Frobenius integrability and Cartan geometries

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Outline of lectures

Lecture one:

(1) Frobenius integrability in 4D conformal structures
(2) Frobenius integrability in 3D conformal structures
- (2,3,4,5)-distribution on the twistor bundle does not have a splitting, thus, does not define a 4th order ODE.

Lecture two:

(1) Frobenius integrability in (2,3,5)-geometries
(2) Integrable (2,3,5)-geometries from scalar 4th order ODEs

Lecture three:

(1) Parabolic quasi-contact cone structures and quasi-contactification
(2) Frobenius integrability in quasi-contactified structures

(2,3,5)-distributions as Cartan geometries

A bracket-generating rank 2 dist in 5D has growth vector (2,3,5) and defines a Cartan geometry $(\pi\colon \mathscr{G} \to M, \psi)$ of type (G_2, P_1) where

$$\psi = \begin{pmatrix} -\frac{2}{3}\phi_0 & \frac{1}{3}\xi_0 & \frac{1}{3}\xi_1 & -\frac{1}{\sqrt{3}}\xi_2 & -\xi_3 & -\xi_4 & 0 \\ \omega^0 & -\frac{1}{3}\phi_0 - \phi_1 & \gamma_1 & -\frac{2}{\sqrt{3}}\xi_1 & \xi_2 & 0 & -\xi_4 \\ \omega^1 & \theta^1 & -\frac{1}{3}\phi_0 + \phi_1 & \frac{2}{\sqrt{3}}\xi_0 & 0 & \xi_2 & \xi_3 \\ \frac{2}{\sqrt{3}}\omega^2 & -\frac{1}{\sqrt{3}}\omega^1 & \frac{1}{\sqrt{3}}\omega^0 & 0 & \frac{2}{\sqrt{3}}\xi_0 & \frac{2}{\sqrt{3}}\xi_1 & -\frac{1}{\sqrt{3}}\xi_2 \\ \omega^3 & -\frac{1}{3}\omega^2 & 0 & \frac{1}{\sqrt{3}}\omega^0 & \frac{1}{3}\phi_0 - \phi_1 & \gamma_1 & -\frac{1}{3}\xi_1 \\ \omega^4 & 0 & -\frac{1}{3}\omega^2 & \frac{1}{\sqrt{3}}\omega^1 & \theta^1 & \frac{1}{3}\phi_0 + \phi_1 & \frac{1}{3}\xi_0 \\ 0 & \omega^4 & -\omega^3 & \frac{2}{\sqrt{3}}\omega^2 & -\omega^1 & \omega^0 & \frac{2}{3}\phi_0 \end{pmatrix}$$

$\subset \mathfrak{so}(3,4)$ wrt $\langle u, w\rangle = u_1 w_7 + u_7 w_1 - u_2 w_6 - u_6 w_2 + u_3 w_5 + u_5 w_3 - u_4 w_4.$

The harmonic invariant is the so-called Cartan quartic $\mathbf{A} \subset \mathrm{Sym}^4(\mathscr{D}^*)$

$$\mathbf{A} = a_4(\omega^0)^4 + 4a_3(\omega^0)^3\omega^1 + 6a_2(\omega^0)^2(\omega^1)^2 + 4a_1\omega^0(\omega^1)^3 + a_0(\omega^1)^4$$

(2,3,5)-geometries: structure equations

One also has the *ternary quartic* $[\mathbf{T}] \subset \mathrm{Sym}^4(\partial\mathscr{D})^*$ defined as

$$\mathbf{T} = \mathbf{A} + \mathbf{B}\omega^2 + \mathbf{C}(\omega^2)^2 + \mathbf{D}(\omega^2)^3 + e(\omega^2)^4,$$

$$\mathbf{B} = b_3(\omega^0)^3 + 3b_2(\omega^0)^2(\omega^1) + 3b_1\omega^0(\omega^1)^2 + b_0(\omega^1)^3,$$

$$\mathbf{C} = c_2(\omega^0)^2 + 2c_1\omega^0\omega^1 + c_0(\omega^1)^2, \qquad \mathbf{D} = d_1\omega^0 + d_0\omega^1.$$

The str group G_0 and its Lie algebra $\mathfrak{g}_0$ acting on $\{\omega^4, \cdots, \omega^0\}^\top$ is

$$G_0 = \begin{pmatrix} \Delta_0 f_1 & \Delta_0 t_1 & 0 & 0 & 0 \\ \Delta_0 g_1 & \Delta_0 f_2 & 0 & 0 & 0 \\ \Delta_0 x_1 & \Delta_0 x_0 & \Delta_0 & 0 & 0 \\ f_1 x_2 + \Delta_1 x_1 & t_1 x_2 + \Delta_1 x_0 & 2\Delta_1 & f_1 & t_1 \\ g_1 x_2 + \Delta_2 x_1 & f_2 x_2 + \Delta_2 x_0 & 2\Delta_2 & g_1 & f_2 \end{pmatrix} \subset \mathrm{CO}(2,3),$$

where $\Delta_0 = f_1 f_2 - g_1 t_1$, $\Delta_1 = \frac{2}{3}(f_1 x_0 - t_1 x_1)$ and $\Delta_2 = \frac{2}{3}(g_1 x_0 - f_2 x_1)$, and

