Symmetric trilinear forms and Einstein-like equations: from affine spheres to Griess algebras

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General scheme: hierarchy of coupled equations

- $SO(n)$-irreducible $\mathbb{W}$. Associated module of trace-free tensors.
- $\mathcal{MC}$ = curvature tensors.
- Symmetric $SO(n)$-map $\phi : \mathbb{W} \times \mathbb{W} \to \mathcal{MC}$, $\mathrm{tr}_h\, \rho(\phi(\omega, \omega)) = |\omega|^2$.
- Generalized gradients $\mathcal{A}_1, \dots, \mathcal{A}_N$, symbols $\sigma_{\mathcal{A}_i}(X)(\omega)$.
- Stress-energy tensor: $\mathcal{T}(\omega) = \rho(\phi(\omega, \omega)) + \frac{1}{2r}|\omega|^2 h$, some $r \in \mathbb{R}$.

$$h(X, \mathrm{div}\,(\mathcal{T})) = \sum_{i=1}^{N} \beta_i h(\sigma_{\mathcal{A}_i}(X)(\omega), \mathcal{A}_i(\omega)).$$

- Coupled projective flatness:

$$\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N, \quad Riem - \epsilon\phi(\omega, \omega) = -\tfrac{\kappa}{n(n-1)} h \owedge h, \quad \epsilon \in \{\pm 1\}.$$

- Coupled Einstein equations:

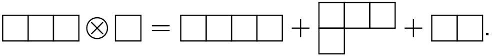
$$\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N, \quad G + \Lambda h = \epsilon\mathcal{T} \iff Ric - \epsilon\, \rho(\phi(\omega, \omega)) = \tfrac{\kappa}{n} h.$$

- $\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N$, and constancy of $\Lambda = \frac{n-2}{2n}\left(s + \epsilon\frac{n+2r}{r(n-2)}|\omega|^2\right)$.

Coupled equations for trace-free Codazzi tensors

Suppose $\omega_{i_1\dots i_k} = \omega_{(i_1\dots i_k)}$ is trace-free. $\operatorname{div}(\omega)_{i_1\dots i_{k-1}} = D^p\omega_{pi_1\dots i_{k-1}} = 0$, $\mathcal{C}(\omega)_{iji_1\dots i_{k-1}} = D_{[i}\omega_{j]i_1\dots i_{k-1}}$.

$$\mathcal{K}(\omega)_{iji_1\dots i_{k-1}} = \mathcal{C}(\omega)_{iji_1\dots i_{k-1}} - \frac{1}{n+k-3}\sum_{s=1}^{k-1} h_{i_s[i}\operatorname{div}(\omega)_{j]i_1\dots\hat{i}_s\dots i_{k-1}}.$$



- Coupled projective flatness:

$$\mathcal{K}(\omega) = 0, \quad \operatorname{div}(\omega) = 0, \quad Riem - \epsilon(\omega \owedge \omega) = -\tfrac{\kappa}{n(n-1)}(h \owedge h) \quad \epsilon \in \{\pm 1$$

- Coupled Einstein equations:

$$\mathcal{K}(\omega) = 0, \quad \operatorname{div}(\omega) = 0, \quad G + \Lambda h = \epsilon\mathcal{T}(\omega) \iff Ric - \epsilon\,\rho(\omega \owedge \omega) = \tfrac{\kappa}{n}$$

- $\mathcal{K}(\omega) = 0, \operatorname{div}(\omega) = 0$ and constancy of $\Lambda = \frac{n-2}{2n}(s - \epsilon|\omega|^2) = \frac{n-2}{2n}\kappa$.

Weitzenböck machinery for Codazzi tensors

- h a Riemannian metric on M^n, $n > 2$, and $\omega \in \Gamma(S^k_0(T^*M))$.
- $\mathcal{L}$ the trace-free part of the symmetrized covariant derivative.

