Frobenius integrability and Cartan geometries

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Outline of lectures

Lecture one:

(1) Frobenius integrability in 4D conformal structures
(2) Frobenius integrability in 3D conformal structures

Lecture two:

(1) Frobenius integrability in (2,3,5)-geometries
(2) Integrable (2,3,5)-geometries from scalar 4th order ODEs

Lecture three:

(1) Parabolic quasi-contact cone structures and quasi-contactification
(2) Frobenius integrability in quasi-contactified structures

Review: Variational 4th order ODE $\iff$ (2,3,5)-geometry + infin symm

Scalar 4th ODEs define Cartan geom $(\mathscr{G}^8 \to M^5, \psi)$ of type $(\mathrm{GL}_2 \ltimes \mathbb{R}^4, B)$. (Following Morimoto, Doubrov, Komrakov, Čap, The.)

Their fundamental invariants are four scalars $w_0, w_1, c_0, c_1.$

Furthermore, M^5 has an *almost conformally quasi-symplectic structure* $[\rho], \rho \wedge \rho \neq 0$ for which the solutions curves are characteristics.

An ACQS structure is *conf q-sympl (CQS)* if $[\rho]$ has a closed representative which for 4th order ODEs implies $c_1 = w_1 = 0.$

Locally, choosing a symp 2-form $\rho_0 \in [\rho]$ one has

$$\mathrm{d}\rho_0 = 0 \Rightarrow \rho_0 = \mathrm{d}\omega^4.$$

On $\tilde{M}^6 = M \times \mathbb{R}$ define $\tilde{\omega}^4 = \mathrm{d}t + \pi^* \omega \Rightarrow \omega^4$ which is *quasi-contact* on $\tilde{M}$. Define $\tilde{\mathscr{G}}^9 \to \tilde{M}^6$ as the *pull-back bundle* of $\mathscr{G}^8 \to M^5$. By the scaling action induced on $\tilde{\omega}^4$, we lift $\tilde{\omega}^4$ to $\tilde{\mathscr{G}}$ and $(\tilde{\omega}, \tilde{\psi})$ gives an $\{e\}$-str on $\tilde{\mathscr{G}}$ for a Cartan geometry of type (P_2, B), where $P_2 \subset G_2$.

Review: Variational 4th order ODE $\Longleftrightarrow$ (2,3,5)-geometry + infin symm

The underlying str of 4th order ODEs is (2,3,4,5)-dist $\mathscr{D}$ with a splitting:

$$\mathfrak{g} = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}.$$

$\mathfrak{g}_-$ agrees with the underlying str of *regular* (G_2, P_{12}) *truncated* at $\mathfrak{g}_{-5}$.

Extend $\tilde{\mathscr{G}}$ to a (G_2, P_{12}) geometry which always descends to a (2,3,5)-geometry and $\tilde{M}^6$ can be viewed as the *cone structure* $\mathbb{P}\mathscr{D} \to N^5$.

By viewing $\mathfrak{g} = \mathfrak{gl}_2 \oplus. \mathbb{R}^4$ as a subspace in $\tilde{\mathfrak{g}} = \mathfrak{g}_2$ we can pull back the inner product (Killing+involution) to $\mathfrak{g}$.

Following Morimoto-Doubrov-Čap-The, use this inner product to obtain a codifferential operator ∂^* as the adjoint of the Lie algebra cohomology diff ∂ to define regular normal Cartan conn.

By this recipe the only modification of the normal Cartan conn of $(2,3,5)$ pulled-back to $\tilde{\mathcal{G}}^9$ involves the quasi-contact form $\tilde{\omega}^4$:

$$\tilde{\omega}^3=\omega^3,\quad \tilde{\omega}^2=\omega^2,\quad \tilde{\omega}^1=\omega^1,\quad \tilde{\theta}^1=\theta^1\quad \tilde{\omega}^0=\omega^0-\tfrac{3}{16}c_0\tilde{\omega}^4,$$

$$\tilde{\phi}_0=\phi_0+\tfrac{9}{64}c_{0;0}\tilde{\omega}^4,\quad \tilde{\phi}_1=\phi_1-\frac{9}{64}c_{0;0}\tilde{\omega}^4,\quad \tilde{\xi}_0=\xi_0-(\frac{9}{32}c_{0;00}+\frac{1}{8}w_{0;\underline{11}})\tilde{\omega}^4.$$