$$\mathfrak{g}_0 = \begin{pmatrix} \phi_0 + \phi_1 & \theta^1 & 0 & 0 & 0 \\ \gamma_1 & \phi_0 - \phi_1 & 0 & 0 & 0 \\ \xi_1 & \xi_0 & \frac{2}{3}\phi_0 & 0 & 0 \\ \xi_2 & 0 & \frac{4}{3}\xi_0 & \frac{1}{3}\phi_0 + \phi_1 & \theta^1 \\ 0 & \xi_2 & -\frac{4}{3}\xi_1 & \gamma_1 & \frac{1}{3}\phi_0 - \phi_1 \end{pmatrix} \subset \mathfrak{co}(2,3),$$

wrt to the inner product $\langle u, w \rangle = u_1 w_5 + u_5 w_1 - u_2 w_4 - u_4 w_2 + \frac{4}{3} u_3 w_3$

3D Twistor correspondence

The projectivized null cone $\mathscr{C} \subset \mathbb{P}TM$ of the conformal str $[g]$ where

$$g = \omega^0 \circ \omega^4 - \omega^1 \circ \omega^3 + \tfrac{2}{3}\omega^2 \circ \omega^2$$

at each point is the 3D *indefinite quadric*

$$\mathscr{C}_x = \{[v] \in \mathbb{P}T_xM \mid g(v,v) = 0\}$$

Thus, a null plane in T_xM is a null line in $\mathscr{C}_x$. Recall the double fibration

$$\begin{array}{ccccc} \mathrm{SO}^+(2,3)/P_1 \cong \mathbb{Q}^3 & \xleftarrow{\ \ell\ } & \mathbb{F} & \xrightarrow{\ \nu\ } & \mathbb{G}_0^3 \cong \mathrm{SO}^+(2,3)/P_2 \\ \wr\Vert & & & & \wr\Vert \\ \mathrm{Sp}(4)/\tilde{P}_2 \cong \mathbb{L}^3 & & & & \mathbb{P}^3 \cong \mathrm{Sp}(4)/\tilde{P}_1 \end{array}$$

$\mathbb{L}^3 \cong \mathbb{Q}^3$ is via Plücker embedding $\mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}(\bigwedge^2\mathbb{R}^4)$ i.e. define $(\mathbb{R}^4, \rho)$

$$\rho = \rho_1 \wedge \rho_4 + \tfrac{4}{3}\rho_2 \wedge \rho_3, \quad \textstyle\bigwedge_0^2\mathbb{R}^4 = \{z \in \bigwedge^2\mathbb{R}^4 \mid \rho(z) = 0\}$$

Then $\mathbb{Q}^3 \subset \mathbb{P}\bigwedge_0^2\mathbb{R}^4$ is the null cone of $\langle\cdot,\cdot\rangle$ defined as

$$\langle z_1, z_2\rangle = -(\rho \wedge \rho)(z_1 \wedge z_2).$$

(2,3,5)-geometries and their space of null planes

Take the basis $\{e_i\}_{i=1}^4$ and $\{\mathrm{v}_i\}_{i=1}^5$ for $(\mathbb{R}^4,\rho)$ and $\wedge_0^2\mathbb{R}^4$

$$\mathrm{v}_1 = e_1\wedge e_3,\quad \mathrm{v}_2 = e_1\wedge e_2,\quad \mathrm{v}_3 = \frac{\sqrt{3}}{3}e_1\wedge e_4 - \frac{\sqrt{3}}{4}e_2\wedge e_3,\quad \mathrm{v}_4 = e_3\wedge e_4,\quad \mathrm{v}_5 = e_2\wedge e_4$$

one finds the isomorphism $\phi\colon \mathfrak{sp}(4)\to\mathfrak{so}(2,3)$

$$\phi\begin{pmatrix} s_1 & s_2 & t_2 & t_3\\ s_3 & s_4 & t_1 & \frac{4}{3}t_2\\ r_2 & r_1 & -s_4 & -\frac{4}{3}s_2\\ r_3 & \frac{4}{3}r_2 & -\frac{4}{3}s_3 & -s_1\end{pmatrix} = \begin{pmatrix} s_1+s_4 & t_1 & 2t_2 & 2t_3 & 0\\ r_1 & s_1-s_4 & -2s_2 & 0 & 2t_3\\ r_2 & -s_3 & 0 & -\frac{3}{2}s_2 & -\frac{3}{2}t_2\\ \frac{1}{2}r_3 & 0 & -\frac{4}{3}s_3 & -s_1+s_4 & t_1\\ 0 & \frac{1}{2}r_3 & -\frac{4}{3}r_2 & r_1 & -s_1-s_4\end{pmatrix}$$

with respect to matrices

$$\mathbf{r} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & \frac{4}{3} & 0\\ 0 & -\frac{4}{3} & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\qquad \mathbf{h} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & \frac{4}{3} & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{pmatrix}$$

