0 & -\Xi_1 & -\Xi_2 & -\Xi_3 & 0 \\ 0 & -\Phi_1 & \Gamma_1 & \Gamma_2 & 0 & \Xi_3 \\ 0 & \Theta^1 & -\Phi_2 & 0 & \Gamma_2 & -\Xi_2 \\ 0 & \Theta^2 & 0 & \Phi_2 & \Gamma_1 & -\Xi_1 \\ 0 & 0 & \Theta^2 & \Theta^1 & \Phi_1 & \Xi_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Weyl curvature and Cotton-York “tensor” of $[g]$ are given by

$$\begin{aligned}
\Theta^1 &= -a_4\omega^0\wedge\omega^2 - a_3\omega^0\wedge\omega^3 - a_3\omega^1\wedge\omega^3 - a_2\omega^1\wedge\omega^3 \\
\tfrac{1}{2}(\Phi_2-\Phi_1) &= a_3\omega^0\wedge\omega^2 + a_2\omega^0\wedge\omega^3 + a_2\omega^1\wedge\omega^2 + a_1\omega^1\wedge\omega^3 \\
\Gamma_1 &= a_2\omega^0\wedge\omega^2 + a_1\omega^0\wedge\omega^3 + a_1\omega^1\wedge\omega^2 + a_0\omega^1\wedge\omega^3 \\
\Theta^2 &= -b_4\omega^0\wedge\omega^1 - b_3\omega^0\wedge\omega^3 + b_3\omega^1\wedge\omega^2 - b_2\omega^2\wedge\omega^3 \\
-\tfrac{1}{2}(\Phi_1+\Phi_2) &= b_3\omega^0\wedge\omega^1 + b_2\omega^0\wedge\omega^3 - b_2\omega^1\wedge\omega^2 + b_1\omega^2\wedge\omega^3 \\
\Gamma_2 &= b_2\omega^0\wedge\omega^1 + b_1\omega^0\wedge\omega^3 - b_1\omega^1\wedge\omega^2 + b_0\omega^2\wedge\omega^3 \\
\Xi_i &= \tfrac{1}{2}C_{ijk}\omega^j\wedge\omega^k, \qquad C_{[ijk]} = C^i_{ij} = 0,\ C_{ijk} = -C_{ikj}
\end{aligned}$$

Twistor bundles: $\mathbb{S}^1$-bundles of null planes

A 2-plane $p \subset T_xM$ is called null if $g|_p = 0$.

$$g|_p = 0 \Rightarrow \eta^0\eta^3 - \eta^1\eta^2 = 0 \Rightarrow \eta^0\eta^3 = \eta^1\eta^2.$$

The null cone of g is the indefinite quadric which is *doubly ruled*.
More explicitly, there are two families of null planes

$$\alpha\text{-planes (ASD)}: \frac{\eta^0}{\eta^1} = \frac{\eta^2}{\eta^3} = \alpha \in \mathbb{R} \cup \{\infty\} \Rightarrow p_\alpha = \ker I_\alpha, \ I_\alpha := \{\eta^0 - \alpha\eta^1, \eta^2 - \alpha\eta^3\}$$

$$\beta\text{-planes (SD)}: \frac{\eta^0}{\eta^2} = \frac{\eta^1}{\eta^3} = \beta \in \mathbb{R} \cup \{\infty\} \Rightarrow p_\beta = \ker I_\beta, \ I_\beta := \{\eta^0 - \beta\eta^2, \eta^1 - \beta\eta^3\}.$$

Similarly, on $\mathscr{G}$ defining

$$\bar{I}_\alpha := \{\omega^0 - \alpha\omega^1, \omega^2 - \alpha\omega^3\}, \qquad \bar{I}_\beta := \{\omega^0 - \beta\omega^2, \omega^1 - \beta\omega^3\},$$

one can regard I_α and I_β and $s^*\bar{I}_\alpha$ and $s^*\bar{I}_\beta$ for some section $s\colon M \to \mathscr{G}$.
i.e. p_α, p_β are the projection of corank 2 planes $\bar{p}_\alpha = \ker \bar{I}_\alpha$, $\bar{p}_\beta = \ker \bar{I}_\beta$.