The Cartan quartic is given by

$$a_4=w_0,\;\; a_3=\tfrac{1}{4}w_{0;\underline{1}},\;\; a_2=\tfrac{1}{12}w_{0;\underline{11}},\;\; a_1=\tfrac{1}{24}w_{0;\underline{111}},\;\; a_0=\tfrac{1}{24}w_{0;\underline{1111}},\;\; w_{0;\underline{11111}}=0$$

where $f_{;\underline{1}}$ denotes differentiation along the vertical v.f. wrt $\mathbb{P}\mathcal{D}\to N$.
Moreover, wrt the conformal metric

$$g=\tilde{\omega}^4\tilde{\omega}^0-\tilde{\omega}^1\tilde{\omega}^3+\tfrac{4}{3}(\tilde{\omega}^2)^2$$

the transversal infinitesimal symmetry $\frac{\partial}{\partial\tilde{\omega}^4}$ is *null* iff $c_0=0.$

- If $c_0=0\Rightarrow w_{0,\underline{111}}=0$ then the (2,3,5)-geom is 3-integrable with null symmetry and its Cartan quartic has type II.
- If $c_0=w_{0;\underline{11}}=0$ then (2,3,5)-geom has holonomy reduction to P_2 with null symmetry and its Cartan quartic has type III.

Conversely, if there is a transv. infin. symm. v then there is reduction of the (G_2, P_{12}) to a (P_2, B)-geometry. In particular, choose a q-cont form s.t. $\theta(v) = 1$. Being transversal infin. symm implies for all $w \in \ker(\theta)$

$$0 = \theta([v,w]) = \mathrm{d}\theta(v,w) \Rightarrow \iota_v \mathrm{d}\theta = 0 \Rightarrow \mathscr{L}_v \mathrm{d}\theta = 0 \Rightarrow \mathrm{d}\theta = \pi^* \rho$$

where $\rho \in \bigwedge^2 T^* M$ is CQS and M is the leaf space of v. Identifying the q-contact distribution with TM, one can show that $\mathscr{D}$ equips M with a CQS scalar 4th order ODE.

Remarks : Fels showed CQS 4th order ODEs are variational i.e. they are the EL eqns for a 2nd order Lagrangian $L(x, y, y'')$.

Following Doubrov-Zelenko treatment (2011) of variational scalar ODEs of order ≥ 6, divergence equiv classes of Lags $L(x, y, y', y'')$ define (2,3,5)-strs from Monge eqns $z' = L(x, y, y', y'')$ and vice versa.

Ivey showed that there is a one to one correspondence between variational 4th order ODEs and *sub-Finsler strs* on contact 3-flds.

To quasi-contactify 4th order ODEs to (G_2, P_{12}) we mainly used

- The quasi-contact str $[\tilde{\omega}]$ on $\tilde{M}^6$.
- The vertical direction of the cone bundle $\mathbb{P}\mathscr{D} \to N$ are *transversal* to the characteristic direction of $[\tilde{\omega}]$ and give a splitting of $\mathfrak{g}_{-1}$.

To generalize this construction define parabolic quasi-contact cone structures as $|k|$-graded parabolic geometries s.t.

- $\mathfrak{g}_{-k}$ is 1D and corresponds to a quasi-contact structure.
- $\mathfrak{g}_{-1}$ has a splitting into an abelian subalgebra of corank 1 and the characteristic of the the quasi-contact structure.

Using the classification of non-rigid parabolic geometries, parabolic quasi-contact cone structures are the following.