Bundle of null planes

In the double fibration define

$$\mathbb{L}^3 \cong \mathbb{Q}^3 \xleftarrow{\ell} \mathbb{F} \xrightarrow{\nu} \mathbb{G}_0^3 \cong \mathbb{P}^3, \quad \hat{p} = \nu \circ \ell^{-1}(p) \subset \mathbb{P}^3, \ \tilde{v} = \ell \circ \nu^{-1}(v) \subset \mathbb{Q}^3,$$

$p_1, p_2 \in \mathbb{Q}^3$ are *null-separated* if they lie on a null line $v \subset \mathbb{Q}^3 \Rightarrow \hat{p}_1 \cap \hat{p}_1 = \hat{v}$

$v_1, v_2 \in \mathbb{P}^3$ are *contact-sep.* if are on a contact line $p \subset \mathbb{P}^3 \Rightarrow \tilde{v}_1 \cap \tilde{v}_2 = \tilde{p}$

Given a null line $v = [z] \in \mathbb{P}^3$, then $\tilde{v} = [z \wedge w] \subset \mathbb{Q}^3$ where $w = z^\perp$
e.g. the 2-plane corresponding to $[e_4]$ is

$$[e_4 \wedge e_4^\perp] = [e_4 \wedge (\mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4)] = \mathrm{span}\{\mathrm{v}_4, \mathrm{v}_5\} = \mathscr{D}$$

These are 3D analogues of incident relations in classical twistor theory.
(c.f. Ward-Wells, *Twistor geometry and field theory*,1991)

Bundle of null planes

Recall that $\mathfrak{g}_0 \subset \mathfrak{co}(2,3)$, and

$$\phi(\mathfrak{g}_0 \cap \mathfrak{so}(2,3)) = \begin{pmatrix} \frac{1}{3}\phi_0 & 0 & 0 & 0 \\ -\xi_0 & \phi_1 & \theta^1 & 0 \\ \xi_1 & \gamma_1 & -\phi_1 & 0 \\ 2\xi_2 & \frac{4}{3}\xi_1 & \frac{4}{3}\xi_0 & -\frac{1}{3}\phi_0 \end{pmatrix} \subset \mathfrak{sp}(4).$$

Proposition : The space of null planes at each point of a (2,3,5)-geometry is equipped with an *invariant flag*

$$\{p_0\} \subset \mathbb{P}^2 \subset \mathbb{P}^3,$$

and G_0 acts transitively on $\mathbb{P}^3 \backslash \mathbb{P}^2$ and $\mathbb{P}^2 \backslash \{p_0\}$, where $p_0 = [e_4] \in \mathbb{P}^3$ is the rank 2 dist $\mathscr{D}$ and $\mathbb{P}^2 = \mathrm{span}\{[e_2], [e_3], [e_4]\}$ is the 2D space of null planes with non-empty intersection with $\mathscr{D}$ i.e. contact-separated from $\mathscr{D}$.

Twistor bundle $\mathscr{N}$ as $\mathbb{P}^2$-bundle of special null planes

Let $[\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3]$ be homog coords on $\mathbb{P}^3$. Let $\mathbb{P}^2$ be represented by $[\lambda_1 e_2 + \lambda_2 e_3 - 2\sqrt{2}\lambda_3 e_4]$ then the corresponding null planes are

$$p = Z \wedge (\lambda_1 e_2 + \lambda_2 e_3 - 2\sqrt{2}\lambda_3 e_4), \qquad \rho(p) = 0 \Rightarrow \tfrac{3\sqrt{2}}{2} z_1 \lambda_3 - z_2 \lambda_2 + z_3 \lambda_1 = 0$$

Working on the *affine chart* $\lambda_2 = 1,$ one obtains

$$Z = t_1 e_1 + \tfrac{3\sqrt{2}}{2} t_1 \lambda_3 e_2 - t_2 e_4,$$

for two param t_1 and t_2, and p corresponds to $\mathrm{span}\{V_1, V_2\}$ where

$$V_1 = \mathsf{v}_1 + \lambda_1 \mathsf{v}_2 - 3\lambda_3 \mathsf{v}_3 - 6\lambda_3^2 \mathsf{v}_5 \qquad V_2 = \mathsf{v}_4 + \lambda_1 \mathsf{v}_5$$

Similarly one can treat other two affine charts.

Twistor bundle $\mathscr{N}$ as $\mathbb{P}^2$-bundle of special null planes

The *special null planes* in T_xM are the $\mathbb{P}^2$-bundle with non-empty intersection with $\mathscr{D}$ and can be parametrized as $\langle\beta^1,\beta^2,\beta^3\rangle^\perp$ where

$$\beta^1=\omega^3-\lambda_1\omega^4,\qquad \beta^2=\omega^0-6\lambda_3^2\omega^4-\lambda_1\omega^1-4\lambda_3\omega^2,\qquad \beta^3=\omega^2+3\lambda_3\omega^4,$$