The *twistor bundles* $\mathscr{N}_\alpha$ ($\mathscr{N}_\beta$) are the $\mathbb{S}^1$-bundles of α-planes(β-planes).
We will switch between I and $\bar{I}$ as they will be interchangeable for us.

Twistor bundles from the structure bundle

Affine parameters α and β represent two parameters in $G_0 = \mathrm{CO}(2,2)$:

$$G_0 = \begin{pmatrix} \frac{f_0}{f_1} & g_1 & g_2 & g_1 g_2 \frac{f_1}{f_0} \\ t_1 & \frac{f_0}{f_2} & t_1 g_2 \frac{f_1}{f_0} & g_2 \frac{f_1}{f_2} \\ t_2 & g_1 t_2 \frac{f_1}{f_0} & f_0 f_2 & g_1 f_1 f_2 \\ t_1 t_2 \frac{f_1}{f_0} & t_2 \frac{f_1}{f_2} & t_1 f_1 f_2 & f_0 f_1 \end{pmatrix}$$

with Lie algebra

$$\mathfrak{g}_0 = \begin{pmatrix} \phi_0 - \phi_1 & \gamma_1 & \gamma_2 & 0 \\ \theta^1 & \phi_0 - \phi_2 & 0 & \gamma_2 \\ \theta^2 & 0 & \phi_0 + \phi_2 & \gamma_1 \\ 0 & \theta^2 & \theta^1 & \phi_0 + \phi_1 \end{pmatrix}$$

Let A_{g_1} be the 1-dimensional subgroup parameterized by g_1.
A_{g_1} acts on $\mathscr{G}$ and $\psi \to A_{g_1}^{-1} \psi A_{g_1}$

$$(\omega^0, \omega^2) \to (\omega^0 - g_1 \omega^1, \omega^2 - g_1 \omega^3), \quad (\omega^1, \omega^3) \to (\omega^1, \omega^3)$$

So g_1 parameterizes α-planes. Similarly, g_2 parameterizes β-planes.

Projective structure on the fibers of the twistor bundles

Consequently, $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ can be regarded as the *leaf space* of $\langle\omega^0,\cdots,\omega^3,\gamma_1\rangle$ and $\langle\omega^0,\cdots,\omega^3,\gamma_2\rangle$, respectively.

If $1/\alpha$ and $1/\beta$ are taken as parameters, one can identify $\mathcal{N}_{1/\alpha}$ and $\mathcal{N}_{1/\beta}$ as the leaf space of $\langle\omega^0,\cdots,\omega^3,\theta^1\rangle$ and $\langle\omega^0,\cdots,\omega^3,\theta^2\rangle$.

The $\mathbb{S}^1$ fibers of $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ are the integral curves of $I_\omega=\{\omega^0,\cdots,\omega^3\}$ over which γ_1 and γ_2 are a *differential*.
Using the structure equations modulo I_ω one obtains

$$\mathrm{d}\gamma_1\equiv-(\phi_2-\phi_1)\wedge\gamma_1,\quad \mathrm{d}(\phi_2-\phi_1)\equiv-\gamma_1\wedge\theta^1,\quad \mathrm{d}\theta_1\equiv(\phi_2-\phi_1)\wedge\theta^1,$$

i.e. each fiber has a *projective structure* $\begin{pmatrix}\frac{1}{2}(\phi_1-\phi_2) & \theta^1\\ \gamma_1 & -\frac{1}{2}(\phi_1-\phi_2)\end{pmatrix}$

Simiarly, fibers of $\mathcal{N}_\beta$ have projective structure $\begin{pmatrix}\frac{1}{2}(\phi_1+\phi_2) & \theta^2\\ \gamma_2 & -\frac{1}{2}(\phi_1+\phi_2)\end{pmatrix}$

Tautological rank 2 distribution on the twistor bundles

$\mathcal{N}_\alpha$ has a tautologically induced rank 2 distribution $\mathscr{D}$.