Parabolic quasi-contact cone structures

- (G_2, P_{12}) which descend to (2,3,5)-geometries.
- (B_3, P_{23}) which descend to (3,6)-geometries.
- $(B_n, P_{12}), n \geq 2$ i.e. causal structures which can descend to odd-dimensional conformal or Lie contact structures.
- $(D_n, P_{12}), n \geq 4$ i.e. causal structures which can descend to even-dimensional conformal or Lie contact structures.
- (D_3, P_{123}) i.e. so called XXX-geometries which can descend to 4D conformal, 3D path, 5D Leg contact, ...

Remarks : Using Doubrov-Zelenko's notion of *bigraded regularity*, the first two cases can represent more general hybrid geometries e.g. in the case of (G_2, P_{12}) the reduced structure can descend to 4th order ODEs or GL_2-structures.

Parabolic almost conformally quasi-symplectic structures (PACQSS)

Similar to 4th order ODEs, we define parabolic almost conf q-sympl str (PACQSS) on odd-dimensional manifold as almost conformally quasi-symplectic structures $[\rho]$ whose underlying structure is that of a parabolic quasi-contact cone structure *truncated* at $\mathfrak{g}_{-k}$.

$\mathfrak{g}_0$ of PACQSS coincides with $\mathfrak{g}_0$ of the corr parab q-cont cone str.
$\mathfrak{g}_+$ of PACQSS is 1D.

PACQSS are

- 4th order ODEs under contact equivalence.
- Pairs of 3rd order ODEs under contact equivalence.
- Orthogonal path geometries i.e path geometries where the vertical bundle of $\mathbb{P}TM \to M$ is augmented with a conformal class of an inner product.
- Reduced XXX-geometries.

Orthogonal path geometries (orthopath geometries)

Now we study 3D CQS orthopath strs by reducing path geometries. (One can study them à la Morimoto et al.)

Path geometries on S^5 is locally modeled on $S \subset \mathbb{P}TM \to M^3$ with

- Fibration $S_x \hookrightarrow S \to M$ with 2D fibers. Let $\mathrm{span}\{v_1, v_2\}$ give the vertical tangent bundle for $S \to M$.
- A line field $\ell = \mathrm{span}\{v_0\}$ transversal to fibers S_x.
- A *multi-contact* structure i.e.

$$T_pS = \mathrm{span}\{v_0, v_1, v_2, [v_0, v_1], [v_0, v_2]\}$$

Let $\langle \omega^0, \omega^1, \omega^2 \rangle$, $\langle \omega^1, \omega^2, \theta^1, \theta^2 \rangle$ be integ Pfaffian systems for S_x and ℓ s.t.

$$\mathrm{d}\omega^a \equiv \theta^a \wedge \omega^0 \mod \langle \omega^1, \omega^2 \rangle$$

Their Cartan geometry $(\mathscr{G} \to M, \psi)$ is of type (A_3, P_{12}) and

$$\psi = \begin{pmatrix} s & \xi_0 & \xi_b \\ \omega^0 & \psi^0_0 + s & \gamma_b \\ \omega^a & \theta^a & \psi^a_b + s\delta^a_b \end{pmatrix}, \qquad s = -\tfrac{1}{n+1}(\psi^0_0 + \psi^a_a)$$

Orthopath structures

The curvature 2-form is given by (Grossman's thesis)

$$\mathrm{d}\psi + \psi \wedge \psi = \begin{pmatrix} 0 & R_0 & R_b + C_b \\ 0 & R^0_0 & R^0_b + C^0_b \\ 0 & R^a_0 & R^a_b + C^a_b \end{pmatrix}$$

where the harmonic parts are $\mathbf{T} = R^a_{00b}\omega^0 \wedge \omega^a$ and $\mathbf{C} = C^a_{bcd}\omega^c \wedge \theta^d$.

An orthopath geom is a path geom augmented by the conf class of $h \in \mathrm{Sym}^2(T^* S_x)$. Assume h has sign (p,q) i.e. wrt some coframe

$$h = \varepsilon_{ab}\theta^a \circ \theta^b.$$

Preserving $[h]$ reduces G_0 from $\mathbb{R} \times \mathrm{GL}_2$ to $\mathbb{R} \times \mathrm{CO}(p,q), p+q=2.$
Remarks : Equivalently, one can define $[\tilde{h}]$, $\tilde{h} = \varepsilon_{ab}\omega^a \circ \omega^b$ on S where $h \in \mathrm{Sym}^2(\mathrm{Ann}(TV)$ where $V = \ell \oplus S_x$.