General formulas:

$$\begin{aligned}
\widehat{\mathcal{R}}(\omega)_{i_1\dots i_k} &= k\,\rho(\mathcal{R})_{p(i_1}\omega_{i_2\dots i_k)}{}^p - k(k-1)\mathcal{R}_{p(i_1 i_2}{}^q\omega_{i_3\dots i_k)q}{}^p,\\
\mathcal{Q}_{\mathcal{R}}(\omega) &= \langle \omega, \widehat{\mathcal{R}}(\omega)\rangle = k\langle \rho(\omega \owedge \omega), \rho(\mathcal{R})\rangle + \binom{k}{2}\langle \omega \owedge \omega, \mathcal{R}\rangle,\\
\Delta\omega &= \tfrac{1}{k}\widehat{\mathcal{R}}(\omega) + \tfrac{n+2(k-2)}{n+k-3}\mathcal{L}\mathrm{div}\,(\omega) + 2\mathcal{K}^*\mathcal{K}(\omega).
\end{aligned}$$

Weitzenböck formula:

$$\tfrac{1}{2}\Delta|\omega|^2 = |D\omega|^2 + \tfrac{n+2(k-2)}{n+k-3}\langle \omega, \mathcal{L}\mathrm{div}\,(\omega)\rangle + 2\langle \omega, \mathcal{K}^*\mathcal{K}(\omega)\rangle + \tfrac{1}{k}\mathcal{Q}_{\mathcal{R}}(\omega).$$

With refined Kato $\implies$ if $\omega \in \ker \mathcal{C} \cap \ker \mathcal{C}^*$, wherever $\omega \neq 0$,

$$|\omega|^{(n+2(k-1))/(n+k-2)}\Delta_h|\omega|^{(n-2)/(n+k-2)} \geqslant \tfrac{n-2}{k(n+k-2)}\mathcal{Q}_{\mathcal{R}}(\omega).$$

AH structures

An AH structure on M^n is a pair $([\nabla], [h])$ comprising a projective structure, $[\nabla]$, and a conformal structure, $[h]$, such that for each $\nabla \in [\nabla]$ and each $h \in [h]$ there is a one-form γ_i such that $\nabla_{[i}h_{j]k} = 2\gamma_{[i}h_{j]k}$.

- Identify AH structure with the pair $(\nabla, [h])$ where $\nabla \in [\nabla]$ is unique aligned representative satisfying $h^{pq}\nabla_i h_{pq} = nh^{pq}\nabla_p h_{qi}$.
- Faraday primitive: one-form $\gamma_i = \frac{1}{2n}h^{pq}\nabla_i h_{pq}$.
- Faraday curvature $F_{ij} = -d\gamma_{ij}$ does not depend on h and equals curvature of connection induced on $-1/n$ densities.
- AH structure is closed if $F_{ij} = 0$.
- AH structure is exact if there is $h \in [h]$ for which γ vanishes.
- Such h is distinguished (it is determined up to homothety).

Relation between AH and statistical structures

Definition: A statistical structure is a pair (∇, h) such that $\nabla_{[i}h_{j]k} = 0$.

- A statistical structure is special if $\nabla_i |\det h| = 0$.
- For statistical (∇, h), ∇ need not be aligned with respect to $[h]$.
- exact AH structure = homothety class of special statistical structures:

Lemma: The aligned representative of the AH structure $([\nabla], [h])$ generated by the statistical structure (∇, h) is $\tilde{\nabla} = \nabla + 2\sigma_{(i}\delta_{j)}{}^{k}$, where $\sigma_i = \frac{1}{n+2}h^{pq}\nabla_i h_{pq}$. In particular, the following are equivalent:

- ∇ is the aligned representative of $([\nabla], [h])$.
- The statistical structure (∇, h) is special.
- The AH structure $([\nabla], [h])$ is exact.

AH structures are locally statistical

Locally statistical structure: $([\nabla],[h])$ such that every $p \in M$ contained in an open $U \subset M$ on which there are $\nabla \in [\nabla]$ (not necessarily aligned) and $h \in [h]$ such that (∇, h) is a statistical structure on U.

Lemma: The following are equivalent.

- $([\nabla],[h])$ is locally statistical.
- For the aligned representative $\nabla \in [\nabla]$ and any $h \in [h]$, there is a one-form τ_i such that $\nabla_{[i}h_{j]k} = \tau_{[i}h_{j]k}$.
- For any $h \in [h]$ there is $\tilde{\nabla} \in [\nabla]$ such that $(\tilde{\nabla}, h)$ is statistical.
- $([\nabla],[h])$ is an AH structure.

A closed AH structure is locally statistical for the aligned $\nabla \in [\nabla]$.

Flat AH structures

- Curvature of AH $([\nabla], [h])$ = curvature of aligned $\nabla \in [\nabla]$.
- Curvature of statistical (∇, h) means curvature of ∇.