At $p \in \mathcal{N}_\alpha$, $\mathscr{D}_p :=$ The *horizontal lift* of $p \subset T_x M$ wrt any Weyl connection.

A coframe $\{\zeta^1, \zeta^2, \zeta^3, \eta^1, \eta^3\}$ on $\mathcal{N}_\alpha$ adapted to $\mathscr{D}$ is

$$\zeta^1 = \eta^0 - \alpha\eta^1, \quad \zeta^2 = \eta^2 - \alpha\eta^3, \quad \zeta^3 = \mathrm{d}\alpha - \alpha^2\theta^1 - \alpha(\phi_1 - \phi_2) + \gamma_1$$

where $\mathscr{D} = \ker I_\zeta$, $I_\zeta = \langle \zeta^1, \zeta^2, \zeta^3 \rangle$, $\mathrm{d}\zeta^1 \equiv \eta^1 \wedge \zeta^3, \mathrm{d}\zeta^2 \equiv \eta^3 \wedge \zeta^3$ mod $\langle \zeta^1, \zeta^2 \rangle$

$$\mathrm{d}\zeta^3 \equiv (a_4\alpha^4 + 4a_3\alpha^3 + 6a_2\alpha^2 + 4a_1\alpha + a_0)\eta^1 \wedge \eta^3 \quad \mathrm{mod} \ \ I_\zeta$$

Wherever $C_\alpha = a_4\alpha^4 + 4a_3\alpha^3 + 6a_2\alpha^2 + 4a_1\alpha + a_0 \neq 0$, $\mathscr{D}$ is (2,3,5).

Alternatively, viewing $\mathcal{N}_\alpha$ as leaf space of $\langle \omega^0, \omega^1, \omega^2, \omega^3, \gamma_1 \rangle$, one has

$$\mathrm{d}\omega^0, \mathrm{d}\omega^2 \equiv 0 \quad \mathrm{d}\gamma_1 \equiv a_0\omega^1 \wedge \omega^3, \quad \mathrm{mod} \ \ \langle \omega^0, \omega^2, \gamma_1 \rangle$$

$\mathscr{D} := \ker\{\omega^0, \omega^2, \gamma_1\}$ is (2,3,5) when $a_0 \neq 0$ i.e. $\alpha = 0$ is not a root of C_α.

The structure group acts transitively on the roots of C_α.

The condition $a_0 = 0$ gives additional coframe adaptation $C_\alpha|_{\alpha=0} = 0$.

Coframe on $\mathcal{N}_\alpha$ arise from $\psi \to A_{g_1}^{-1}\psi A_{g_1} + A_{g_1}^{-1}\mathrm{d}A_{g_1}$ using $s\colon M \to \mathscr{G}$.

Principal null planes and self-duality

$p \in \mathcal{N}_\alpha$ is principal null plane if $\mathcal{D}$ is not bracket generating at p.
Alternatively, $p \in \mathcal{N}_\alpha$ is principal if $s^* W^-(\mathcal{D}_p) = 0$, where $s\colon \mathcal{N}_\alpha \to \mathcal{G}$

$$W^- = a_0(\omega^1)^4 + 4a_1(\omega^0)(\omega^1)^3 + 6a_2(\omega^0)^2(\omega^1)^2 + 4a_3(\omega^0)^3\omega^1 + a_4(\omega^0)^4.$$