CQS orthopath structures

Using ε_{ab} to raise and lower indices we have

$$\psi^a_b - \psi^0_0\delta^a_b = \phi^a_b + \sigma^a_b + \phi_1\delta^a_b, \quad \sigma_{ab} = \sigma_{ba}, \quad \phi_{ab} = -\phi_{ba}$$

where ϕ^a_b is $\mathfrak{co}(1,1)$-valued and

$$\sigma_{ab} = F_{abc}\theta^c + f_{abc}\omega^c.$$

One can further reduce the structure bundle by $\dim(\mathfrak{g}_+) - 1$ dimensions by restricting to coframes wrt which

$$\varepsilon^{ab}F_{abc}, \varepsilon^{ab}f_{abc} = 0 \Rightarrow \xi_a, \gamma_a \equiv 0 \mod \langle\omega^0, \omega^a, \theta^a\rangle$$

Now, the ACQS structure $[\rho]$, $\rho = \varepsilon_{ab}\theta^a \wedge \omega^a$ is well-defined and has characteristics along $\frac{\partial}{\partial\omega^0}$ i.e. the solution curves of the system of second order ODEs that locally defines the path geometry.

Requiring ρ to be CQS implies, in particular,

$$F_{abc} = F_{(abc)}, \quad f_{abc} = f_{(abc)}, \quad R_{ab} = R_{ba}$$

where $R_{ab} = \varepsilon_{ab}R^a_{00b}$ is the torsion of the path geometry.

CQS orthopath structures

This reduction defines CQS orthopath geometries as Cartan geoms $(\mathscr{G} \to M, \psi)$ of type $(\mathbb{R}^4 \rtimes (\mathrm{GL}_2 \otimes \mathrm{O}(p,q)), B \otimes O(p,q))$ where $B \subset \mathrm{GL}_2$ is Borel

$$\psi = \begin{pmatrix} -\phi_0 + \varepsilon & \xi_0 & 0 \\ \omega^0 & -\phi_1 + \varepsilon & 0 \\ \omega^a & \theta^a & \phi^a_b + \varepsilon \delta^a_b \end{pmatrix}, \quad \varepsilon = \frac{1}{n+1}(\phi_0 + \phi_1)$$

and the fundamental invariants are

$$\mathbf{F} = F_{abc}\theta^a \circ \theta^b \circ \theta^c, \quad \mathbf{W} = R_{ab}\omega^a \circ \omega^b, \quad \mathbf{N} = N_{ab}\omega^a \wedge \omega^b, \quad \mathbf{q} = \varepsilon^{ab} Q_{ab}(\omega^0)^2$$

$\mathbf{F}$ and $\mathbf{W}$ are lifted to the harmonic invs of a causal str after quasi-cont.

We quasi-contactify those 3D indefinite orthopath strs that give 4D indefinite conformal strs i.e. $\mathbf{F} = 0$.

The fund inv are reduced to four scalars $R_{11}, R_{12}, N_{12}, q.$
If $N_{12} = 0$ then such orthopath geometries define 3D projective strs.

Conformal structures from CQS orthopath geometries

As before, define $\tilde{M} = \mathbb{R} \times M$ and let $\rho_0 \in [\rho]$ be a quasi-sympl 2-form:

$$\mathrm{d}\rho_0 = 0 \Rightarrow \rho = \mathrm{d}\omega^3.$$

Define $\tilde{\omega}^3 = \mathrm{d}t + \pi^*\omega^3$ and $\tilde{\mathscr{G}}^{10} \to \tilde{M}^6$ as the pull-back of $\mathscr{G}^9 \to M^5$. There is a natural scaling action on $\tilde{\omega}^3$ using which one can lift $\tilde{\omega}^3$ to $\tilde{\mathscr{G}}$ and get a coframe $(\tilde{\omega}^3, \tilde{\psi} = \pi^*\psi)$ on $\tilde{\mathscr{G}}$ as an $\{e\}$-structure for a Cartan geometry of type (P_2, B), $P_2 \subset A_3$, $B \subset \mathrm{GL}_2$ is Borel.