- A Kähler affine structure = flat statistical structure.
- Kähler affine structures are often called Hessian, but seems better to use Hessian for when $h = \nabla dF$ for a global potential F.
- With this convention, Kähler affine structures are locally Hessian.
- Locally Kähler affine $\iff$ projectively flat AH.
- Flat special Hessian = solution of WDVV or associativity equations.
- Monge-Ampère metric: Kähler affine (∇, h) with $\nabla \log |\det h|$ parallel.

Construction of AH structures

Given AH structure $([\nabla], [h])$ and $h \in [h]$ with Faraday primitive γ_i:

- The trilinear form $\mathcal{L}_{ijk} = \nabla_i h_{jk} - 2\gamma_i h_{jk}$ is symmetric and trace-free.
- The cubic torsion $\mathcal{L}_{ij}{}^{k} = h^{kp}\mathcal{L}_{ijp}$ is independent of $h \in [h]$.

Construction

Data: metric h with Levi-Civita connection D, one-form γ_i, and completely symmetric trace-free 3-tensor $\mathcal{L}_{ijk} = \mathcal{L}_{(ijk)}$.
With $[h]$, the connection

$$\nabla = D + \mathcal{L}_{ijp}h^{pk} - 2\gamma_{(i}\delta_{j)}{}^{k} + h_{ij}h^{kp}\gamma_p$$

generates an AH structure for which it is the aligned representative and γ_i is the associated Faraday primitive.

Conjugacy of AH structures

$([\nabla], [h])$ an AH structure with cubic torsion $\mathcal{L}_{ij}{}^{k}$.

- $\bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^{k}$ generates with $[h]$ an AH structure for which it is aligned and having cubic torsion $\bar{\mathcal{L}}_{ij}{}^{k} = -\mathcal{L}_{ij}{}^{k}$.
- The AH structure $([\bar{\nabla}], [h])$ is conjugate to $(\nabla, [h])$.
- Conjugacy is involutive.

The following are equivalent:

- $([\nabla], [h])$ is self-conjugate.
- Cubic torsion vanishes, $\mathcal{L}_{ijk} \equiv 0$.
- $([\nabla], [h])$ is Weyl.

AH structure induced on an affine hypersurface

M^n a cooriented nondegenerate hypersurface in $(N^{n+1}, \hat{\nabla})$.

The pair $(\nabla, [h])$ is an AH structure if there vanishes the trace-free part of the normal curvature of $\hat{\nabla}$ (the normal part of $\hat{R}(X, Y)Z$).

This is always true if $\hat{\nabla}$ is projectively flat.

A cooriented nondegenerate immersed hypersurface M in flat affine space $(\mathcal{A}, \hat{\nabla})$ acquires two conjugate exact AH structures:

- The AH structure $([\nabla], [h])$ induced via the affine normal.
- The pullback $[\bar{\nabla}]$ via the conormal Gauss map $M \to \mathbb{P}^+(\mathcal{A}^*)$ of the flat projective structure on $\mathbb{P}^+(\mathbb{V})$ constitutes with $[h]$ the AH structure conjugate to $([\nabla], [h])$.

- Definition: $([\nabla], [h])$ is conjugate projectively flat.
- Characterizes AH structures locally equivalent to affine hypersurfaces.

Special statistical structures on an affine hypersurface

Alternative formulation.

A cooriented nondegenerate hypersurface in flat affine space acquires a pair of conjugate special statistical structures, one of which is projectively flat.

- Special statisical $\iff$ cubic form $\mathcal{L}_{ijk} = \nabla_i h_{jk}$ symmetric, trace-free.
- $\bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^{k}$ generates with h the conjugate statistical structure $(\bar{\nabla}, h)$ having trilinear form $\bar{\mathcal{L}}_{ijk} = -\mathcal{L}_{ijk}$ and also special.

$$\nabla - \tfrac{1}{2}\mathcal{L}_{ij}{}^{k} = D = \bar{\nabla} + \tfrac{1}{2}\mathcal{L}_{ij}{}^{k}, \quad \bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^{k}.$$

D is Levi-Civita of h. $\mathcal{L}_{ij}{}^{k} = h^{kp}\nabla_p h_{ij}$. $\mathcal{L}_{ip}{}^{p} = 0$.