Exercise: W^- is not well-defined on $\mathcal{N}_\alpha$ but its vanishing set on $\mathcal{D}$ is.
Moreover, W^- is an invariant weighted quartic on $\mathcal{N}_\beta$.
More explicitly, let

$$V = f_0\partial_{\phi_0} + f_1\partial_{\phi_1} + f_2\partial_{\phi_2} + g_2\partial_{\gamma_2} + t_1\partial_{\theta^1} + t_2\partial_{\theta^2} + x^i\partial_{\xi_i}$$

be an infinitesimal generator of a fiber action in $\mathcal{G} \to \mathcal{N}_\alpha$, then

$$\mathcal{L}_V W^-(\mathcal{D}_\alpha) = (2(f_2 + f_1 - f_0)v_1^4 - 4g_2 v_1^3 v_2)C_\alpha.$$

where $\mathcal{D}_\alpha = \mathrm{span}\{V_1, V_2\}$ with $V_1 = v_1(\alpha\partial_{\omega^0} + \partial_{\omega^1}), V_2 = v_2(\alpha\partial_{\omega^2} + \partial_{\omega^3})$

$[g]$ is *self-dual* if $W^- = 0 \Leftrightarrow$ all α-planes are principal $\Leftrightarrow \mathcal{D}$ is integrable.

Then $\mathcal{N}_\alpha$ is foliated by null surfaces parametrized by g_1. The quartic W^+ gives the curvature of the resulting torsion-free path geometry.

Integrability by null surfaces: necessary condition

The existence of an integrable distribution of α-planes means I_{α_0} is Frobenius integrable for some $\alpha_0 \colon M \to \mathcal{N}_\alpha$. Let us find $\langle I_{\alpha_0}\rangle_{diff}$:

$$\mathrm{d}I_{\alpha_0} \equiv 0 \bmod I_{\alpha_0} = \langle \eta^0 - \alpha_0\eta^2, \eta^1 - \alpha_0\eta^3\rangle \Rightarrow \mathrm{d}\alpha_0 \equiv \alpha_0^2\theta^1 + \alpha_0(\phi_1 - \phi_2) - \gamma_1, \quad (1)$$

mod I_{α_0}. Identify α_0 with some g_1 and use $\psi \to A_{g_1}^{-1}\psi A_{g_1} + A_{g_1}^{-1}\mathrm{d}A_{g_1}$ to get

$$\gamma_1 \to \mathrm{d}\alpha_0 - \alpha_0^2\theta^1 - \alpha_0(\phi_1 - \phi_2) + \gamma_1$$

i.e. (1) simply means a reduction of γ_1 for some further adaptation. Another differentiation gives

$$(a_4\alpha_0^4 + 4a_3\alpha_0^3 + 6a_2\alpha_0^2 + 4a_1\alpha_0 + a_0)\eta^1 \wedge \eta^3 = 0.$$

Similarly, the existence of a foliation of M by β-surface implies

$$\mathrm{d}\beta_0 \equiv \beta_0^2\theta^2 + \beta_0(\phi_1 + \phi_2) - \gamma_2, \quad \bmod \ I_{\beta_0}$$
$$(b_4\beta_0^4 + 4b_3\beta_0^3 + 6b_2\beta_0^2 + 4b_1\beta_0 + b_0)\eta^2 \wedge \eta^3 = 0$$

for a section $\beta_0 \colon M \to \mathcal{N}_\beta$.

Integrability by null surfaces: sufficient conditions

Let $\alpha_0 \colon \mathscr{G} \to \mathscr{N}_\alpha$ be a principal α-plane s.t. α_0 is a double root of C_α.
Define $\mathscr{G}_1 \subset \mathscr{G}$ such that $\alpha_0 = 0$ i.e.