Like in 3D, 4D conformal strs, λ_1,λ_3 can be identified with group parameters $g_1,-3x_1$ which correspond to conn forms γ_1,ξ_1.
Hence the twistor bundle $\mathscr{N}^7$ is the leaf space of

$$\langle\omega^0,\omega^1,\omega^2,\omega^3,\omega^4,\gamma_1,\xi_1\rangle$$

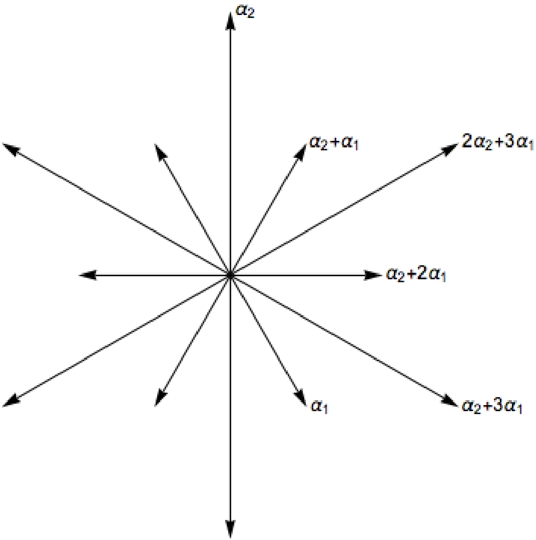
and its 2D fibers $\mathscr{N}_x$ are the 2D normal subgroup $A_{g_1,x_1}\subset P_1$.
Proposition : $\mathscr{N}$ has a canonical rank 2 dist $\mathscr{H}=\ker\{\beta^1,\cdots,\beta^5\}$ where

$$\beta^4=\mathrm{d}\lambda_1+6\lambda_3^3\omega^4-\lambda_1^2\theta^1-2\lambda_1\phi_1+\gamma_1$$
$$\beta^5=\mathrm{d}\lambda_3+\lambda_3^2\omega^1-\tfrac{1}{3}\lambda_3\phi_0-\lambda_3\phi_1-\lambda_3\lambda_1\theta^1-\tfrac{1}{3}\lambda_1\xi_0-\tfrac{1}{3}\xi_1$$

$\mathscr{H}$ is integrable if and only if $\mathscr{D}$ is flat.
Direct computation shows modulo $\langle\beta^1,\cdots,\beta^5\rangle$ $\mathrm{d}\beta^1,\mathrm{d}\beta^2,\mathrm{d}\beta^3\equiv 0$,

$$\mathrm{d}\beta^4\equiv\mathbf{A}(\lambda_1)\omega^1\wedge\omega^4,\qquad \mathrm{d}\beta^5\equiv\tfrac{1}{4}\Big(\mathbf{B}(\lambda_1)+\lambda_3\tfrac{\mathrm{d}}{\mathrm{d}\lambda_1}\mathbf{A}(\lambda_1)\Big)\omega^1\wedge\omega^4$$

Twistor bundle $\mathscr{N}$



Integrability by distinguished null surfaces: necessary condition

Let us find necessary conditions for the existence of $\lambda\colon M \to \mathcal{N}$ for which s^*I is Frobenius integrable where $I = \langle \beta^1, \beta^2, \beta^3 \rangle$.

$$\beta^1 = \omega^3 - \lambda_1\omega^4, \qquad \beta^2 = \omega^0 - 6\lambda_3^2\omega^4 - \lambda_1\omega^1 - 4\lambda_3\omega^2, \qquad \beta^3 = \omega^2 + 3\lambda_3\omega^4.$$

One finds modulo I, $\mathrm{d}I \subset I$ implies $\beta^4, \beta^5 \in s^*I$ where

$$\beta_4 = \mathrm{d}\lambda_1 + 6\lambda_3^3\omega^4 - \lambda_1^2\theta^1 - 2\lambda_1\phi_1 + \gamma_1$$
$$\beta_5 = \mathrm{d}\lambda_3 + \lambda_3^2\omega^1 - \tfrac{1}{3}\lambda_3\phi_0 - \lambda_3\phi_1 - \lambda_3\lambda_1\theta^1 - \tfrac{1}{3}\lambda_1\xi_0 - \tfrac{1}{3}\xi_1$$

i.e. a 2D reduction of $\mathscr{G}$. Another differentiation gives

$$\mathrm{d}\beta^4 \equiv \mathbf{A}(\lambda_1)\omega^1 \wedge \omega^4, \qquad \mathrm{d}\beta^5 \equiv \tfrac{1}{4}\left(\mathbf{B}(\lambda_1) + \lambda_3 \tfrac{\mathrm{d}}{\mathrm{d}\lambda_1}\mathbf{A}(\lambda_1)\right)\omega^1 \wedge \omega^4$$

Integrability by null surfaces: sufficient conditions

Proposition : Let $\lambda\colon \mathscr{G} \to \mathscr{N} \setminus \{\mathscr{D}\}$ be a special null plane such that it is a repeated root of $\mathbf{A}$ and $\mathbf{B}$ of multiplicity $k \geq 2$ and $\geq k$ respectively. Then λ is integrable iff λ is a root of $\mathbf{C}$ of multiplicity $\geq k-1.$

Assume $k=2$. As before, define the co-dim 2 subbundle $\mathscr{G}_2 \subset \mathscr{G}$

$$\mathscr{G}_2 = \{u \in \mathscr{G} \mid a_0(u) = a_1(u) = b_0(u) = b_1(u) = 0)\}.$$