$\tilde{M}^6$ can be viewed as the sky bundle of $[g]$ where $g = \tilde{\omega}^0 \circ \tilde{\omega}^3 - \tilde{\omega}^1 \circ \tilde{\omega}^2$.

Pulling back the normal Cartan conn for 4D conf str to $\tilde{\mathscr{G}}^{10}$ we have

$$\tilde{\omega}^2 = \omega^2, \quad \tilde{\omega}^1 = \omega^1, \quad \tilde{\theta}^1 = \theta^1, \quad \tilde{\theta}^2 = \theta^2, \quad \tilde{\omega}^0 = \omega^0 - \tfrac{1}{2} q\tilde{\omega}^3.$$

$q = 0$ iff the infitesimal symmetry $\frac{\partial}{\partial \tilde{\omega}^4}$ is null.

Flatness, integrability and holonomy reduction

CQS orthopath strs giving conf strs loc depend on 5 fcns of 3 vars.

- When $R_{12} = R_{11} = 0$, then the 4D conformal str is flat. Such orthopath geometries define torsion-free path geometries and their local moduli depends on 5 constants.
- When $q = 0$, both SD and ASD Weyl curv have Petrov type I and there is an integrable distr of ASD and SD null planes with null symm. The local generality is 4 functions of 3 variables.
- When $q = N_{12} = 0$, then $R_{11;\underline{111}}, R_{12;\underline{222}} = 0$ and $R_{11;\underline{11},} = R_{12;\underline{22}}$, then $[g]$ is 3-integ with null symm and both SD and ASD Weyl curv have type II. The local generality is 3 functions of 3 variables.
  In particular, $\langle\omega^0\rangle$ is integ as in fiber equivalent classes of ODEs.
- When $q = N_{12} = R_{11;\underline{11}} = 0$ then $[g]$ is 3-integ with null symm, both SD and ASD Weyl curvs have Petrov type III, with holonomy reduced to P_2. The local generality is 2 functions of 3 variables.

Flatness, integrability and holonomy reduction

- When $R_{11} = 0$, then $[g]$ is SD. The local generality is 6 functions of 2 variables.
- When $R_{11} = q = 0$, then $[g]$ is SD with null symm and the SD Weyl has no repeated root but there is an integrable distribution of SD null planes. The local generality is 4 functions of 2 variables.
- When $R_{11} = q = N_{12} = 0$, then $[g]$ is SD, 3-integ with null symm and SD Weyl curv has type III with holonomy reduced to P_2. The local generality is 3 functions of 2 variables.

Remarks : By Dunajski-West construction (when $R_{11} = q = 0$) the orthopath geom descends further to a surface. Calderbank generalized the construction by not assuming the null vector field to be a symmetry.

Jones-Tod showed all SD conf str with non-null infin. symm. arise from EW strs plus a solution of generalized monopole eqn. If CSQ *implies being variational*, then projective strs defined by EW geoms are variational, most likely wrt to a generalized Randers metric.

Orthopath structure on surfaces

Orthopath geometries on surfaces are the most degenerate case.
Being CQS imposes a 2D reduction i.e.

$$\begin{pmatrix} -\phi_0+\varepsilon & \xi_0 & \xi_1 \\ \omega^0 & -\phi_1+\varepsilon & \xi_2 \\ \omega^1 & \theta^1 & \varepsilon \end{pmatrix} \to \begin{pmatrix} -\phi_0+\varepsilon & \xi_0 & 0 \\ \omega^0 & -\phi_1+\varepsilon & 0 \\ \omega^1 & \theta^1 & \varepsilon \end{pmatrix}, \varepsilon = \frac{1}{3}(\phi_0+\phi_1)$$

The CQS class $[\rho]$ is given by $\rho = \omega^1 \wedge \theta^1.$

The fundamental invariants are two scalars

$$\mathbf{c} = c_0(\omega^0)^3 \otimes \theta^1, \qquad \mathbf{w} = w_0(\theta^1)^2.$$

If $\mathbf{w} = 0$ then CQS orthopath geometries define a subset of fiber equivalent classes of ODEs which contains Painléve equations.
The local generality of such orthopath strs is 1 function of 2 variables.