$$\mathscr{G}_1 = \{u \in \mathscr{G} \mid a_0(u) = a_1(u) = 0\}.$$

α-plane for $\alpha = 0$ is $\ker\{\omega^0, \omega^2\}$ and *preserved* by $G_1 \subset G_0$ where $g_1 = 0$.
We have an $\{e\}$-structure $(\pi \colon \mathscr{G}_1 \to M, \psi_1)$ where $\psi_1 = \psi|_{\gamma_1 = 0}$.
To obtain the structure equations find γ_1 :

$$\mathrm{d}a_0 = a_0(2\phi_0 + 2\phi_1 - 2\phi_2) + 4a_1\gamma_1 + a_{0;i}\omega^i$$
$$\mathrm{d}a_1 = a_0\theta^1 + a_1(2\phi_0 + \phi_1 - \phi_2) + 3a_2\gamma_1 + a_{1;i}\omega^i$$

Restrict to $\mathscr{G}_1$

$$\gamma_1 = -\tfrac{1}{3a_2} a_{1;i}\omega^i \equiv \tfrac{1}{3a_2}(C_{113}\omega^1 + C_{313}\omega^3) \mod \{\omega^0, \omega^2\}$$

where $\Xi_i = \frac{1}{2} C_{ijk}\omega^j \wedge \omega^k$.

$$\Rightarrow \mathrm{d}\omega^0 \equiv \tfrac{1}{3a_2} C_{313}\omega^1 \wedge \omega^3, \qquad \mathrm{d}\omega^2 \equiv -\tfrac{1}{3a_2} C_{113}\omega^1 \wedge \omega^3 \mod \{\omega^0, \omega^2\}$$

Conformal Goldberg-Sachs theorem

The Cotton-York "tensor" is the $\mathbb{R}^4$-valued 2-form $Y := (\Xi_0, \cdots, \Xi_3)$ and

$$Y(\tfrac{\partial}{\partial\omega^1}, \tfrac{\partial}{\partial\omega^3}) = (C_{013}, C_{113}, C_{213}, C_{313}).$$

Restricted to $\mathscr{G}_1$ the vanishing of (C_{113}, C_{313}) is invariantly defined.

Also, there is a unique co-dim 2 subbdle $\mathscr{G}_3 \subset \mathscr{G}_1$ s.t. (C_{013}, C_{213}) is zero.

Theorem (Conformal GS theorem): If $\alpha_0 : M \to \mathscr{N}_\alpha$ is principal null plane, any two of the following conditions implies the third.

(1) α_0 is a repeated root of C_α.

(2) The Cotton-York tensor is degenerate on α_0 for some $g \in [g]$.

(3) α_0 is integrable.

One only needs to show $(2) + (3) \to (1)$. Proof goes by contradiction. Assume α_0 is not repeated. Define

$$\mathscr{G}_1 = \{u \in \mathscr{G} \mid a_0(u) = 0\}.$$

The integrability implies $\gamma_1 \equiv 0$ modulo $\langle\omega^0, \omega^2\rangle$.

Conformal Goldberg-Sachs theorem

Requiring that the Cotton tensor vanishes on p_{α_0} results in a unique 4D reduction to $\mathscr{G}_5 \subset \mathscr{G}_1$ in which

$$\xi_0, \xi_1, \xi_2, \xi_3 \equiv 0 \mod \langle \omega^0, \cdots, \omega^3 \rangle.$$

Thus one obtains a Weyl structure $([g], \nabla)$ for which

$$Ric_{ij} = Ric_{(ij)} + Ric_{[ij]}, \qquad Ric_{[24]} = 20 a_1.$$

$Ric_{[ij]} = 0$ implies that the Cotton tensor corresponds to a metric $g \in [g]$.

Remarks :

(1) In the direction $(1) + (3) \to (2)$ one also needs to require $Ric_{[ij]} = 0$ which can always be done.

(2) In the Akivis-Goldberg book (1996) and Grossman's article (Selecta 2000), there is a theorem claiming that $(1) \to (3)$ in which they only check the necessary conditions!

(3) What are examples of Petrov type I with an α-foliation? Akivis-Goldberg claim recurrent conformal structures of type I are such examples. However, being recurrent seems to imply that Petrov type cannot be generic (c.f McLenaghan-Leroy 1972).