$\mathbf{A}$ and $\mathbf{B}$ sharing equally repeated roots defines a section of $\mathscr{N}$

$$\mathrm{d}a_0 \equiv \tfrac{4}{3}a_0\phi_0 + 4a_0\phi_1 + 4a_1\gamma_1, \quad \mathrm{d}a_1 \equiv a_0\theta^1 + \tfrac{4}{3}a_1\phi_0 + 2a_1\phi_1 + 3a_2\gamma_1$$

$$\mathrm{d}b_0 \equiv \tfrac{4}{3}(a_0\xi_0 - a_1\xi_1) + \tfrac{5}{3}b_0\phi_0 + 3b_0\phi_1 + 3b_1\gamma_1$$

$$\mathrm{d}b_1 \equiv \tfrac{4}{3}(a_1\xi_0 - a_2\xi_1) + b_0\theta^1 + \tfrac{5}{3}b_1\phi_0 + b_1\phi_1 + 2b_2\gamma_1.$$

One obtains a 2D reduction which gives modulo $\langle \omega^0, \omega^2, \omega^3 \rangle$

$$\gamma_1 \equiv -\tfrac{1}{2a_2}h_3\omega^4, \qquad \xi_1 \equiv -\tfrac{9}{8a_2}c_0\omega^1 + (\cdots)\omega^4$$

$$\Rightarrow \mathrm{d}\omega^3 \equiv 0, \qquad \mathrm{d}\omega^2 \equiv \tfrac{9}{8a_2}c_0\omega^1 \wedge \omega^4, \qquad \mathrm{d}\omega^0 = \tfrac{1}{2a_2}h_3\omega^1 \wedge \omega^4, \qquad h_3 = -\tfrac{3}{4}c_{0;1}$$

Goldberg-Sachs theorem

Theorem : Given a special null plane dist $\mathscr{K} : M \to \mathscr{N}$ and $k \geq 2$ any two of the following imply the third

(1) $\mathscr{K}$ is a repeated root of $\mathbf{A}$ and $\mathbf{B}$ with multiplicity k and $\geq k$ resp.

(2) $\mathscr{K}$ is a root of $\mathbf{C}$ with mulitplicity at least $k-1$ everywhere.

(3) $\mathscr{K}$ is integrable.

What is left is to show $(2)+(3) \to (1)$ which is a proof by contradiction.

Remarks :
The set of repeated roots of $\mathbf{A}$ and $\mathbf{B}$ with equal multiplicity defines a finite set of points at each fiber of $\mathscr{N}$ in non-flat case.
Similarly one can investigate Frobenius integrability in (3,6)-geometries by exploiting the isomorphism $\mathfrak{sl}(4) \to \mathfrak{so}(3,3)$.

3-integrability: necessary conditions

Let $\mathcal{N}^* \to M$ be the twistor $\mathbb{P}^2$-bundle of *special null co-planes*. They can be parametrized as the dual to the null planes of $\mathcal{N}$ wrt to the conformal metric i.e. in the affine chart $\lambda_2 = 1$, one has $I = \langle \zeta^1, \zeta^2 \rangle$

$$\zeta^1 = \omega^3 - \lambda_1 \omega^4, \qquad \zeta^2 = \omega^0 - 6\lambda_3^2 \omega^4 - \lambda_1 \omega^1 - 4\lambda_3 \omega^2$$

One finds modulo I, $\mathrm{d}I \subset I$ implies $\zeta^3, \zeta^4, \zeta^5 \in s^* I$ where

$$\begin{aligned}
\zeta^3 =& \mathrm{d}\lambda_1 - 12\lambda_3^3 \omega^4 - \lambda_1^2 \theta^1 - 6\lambda_3^2 \omega^2 - 2\lambda_1 \phi_1 + \gamma_1 \\
\zeta^4 =& \mathrm{d}\lambda_3 + \lambda_3^2 \omega^1 + \tfrac{1}{12}\mu\omega^2 + \tfrac{1}{4}\mu\lambda_3\omega^4 - \tfrac{1}{3}\lambda_3\phi_0 - \lambda_3\phi_1 - \lambda_1\lambda_3\theta^1 - \tfrac{1}{3}\lambda_1\xi_0 - \tfrac{1}{3}\xi_1 \\
\zeta^5 =& \mathrm{d}\mu + \tfrac{3}{4}\mu\lambda_3\omega^1 + \mu^2\omega^4 - \mu(\phi_0 + \phi_1) + (6\lambda_3^3 - \lambda_1\mu)\theta^1 \\
& + 6\lambda_3^2\xi_0 - 3\lambda_3\xi_2 + \lambda_1\xi_3 + \xi_4
\end{aligned}$$

Thus the necessary conditions for 3-integrability defines an 8D twistor bundle which is the leaf space of

$$\langle \omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \gamma_1, \xi_1, \xi_4 \rangle$$

whose fibers are the 3D normal subgroup $A_{\mathrm{g}_1, x_1, x_4} \subset P_1$.