3D causal structures from CQS orthopath surfaces

3D causal strs are parabolic geoms of type (B_2, P_{12}), or equivalently, contact equivalent classes of 3rd order ODEs, have two harmonic invs: The Wünschmann invariant W and the Cartan invariant C.

The same recipe gives 3D causal str as quasi-contactified 2D orthopath str on $\tilde{M}^4 = \mathbb{R} \times M^3$. The Cartan conn pulled-back to $\tilde{\mathscr{G}}^7$ is

$$\tilde{\omega}^1 = \omega^1, \quad \tilde{\theta}^1 = \theta^1, \quad \tilde{\omega}^0 = \omega^0 + \tfrac{1}{2} w_0 \tilde{\omega}^2,$$
$$\tilde{\phi}_0 = \phi_0 + \tfrac{1}{2} w_{0;0} \tilde{\omega}^2, \qquad \tilde{\phi}_1 = \phi_1, \qquad \tilde{\xi}_0 = \xi_0 + \tfrac{1}{2} w_{0;00} \tilde{\omega}^2$$

where $\tilde{\omega}^2$ is the lifted quasi-contact form on $\tilde{\mathscr{G}}$ and

$$W = \pi^*(w_{0;\underline{1}}(\theta^1)^3), \qquad C = \pi^* \mathbf{c}$$

In particular, the infitesimal symmetry is null iff $w_0 = 0$.

Flatness, integrability and holonomy reduction

CQS orthopath geometries locally depend on 1 function of 3 variables.

- When $w_{0;\underline{1}} = c_0 = 0$, then the 3D conformal str is flat. Such orthopath geometries define co-projective structures and their local moduli depends on 3 constants.
- When $c_0 = 0$, 3D causal strs descend to contact proj strs. The local generality is 3 functions of 2 variables.
- When $w_0 = 0$, the 3D conformal structure has Petrov type N for the Cotton-York tensor, and is integrable with holonomy reduced to P_2. The local generality is 1 functions of 2 variables.
- CQS orthopath surfaces that are projectively flat satisfy $c_0 = 0$ and a 2nd order condition on w_0. The local generality is 1 functions of 2 variables.

Variationality of PCQS strs via Griffiths formalism

A variational problem, denoted as (M, I, ϕ), is the study of the functional

$$\Phi\colon \mathscr{V}(I) \to \mathbb{R} \quad \Phi(\gamma) = \int_\gamma \phi$$

where γ is an integral curve of I and $\mathscr{V}(I)$ is the space of smooth immersions of the integral curves of I into M for a fixed interval (a, b). Define $\delta\Phi(\gamma)\colon T_\gamma \mathscr{V}(I) \to \mathbb{R}$ to be a variation of Φ at γ by

$$\delta\Phi(\gamma)[v] = \frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\gamma_s} \phi\right)\Big|_{s=0}$$

where $\gamma_s \in \mathscr{V}(I)$ is any *compactly supported variation* of γ with $\gamma_0 = \gamma$ and v is the the *vartiational vector field* for the deformation $s \to \gamma_s$.

The Euler-Lagrange equations are the conditions that

$$\delta\Phi(\gamma)[v] = 0, \quad \forall v \in T_\gamma \mathscr{V}(I).$$

Integral curves satisfying EL equations are called extemals of Φ.