3D conformal geometry

The conformal geometry of $g = (\eta^2)^2 - 2\eta^1\eta^3$ is a Cartan geometry $(\pi\colon \mathcal{G} \to M, \psi)$ of type $(\mathrm{SO}(2,3), P_1)$ for which

$$\psi = \begin{pmatrix} -\phi_2 & \xi_1 & \xi_2 & \xi_3 & 0 \\ \omega^1 & -\phi_1 & \gamma_1 & 0 & \xi_3 \\ \omega^2 & \theta^1 & 0 & \gamma_1 & -\xi_2 \\ \omega^3 & 0 & \theta^1 & \phi_1 & \xi_1 \\ 0 & \omega^3 & -\omega^2 & \omega^1 & \phi_2 \end{pmatrix}$$

which is $\mathfrak{so}(2,3)$-valued wrt to

$$\langle u, w \rangle = u_1 w_5 + w_1 u_5 - u_2 w_4 - w_2 u_4 + u_3 w_3$$

The conformal class $[\mathbf{g}]$ of $\mathbf{g} = (\omega^2)^2 - 2\omega^1\omega^3 \in S^2 T^*\mathcal{G}$ is well-defined and $s^*\mathbf{g} \in [g]$ for any section $s\colon M \to \mathcal{G}$.

3D conformal geometry: structure equations

The Cartan curvature is given by

$$\Psi = \mathrm{d}\psi + \psi \wedge \psi = \begin{pmatrix} 0 & \Xi_1 & \Xi_2 & \Xi_3 & 0 \\ 0 & 0 & 0 & 0 & \Xi_3 \\ 0 & 0 & 0 & 0 & -\Xi_2 \\ 0 & 0 & 0 & 0 & \Xi_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the Cotten-York tensor is an $\mathbb{R}^3$-valued 2-form (Ξ_1, Ξ_2, Ξ_3)

$$\begin{aligned} \Xi_1 &= 2a_2\omega^2 \wedge \omega^3 - 4a_3\omega^1 \wedge \omega^3 - 4a_4\omega^1 \wedge \omega^2 \\ \Xi_2 &= -2a_1\omega^2 \wedge \omega^3 - 4a_2\omega^1 \wedge \omega^3 - 4a_3\omega^1 \wedge \omega^2 \\ \Xi_3 &= -4a_0\omega^2 \wedge \omega^3 - 2a_1\omega^1 \wedge \omega^3 - 2a_2\omega^1 \wedge \omega^2. \end{aligned}$$

The structure algebra and structure group are:

$$\mathfrak{g}_0 = \begin{pmatrix} \phi_2 - \phi_1 & \gamma_1 & 0 \\ \theta^1 & \phi_2 & \gamma_1 \\ 0 & \theta^1 & \phi_1 + \phi_2 \end{pmatrix} \qquad G_0 = \begin{pmatrix} \frac{f_2}{f_1} & g_1 & \frac{f_1}{2f_2} g_1^2 \\ t_1 & f_2 + \frac{f_1}{2f_2} g_1 t_1 & g_1 f_1 \\ \frac{f_1}{2f_2} t_1^2 & t_1 f_1 & f_1 f_2 \end{pmatrix}$$

Integrability by null surfaces: necessary condition

A 2-plane $p \in T_xM$ is null if $Ann(p) \subset T^*_x M$ is null i.e. $p = \ker \eta_\alpha$ where

$$\eta_\alpha = \eta^1 + \alpha\eta^2 + \tfrac{1}{2}\alpha^2\eta^3, \qquad \alpha \in \mathbb{R} \cup \{\infty\}.$$