3-integrability: necessary conditions

Modulo $\langle\zeta^1,\zeta^2,\zeta^3,\zeta^4\rangle$ one has $\mathrm{d}\zeta^1,\mathrm{d}\zeta^2\equiv 0$ and

$$\begin{aligned}\mathrm{d}\zeta^3 &\equiv \mathbf{A}\omega^1\wedge\omega^4+\left(\mathbf{B}+\lambda_3\mathbf{A}'\right)\omega^2\wedge\omega^4\\ \mathrm{d}\zeta^4 &\equiv \tfrac{1}{4}\left(\mathbf{B}+\lambda_3\mathbf{A}'\right)\omega^1\wedge\omega^4\\ &\qquad+\left(\tfrac{1}{4}\mathbf{C}+\tfrac{2}{3}\lambda_3\mathbf{B}'+\tfrac{1}{3}\lambda_3^2\mathbf{A}''\right)\omega^2\wedge\omega^4+\tfrac{1}{12}(3\lambda_3\omega^4+\omega^2)\wedge\zeta^5\end{aligned}$$

and modulo $\langle\zeta^1,\zeta^2,\zeta^3,\zeta^4,\zeta^5\rangle$ one finds

$$\begin{aligned}\mathrm{d}\zeta^5 \equiv &\left(-\tfrac{1}{4}\mu\mathbf{A}'+\mathbf{H}+\tfrac{9}{8}\lambda_3\mathbf{C}+\tfrac{3}{2}\lambda_3^2\mathbf{B}'+\tfrac{1}{2}\lambda_3^3\mathbf{A}''\right)\omega^1\wedge\omega^4\\ &+\left(-\tfrac{1}{3}\mu\mathbf{B}'+\mathbf{K}+\tfrac{5}{4}\lambda_3\mathbf{D}+\tfrac{4}{3}\lambda_3\mathbf{H}'+\tfrac{9}{2}\lambda_3^2\mathbf{C}'+4\lambda_3^3\mathbf{B}''+\lambda_3^4\mathbf{A}'''\right)\omega^2\wedge\omega^4\end{aligned}$$

where

$$\begin{aligned}\mathbf{H} &= k_3\lambda_1^3+3(h_6-\tfrac{1}{16}d_1)\lambda_1^2+3(h_4-\tfrac{1}{32}d_0)\lambda_1+h_3\\ \mathbf{K} &= k_2\lambda_1^2+2(h_5+\tfrac{3}{64}e)\lambda_1+h_2\end{aligned}$$

3-integrability: sufficient conditions

Using the natural map between special null planes and co-planes direct inspection implies:

Proposition : 3-integrability implies 2-int by special null planes.

The following can be shown using the same method as before.

Proposition : Let $\mathscr{K} : M \to \mathscr{N}^* \setminus \{\partial\mathscr{D}\}$ have the property that it is a repeated root of $\mathbf{A}, \mathbf{B}$ with multiplicity $k \geq 2$ and $\geq k$ and a root of $\mathbf{C}$ with multiplicity $\geq k-1$ everywhere. Then, $\mathscr{K}$ is Frobenius integrable if and only if it is a root of $\mathbf{K}$ with multiplicity at least $k-1$.

Remarks : The 2-integrability is evident above. One is likely to find a Goldberg-Sachs-like theorem for 3-integrability.

Scalar 4th order ODEs as a Cartan geometry

Contact equivalent classes of scalar 4th order ODEs define a Cartan geometry $(\pi\colon \mathcal{G} \to M, \psi)$ of type $(\mathrm{GL}_2 \ltimes \mathbb{R}^4, B)$ (due to Morimoto, Doubrov, Komrakov, Čap, The,...) where ψ is

$$\tilde{\psi} = \begin{pmatrix} -\frac{2}{3}\phi_0 + \varepsilon & \frac{1}{3}\xi_0 & 0 & 0 & 0 \\ \omega^0 & -\frac{1}{3}\phi_0 - \phi_1 + \varepsilon & 0 & 0 & 0 \\ \omega^1 & \theta^1 & -\frac{1}{3}\phi_0 + \phi_1 + \varepsilon & \frac{2}{\sqrt{3}}\xi_0 & 0 \\ \frac{2}{\sqrt{3}}\omega^2 & -\frac{1}{\sqrt{3}}\omega^1 & \frac{1}{\sqrt{3}}\omega^0 & \varepsilon & \frac{2}{\sqrt{3}}\xi_0 \\ \omega^3 & -\frac{1}{3}\omega^2 & 0 & \frac{1}{\sqrt{3}}\omega^0 & \frac{1}{3}\phi_0 - \phi_1 + \varepsilon \end{pmatrix}$$