Variationality of PCQS strs via Griffiths formalism

For the variational problem $(M, \{0\}, \phi)$, γ is an extremal iff

$$v \lrcorner \mathrm{d}\phi|_\gamma = 0,$$

for any compactly supported variation , because

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\gamma_s}\phi\right)\Big|_{s=0} = \int_b^a v \lrcorner \mathrm{d}\phi + \mathrm{d}(v \lrcorner \phi) = \int_b^a v \lrcorner \mathrm{d}\phi + \phi(v)|_b^a = \int_b^a v \lrcorner \mathrm{d}\phi.$$

Thus the EL system of this variational problem is the *Cartan system*

$$\mathscr{C}(\mathrm{d}\phi) = \{v \lrcorner \mathrm{d}\phi \mid v \in C_0^\infty(TM)\}$$

and extremals are the characteristic curves of $\mathrm{d}\phi$ on M.

If $I \neq \{0\}$ Griffiths gives a recipe to *lift* the variational problem (M, I, ϕ) to another variational problem of the form $(Z, \{0\}, \zeta)$. The projection of the extremals of the latter are extremals of the former.
The converse is not necessarily true.

Variationality of PCQS strs via Griffiths formalism

Question: When do the characteristic curves of a PACQSS arise from a variational problem in the sense of Griffiths? When it is CQS.

Question: What is the most natural non-degenerate variational problem with $I \neq \{0\}$ for such geometries?

- Being CQS 4th order ODEs implies that $J^1(\mathbb{R},\mathbb{R})$ has a sub-Finsler structure given by a 2nd order Lagrangian (Ivey).
- Being CQS orhotpath geom implies that the paths are geodesics of a pseudo-Finsler metric compatible with the degen bilinear form i.e. 1st order Lagrangian with prescribed vertical Hessian.
- Being CQS pair of third order ODEs seems to imply that there is a degenerate 2nd order Lagrangian whose extremals on $J^2(\mathbb{R},\mathbb{R}^2)$ are the solution curves. These may be Finslerian conformal geodesics in 3D. There is more work to be done here.
- In the case of reduced XXX-geometry an interpretation is unclear.

Remarks : Alternatively, following Fels' work, one can solve the *variational multiplier problem* for PACQS structures.

Perspectives and speculations

(1) Find more general notions of contactification and relation to variational/integrable geometries of other types and versions of (strong) Goldberg-Sachs and Kerr theorems e.g. having null symmetry implies integrability and Calderbank's generalization of Dunajski-West construction.
(2) Do the class of variational pairs of third order ODEs contain all conformal geodesics equations in 3D?
(3) Can one extend Doubrov-Zelenko result to "variational" pairs (or systems) of ODEs and certain rank 3 (or $k \geq 4$) distributions?
(4) How to relate EW strs+gen monopole eqns to orthopath geometries reduced from a SD conf str? Derive the corresponding pseudo-Finsler (Randers) metric from an EW str+gen monop eqn.
(5) Give examples of cone structures with a transversal infinitesimal symmetry arising as VMRTs.
(6) Study BGG operators for PCQS structures.
(7) Are the path geometries of Finsler metrics of scalar flag curvature and constant flag curvatures are equivalent?

- O. Makhmali, K. Sagerschnig, Parabolic quasi-contact cone structures with transversal infinitesimal symmetry. in preparation
- A. Čap, B. Doubrov. D. The, On C-class equations, CAG
- D. M. Calderbank, Selfdual 4-manifolds, projective surfaces, and the Dunajski-West construction, SIGMA, 2014
- B. Doubrov, I Zelenko, Equivalence of variational problems of higher order, DGA, 2011
- M. Dunajski, S. West, Anti-self-dual conformal structures with null Killing vectors from projective structures CMP, 2007
- M. Fels, The inverse problem of the calculus of variations for scalar fourth-order ordinary differential equations. TAMS, 1996.
- P. A. Griffiths, *Exterior differential systems and the calculus of variations*, Birkhäuser, 1983.
- D. A. Grossman, Path geometries and second-order ordinary differential equations. Thesis, Princeton Uni, 2000
- T. Ivey, An inverse problem from sub-Riemannian geometry, Pacific J. Math., 2003.
- O. Jones, P. Tod, Minitwistor spaces and Einstein-Weyl spaces, CQG, 1985