If $\alpha_0 \colon M \to \mathcal{N}_\alpha$ is integrable then

$$\mathrm{d}\eta_{\alpha_0} \equiv 0 \mod \eta_{\alpha_0} \Rightarrow \mathrm{d}\alpha_0 + \tfrac{1}{2}\alpha_0^2\theta^1 - \alpha_0\phi_1 - \gamma_1 - x_3(\eta^2 + \alpha\eta^3) \equiv 0 \mod \langle \eta_{\alpha_0} \rangle,$$

for some x_3. This is the transformation of γ_1 under change of gauge $\psi \to A^{-1}\psi A + A^{-1}\mathrm{d}A$ arising from the action of the 2D normal subgroup $A = A_{g_1,x_3} \subset P_1$, where $g_1 = \alpha_0$ and g_1, x_3 correspond to conn forms γ_1, ξ_3. Two more differentiation gives

$$\mathrm{d}x_3 - x_3(\phi_2 + \phi_1) + \alpha_0 x_3\theta^1 - x_3^2\eta^3 + \tfrac{1}{2}\alpha_0^2\xi_1 - \alpha_0\xi_2 + \xi_3 \equiv 0 \mod \langle \eta_{\alpha_0} \rangle$$
$$(a_4\alpha_0^4 + 4a_3\alpha_0^3 + 6a_2\alpha_0^2 + 4a_1\alpha_0 + a_0)\eta^2 \wedge \eta^3 = 0.$$

The first of which is the change in ξ_3 after acting by A_{g_1,x_3}.

Twistor bundle of null planes

Let $\mathscr{T} \to M$ be the 5D leaf space of $\langle \omega^1, \omega^2, \omega^3, \gamma_1, \xi_3 \rangle$ whose 2D fibers over M is the normal subgroup of P_1 parametrized by g_1, x_3.

The parameter g_1 can be identified with α.

At each point of $\mathscr{T}$, define adapted coframe $\{\zeta_1, \zeta_2, \zeta_3, \eta^2, \eta^3\}$ where

$$\zeta_1 = \eta^1 + \alpha\eta^2 + \tfrac{1}{2}\alpha^2\eta^3, \quad \zeta_2 = \mathrm{d}\alpha + \cdots, \quad \zeta_3 = \mathrm{d}x_3 + \cdots,$$

$\mathscr{T}$ has a rank 2 distribution $\mathscr{D} = \ker I_\zeta$, $I_\zeta := \{\zeta_1, \zeta_2, \zeta_3\}$ satisfying

$$\mathrm{d}\zeta_1, \mathrm{d}\zeta_2 \equiv 0, \quad \mathrm{mod} \ \ I_\zeta$$
$$\mathrm{d}\zeta_3 \equiv C_\alpha \eta^2 \wedge \eta^3 \quad \mathrm{mod} \ \ I_\zeta, \quad C_\alpha = a_4\alpha^4 + 4a_3\alpha^3 + 6a_2\alpha^2 + 4a_1\alpha + a_0$$

Wherever $C_\alpha \neq 0$, $\mathscr{D}$ has growth $(2,3,4,5)$ for a 4th order ODE. Alternatively, this can be seen form the str eqns for $(\omega^1, \omega^2, \omega^3, \gamma_1, \xi_3)$.

$p_\alpha \in \mathscr{N}$ is principal if $C_\alpha = 0$. If all null planes are principal, $[g]$ is flat.

Integrability by null surfaces: sufficient conditions

Let C_α have type III or N and for $k \geq 2$ and define

$$\mathscr{G}_1 = \{u \in \mathscr{G} \mid a_0(u) = \cdots = a_k(u) = 0\}.$$

This gives a 1D reduction of $\mathscr{G}$. By $\mathrm{d}a_2 = 0$ and $\mathrm{d}^2 = 0$ for $k = 2$ one has

$$\gamma_1 = -\frac{1}{a_3}(a_{3;2} + a_{4;3})\omega^1 - \frac{1}{2a_3}a_{3;3}\omega^2.$$

One can easily verify modulo $\langle \omega^1 \rangle$

$$\mathrm{d}\omega^1 \equiv 0, \quad \text{and} \quad \gamma_1 \equiv 0$$

because $\mathrm{d}\left(\frac{a_{3;3}}{2a_3}\right) \equiv \frac{a_{3;3}}{2a_3}(\phi_1 + \phi_2) - \xi_3$ mod I_ω. Similarly for type N.