Scalar 4th order ODEs as a Cartan geometry

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$$\begin{pmatrix}
-\frac{2}{3}\phi_0 & \frac{1}{3}\xi_0 & \frac{1}{3}\xi_1 & -\frac{1}{\sqrt{3}}\xi_2 & -\xi_3 & -\xi_4 & 0 \\
\omega^0 & -\frac{1}{3}\phi_0 - \phi_1 & \gamma_1 & -\frac{2}{\sqrt{3}}\xi_1 & \xi_2 & 0 & -\xi_4 \\
\omega^1 & \theta^1 & -\frac{1}{3}\phi_0 + \phi_1 & \frac{2}{\sqrt{3}}\xi_0 & 0 & \xi_2 & \xi_3 \\
\frac{2}{\sqrt{3}}\omega^2 & -\frac{1}{\sqrt{3}}\omega^1 & \frac{1}{\sqrt{3}}\omega^0 & 0 & \frac{2}{\sqrt{3}}\xi_0 & \frac{2}{\sqrt{3}}\xi_1 & -\frac{1}{\sqrt{3}}\xi_2 \\
\omega^3 & -\frac{1}{3}\omega^2 & 0 & \frac{1}{\sqrt{3}}\omega^0 & \frac{1}{3}\phi_0 - \phi_1 & \gamma_1 & -\frac{1}{3}\xi_1 \\
\omega^4 & 0 & -\frac{1}{3}\omega^2 & \frac{1}{\sqrt{3}}\omega^1 & \theta^1 & \frac{1}{3}\phi_0 + \phi_1 & \frac{1}{3}\xi_0 \\
0 & \omega^4 & -\omega^3 & \frac{2}{\sqrt{3}}\omega^2 & -\omega^1 & \omega^0 & \frac{2}{3}\phi_0
\end{pmatrix}$$

Scalar 4th order ODEs as a Cartan geometry

Contact equivalent classes of scalar 4th order ODEs define a Cartan geometry $(\pi\colon \mathcal{G} \to M, \psi)$ of type $(\mathrm{GL}_2 \ltimes \mathbb{R}^4, B)$ (due to Morimoto, Doubrov, Komrakov, Čap, The,...) where ψ is

$$\tilde{\psi} = \begin{pmatrix} -\frac{2}{3}\phi_0+\varepsilon & \frac{1}{3}\xi_0 & 0 & 0 & 0 \\ \omega^0 & -\frac{1}{3}\phi_0-\phi_1+\varepsilon & 0 & 0 & 0 \\ \omega^1 & \theta^1 & -\frac{1}{3}\phi_0+\phi_1+\varepsilon & \frac{2}{\sqrt{3}}\xi_0 & 0 \\ \frac{2}{\sqrt{3}}\omega^2 & -\frac{1}{\sqrt{3}}\omega^1 & \frac{1}{\sqrt{3}}\omega^0 & \varepsilon & \frac{2}{\sqrt{3}}\xi_0 \\ \omega^3 & -\frac{1}{3}\omega^2 & 0 & \frac{1}{\sqrt{3}}\omega^0 & \frac{1}{3}\phi_0-\phi_1+\varepsilon \end{pmatrix}$$

with $\varepsilon = \frac{1}{5}(\phi_0+\phi_1)$.

The fund invs are four scalars: two Wilczyński invariants w_1, w_0 of homog 3,4 and Bryant invariants c_1, c_0 of homog 3,4.

Scalar 4th order ODEs: structure equations

A linear scalar ODE of order $n+1$ expressed as

$$y^{(n+1)} = p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots p_0(x)y(x)$$

where $y(x)$ is a $\mathbb{R}^{n+1}$-valued function, is in Laguerre-Forsyth normal form if $p_n = p_{n-1} = 0$. In our case we have $n+1=4$.

The soln space $\mathscr{S}$ is leaf space of the solutions curves tangent to $\frac{\partial}{\partial\omega^0}$.
At $s \in \mathscr{S}$, define the co-vector function $y = \omega^3$ where $\omega^3 \in T^*_s\mathscr{S}$ and

$$y^{(k)} = \mathscr{L}^k_{\frac{\partial}{\partial\omega^0}}\omega^3.$$

Using the structure equations one obtains

$$y' = -\omega^2, \quad y'' = \omega^1, \quad y''' = w_1\omega^3 + \theta^1.$$

Thus, the linearization of the ODEs along the solution $s \in \mathscr{S}$ is

$$y'''' = 2w_1y' - (w_0 + w_{1;0})y.$$

Almost conformally quasi-symplectic structures

A quasi-symplectic 2-form on M^5 is the odd-dimensional analogue of symplectic 2-form i.e.

$$\rho \subset \bigwedge^2 T^* M: \qquad \rho \wedge \rho \neq 0, \qquad \mathrm{d}\rho = 0.$$

Any 4th order order ODE uniquely determines a 2-form up to scale, $[\rho]$,

$$\rho = -\theta^1 \wedge \omega^3 - \omega^1 \wedge \omega^2$$

which has *maximal rank* but not closed, referred to as an *almost conformally quasi-symplectic structure* (ACQS), with its degenerate direction tangent to the solution curves i.e. $\frac{\partial}{\partial \omega^0}$.

ACQS structure is called CQS if the $[\rho]$ has a closed representative.

Proposition : Being CQS is equivalent to the induced connection on the line bundle $[\rho] \subset \bigwedge^2 T^* M$ being torsion-free and flat.

$$\mathrm{d}\rho = -(\phi_0 + \phi_1) \wedge \rho \quad \& \quad \mathrm{d}(\phi_0 + \phi_1) = 0 \Rightarrow w_1, c_1 = 0.$$

Conf q-symplectic 2-forms and quasi-contactification

Given a CQS 4th order ODE on M^5, let $\tilde{M}^6 = M \times \mathbb{R}$ with $\pi: \tilde{M} \to M$. Take a representative $\rho_0 \in [\rho]$ such that locally

$$\mathrm{d}\rho_0 = 0 \Rightarrow \rho_0 = \mathrm{d}\omega^4$$

Define $\tilde{\omega}^4 = \pi^*\omega^4 + \mathrm{d}t$ on $\tilde{M}$. Then $\mathrm{d}\tilde{\omega}^4 = \pi^*\rho_0$, and it defines a *quasi-contact* structure on $\tilde{M}^6$ i.e. $\ker\tilde{\omega}^4$ is maximally non-integ and $\tilde{\omega}^4$ has a degenerate direction.