- Conjugate projectively flat $\implies$ locally an affine hypersurface.

Curvature of special statistical structures

$R_{ijkl} = R_{ijk}{}^{p}h_{pl}$ curvature of ∇. $\bar{R}_{ijkl} = \bar{R}_{ijk}{}^{p}h_{pl}$ curvature of $\bar{\nabla}$.

$\bar{R} - R = \mathcal{C}(\mathcal{L})$ and $\rho(\bar{R}) - \rho(R) = \operatorname{div}(\mathcal{L})$.

- Curvature is self-conjugate $\iff$ $\mathcal{L}$ is Codazzi.
- Ricci curvature is self-conjugate $\iff$ $\mathcal{L}$ is divergence free.

Proof:
$$-2R_{ij(kl)} = 2\nabla_{[i}\nabla_{j]}h_{kl} = 2\nabla_{[i}\mathcal{L}_{j]kl} = 2D_{[i}\mathcal{L}_{j]kl} = \mathcal{C}(\mathcal{L})_{ijkl},$$
$$\implies \bar{R}_{ijkl} = R_{ijkl} + 2\nabla_{[i}\mathcal{L}_{j]kl} = -R_{ijlk},$$
$$\implies R_{ip}{}^{p}{}_{j} = \rho(\bar{R})_{ij} \implies \operatorname{tr}_h \rho(R) = \operatorname{tr}_h \rho(\bar{R}).$$

Curvature tensor of D: $Riem = R + \frac{1}{4}\mathcal{L} \owedge \mathcal{L} + \frac{1}{2}\mathcal{C}(\mathcal{L})$,

Ricci curvature of D: $Ric = \rho(R) + \frac{1}{4}\rho(\mathcal{L} \owedge \mathcal{L}) + \frac{1}{2}\operatorname{div}(\mathcal{L})$.

Einstein equations for special statistical structures

"Natural" notion of Einstein

A special statistical structure (∇, h) is Einstein if:

- (Naive Einstein) $\rho(R)$ and $\rho(\bar{R})$ are multiples of h.
- (Conservation) The scalar curvature $\mathrm{tr}_h\, \rho(R) = \mathrm{tr}_h\, \rho(\bar{R})$ is constant.

- Recover usual Einstein equations if $\nabla = D$.
- If Weyl curvature is self-conjugate, naive Einstein $\implies$ conservation.
- If $\mathcal{L}$ is Codazzi, constancy of s follows from naive Einstein condition.

Better notion (?)

(∇, h) Einstein + self-conjugate curvature $\iff$ $(h, \mathcal{L})$ coupled Einstein.

(∇, h) has $R = \alpha h \owedge h \iff (h, \mathcal{L})$ coupled projectively flat.

- Key point is $R = \alpha h \owedge h \implies \bar{R} = \alpha h \owedge h \implies \mathcal{L}$ Codazzi.

Example: naive Einstein not Einstein

There exist naive Einstein structures that are not Einstein.

- $h = \sum_{i=1}^{3} dx^i \otimes dx^i$ Euclidean with Levi-Civita connection D.
- $dx^{ijk} = \sum_{\sigma \in S_3} dx^{\sigma(i)} \otimes dx^{\sigma(j)} \otimes dx^{\sigma(k)}$.

$$L = (x_1 + x_3)dx^{112} + (x_1 - x_3)dx^{123} - (x_1 + x_3)dx^{233}$$

- $L^\sharp$ defined by $h(L^\sharp(u,v),w) = L(u,v,w)$.
- $\nabla = D + L^\sharp$.
- L is trace-free and divergence free.
- $(\nabla, [h])$ is naive Einstein special statistical but not Einstein. Its scalar curvature is a multiple of $x_1^2 + x_3^2$.

Einstein special statistical structures and affine spheres

A nondegenerate $M^n \subset \mathcal{A}^{n+1}$ is an affine sphere if its affine normals meet in a point (its center) or are parallel (center at infinity).

Theorem (Fox 2009): For a nondegenerate cooriented $M^n \subset \mathcal{A}^{n+1}$ with induced conjugate special statistical structures (∇, h) and $(\bar{\nabla}, h)$ the following are equivalent:

- M is an affine sphere.
- (∇, h) and $(\bar{\nabla}, h)$ have self-conjugate curvature.
- (∇, h) and $(\bar{\nabla}, h)$ are Einstein.