Proposition : Repeated principal null planes for 3D conformal structure of type III and N are always integrable and the local generality of such structures depends on 3 and 2 functions of 2 variables, respectively.

Type II and D

For type II and D the vanishing of one and two scalars are required to ensure integrability. The local generality of such structures is given by 4 functions of 2 variables and 3 constants, respectively.

For type D assume there is a double root at 0 and ∞. The str bdle reduces by 2D and the str eqns become a closed system after two prolongations with 8 scalars. Assuming genericity, reduce further to

$$\mathrm{d}\omega^1 = 0, \quad \mathrm{d}\omega^2 = z_1\omega^1 \wedge \omega^3, \quad \mathrm{d}\omega^3 = z_2\omega^1 \wedge \omega^3$$
$$\mathrm{d}z_1 = \tfrac{2}{3}\omega^3 + \tfrac{4}{3}z_3\omega^3, \quad \mathrm{d}z_3 = z_2 z_3\omega^1 + \tfrac{1}{2}z_2\omega^3$$
$$\mathrm{d}z_2 = -\tfrac{1}{3z_1}z_3(3z_1^3 - 2z_2 - 12)(2\omega^1 + \omega^3).$$

It can be checked that they have have 2D symmetry i.e.

$$\mathscr{L}_V\omega^i = 0 \Rightarrow V = v_1\tfrac{\partial}{\partial\omega^1} - 2v_1 z_3\tfrac{\partial}{\partial\omega^3} + v_2\tfrac{\partial}{\partial\omega^2}, \quad \mathrm{d}v_1 = 0, \quad \mathrm{d}v_2 = -v_1 z_1(2z_3\omega^1 + \omega^3)$$

Such conformal structures depend on 3 constants by an application of the Frobenius theorem:

Frobenius theorem for a closed system

On $M \times \mathbb{R}^3$, where (z_1, z_2, z_3) are coordinates for $\mathbb{R}^3$, define

$$(\omega^1, \omega^2, \omega^3, \zeta_1, \zeta_2, \zeta_3)$$

where

$$\zeta_1 = \mathrm{d}z_1 - \tfrac{2}{3}\omega^3 - \tfrac{4}{3}z_3\omega^3, \quad \zeta_3 = \mathrm{d}z_3 - z_2 z_3 \omega^1 - \tfrac{1}{2}z_2\omega^3$$
$$\zeta_2 = \mathrm{d}z_2 + \tfrac{1}{3z_1} z_3 (3z_1^3 - 2z_2 - 12)(2\omega^1 + \omega^3).$$

By str eqns for ω^i's, the ideal $I_z = \langle \zeta_1, \zeta_2, \zeta_3 \rangle$ is Frobenius.

Thus tere is a local coordinate chart in which $I_z = \langle \mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3 \rangle$ and the corresponding integrable conformal structures of type D are locally parametrized by

$$x_1 = \text{const}, \quad x_2 = \text{const}, \quad x_3 = \text{const}.$$

References

- A. R. Gover, C. D. Hill, and P. Nurowski. Sharp version of the Goldberg-Sachs theorem. *Ann. Mat. Pura Appl.* (4), 190(2):295340, 2011.
- P. Nurowski and A. Taghavi-Chabert. A Goldberg-Sachs theorem in dimension three. *Class. Q. Gravity* 32.11 (2015), pp. 115009, 36.
- M. A. Akivis and V. V. Goldberg. *Conformal differential geometry and its generalizations*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1996. A Wiley-Interscience Publication.