The scaling action on ρ_0 naturally extends to $\tilde{\omega}^4$ and one obtains $[\tilde{\omega}^4]$.

Let $\tilde{\mathscr{G}} \to \tilde{M}$ be the pull-back of the structure bundle $\mathscr{G} \to M$, and $\tilde{\psi}$ the pull-back of ψ with $\tilde{\omega}^4$ as the lifted 1-form.

On $\tilde{M}$ the rank 2 distribution $\ker\{\tilde{\omega}^1, \cdots, \tilde{\omega}^4\}$ has a splitting and defines a (P_2, B) geometry where $P_2 \subset \mathrm{G}_2$.

(2,3,5)-geometries from 4th order ODEs

One can find the corresponding (G_2, P_{12}) Cartan geometry which is *regular* and therefore defines a (2,3,5)-geometry.

The normal connection of the (2,3,5)-geometry is

$$\tilde{\omega}^3 = \omega^3, \quad \tilde{\omega}^2 = \omega^2, \quad \tilde{\omega}^1 = \omega^1, \quad \tilde{\theta}^1 = \theta^1 \quad \tilde{\omega}^0 = \omega^0 - \tfrac{3}{16} c_0 \tilde{\omega}^4,$$
$$\tilde{\phi}_0 = \phi_0 + \tfrac{9}{64} c_{0;0} \tilde{\omega}^4, \quad \tilde{\phi}_1 = \phi_1 - \tfrac{9}{64} c_{0;0} \tilde{\omega}^4, \quad \tilde{\xi}_0 = \xi_0 - (\tfrac{9}{32} c_{0;00} + \tfrac{1}{8} w_{0;\underline{11}}) \tilde{\omega}^4.$$

where $w_{0;\underline{1}} = \frac{\partial}{\partial \theta^1} w_0$.

The scalar harmonic inv of (G_2, P_{12}) geometry is $\pi^* w_0 (\omega^0)^4$ and

$$a_4 = w_0, \ \ a_3 = \tfrac{1}{4} w_{0;\underline{1}}, \ \ a_2 = \tfrac{1}{12} w_{0;\underline{11}}, \ \ a_1 = \tfrac{1}{24} w_{0;\underline{111}}, \ \ a_0 = \tfrac{1}{24} w_{0;\underline{1111}},$$

and, $w_{0;\underline{11111}} = 0$.

Flatness, integrability and holonomy reduction

4th order ODE locally are given by 1 function of 5 variables.
CQS 4th order ODEs locally depend on 1 function of 4 varialbes.

We consider the following cases.

- When $w_0 = 0,$ then the (2,3,5)-geometry is flat. Such fourth order ODEs define torsion-free GL_2-structures with symmetric Ricci and their local moduli depends on 5 constants.
- When $c_0 = 0,$ then $w_{0;\underline{111}} = 0,$ i.e. the Cartan quartic is type $II,$ and (2,3,5)-geometries are 3-integrable i.e. $\langle \tilde{\omega}^0, \tilde{\omega}^4 \rangle$ is Frobenius. The local generality is 2 functions of 3 variables.
- If $c_0 = 0$ and $w_{0;\underline{11}} = 0$ i.e. the Cartan quartic has type III, then one has a holonomy reduction of the (2,3,5)-geometry to P_2, the local generality is 1 function of 3 variables.
- If $c_0 = 0$ and $w_{0;\underline{1}} = 0$ i.e. the Cartan quartic has type N the local generality is 2 function of 2 variables.

Variational 4th order ODEs ⇔ (2,3,5)-dist + infin symmetry

Theorem : (Fels 1996) A 4th order ODE is CQS iff it is variational.

Corollary : Variational 4th order ODEs ⇔ (2,3,5)-dist + infin symmetry

Given a (2,3,5)-geometry, consider the corresponding (G_2, P_{12}) geom. If it arises from a 4th order ODE then $\frac{\partial}{\partial t}$ is an infin symmetry that is transversal to the quasi-contact distribution.

Conversely, if there is a transv. infin. symm. v then there is reduction of the (G_2, P_{12}) to a (P_2, B)-geometry. In particular, choose a q-cont form s.t. $\theta(v) = 1$. Since $\mathscr{L}_v\theta = \mathscr{L}_v \mathrm{d}\theta = 0,$ one has $\mathrm{d}\theta(v, .) = 0 \Rightarrow \mathrm{d}\theta = \pi^* \rho$ where $\rho \in \bigwedge^2 T^* M$ is CQS where M is the leaf space of v. Identifying the q-contact distribution with TM, one can show that it is equipped with a CQS 4th order ODE.

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