These conditions imply and, if $n > 2$, are implied by

- (∇, h) and $(\bar{\nabla}, h)$ are projectively flat.

Yields lots of solutions of the coupled projectively flat equations.

Generalizing constant scalar curvature

AH structure $(\nabla, [h])$ on M^n. For $h \in [h]$ and $s = h^{ij}Ric(\nabla)_{ij}$, the one-forms $ds_i + 2s\gamma_i$ and $h^{pq}\nabla_p d\gamma_{qi}$ rescale under change of h, and

$$\tfrac{n-2}{n}\left(ds_i + 2s\gamma_i - nh^{pq}\nabla_p F_{qi}\right) - 2\gamma^p(S_{ip} + \bar{S}_{ip}) - h_{ia}h^{pq}\left(\nabla_p S_q{}^a + \bar{\nabla}_p \bar{S}_q{}^a\right)$$
$$= \mathcal{L}^{abc}R_{i(abc)} = -\mathcal{L}^{abc}\mathcal{K}(\mathcal{L})_{iabc} - \tfrac{1}{n}\mathcal{L}_i{}^{pq}U_{pq},$$

with $\gamma^i = h^{ip}\gamma_p$, $S_{ij} = R_{(ij)} - \frac{s}{n}h_{ij}$, $S_i{}^j = h^{jb}S_{ib}$, $U_{ij} = \bar{R}_{ij} - R_{ij} = \bar{S}_{ij} - S_{ij}$.

- Proof: Adaptation of the usual argument showing the Einstein tensor $R_{ij} - \frac{1}{2}sh_{ij}$ is divergence free by tracing $2\nabla_{[i}R_{j]k} = \nabla_p R_{ijk}{}^p$.
- If $n > 2$: self-conjugate curvature and vanishing of $S_{ij} = \bar{S}_{ij} \implies$

  $$(\star) \qquad 0 = ds_i + 2s\gamma_i + nh^{pq}\nabla_p d\gamma_{qi}.$$

  $(\star)$ generalizes constancy of the scalar curvature of an Einstein metric.
- D. Calderbank: $(\star)$ as definition of Einstein for 2-D Weyl structures.

Einstein AH structures

Definition

An AH structure $(\nabla, [h])$ with conjugate $(\bar{\nabla}, [h])$ is:

- naive Einstein if for any $h \in [h]$, $R_{(ij)} = \frac{s}{n}h_{ij} = \bar{R}_{(ij)}$.
- Einstein if it is naive Einstein and for every $h \in [h]$ with associated one-form γ_i and scalar curvature s, there holds

$$(\star) \qquad 0 = ds_i + 2s\gamma_i + nh^{pq}\nabla_p d\gamma_{qi}.$$

- Recovers usual notion of Einstein-Weyl.
- By definition, $(\nabla, [h])$ is Einstein if and only if $(\bar{\nabla}, [h])$ is Einstein.
- When $n > 2$, self-conjugate curvature + naive Einstein $\implies$ Einstein.
- When $n = 2$, $R_{ijkl} = 2h_{l[i}S_{j]k} + sh_{l[i}h_{j]k} + F_{ij}h_{kl}$ and
  - $\bar{S}_{ij} + S_{ij} = 0$ is automatic
  - the naive Einstein equations are equivalent to self-conjugate curvature.
  - the vanishing of $ds_i + 2s\gamma_i + 2h^{pq}\nabla_p d\gamma_{qi}$ has to be required.

Bach tensor

- Bach tensor $\mathcal{O}_{ij}$ of a Weyl structure $([\nabla], [h])$ is the trace-free symmetric 2-tensor defined by

$$\mathcal{O}_{ij} = \nabla^p W_{pij} + W^{pq} W_{pijq} = \frac{1}{3-n} \nabla^p \nabla^q W_{pijq} + W^{pq} W_{pijq}.$$

  where W_{ijkl} and W_{ijk} are conformal Weyl and Cotton tensor, W_{ij} is conformal Schouten tensor plus $\frac{1}{2} F_{ij}$.
    - When $n = 4$, $\mathcal{O}_{ij}$ is divergence free and vanishes for closed Einstein Weyl structures and (anti)-self-dual conformal structures are Bach flat
    - On a compact 4-manifold, $\mathcal{O}_{ij}$ is first variation of $\int_M |W|^2$.
    - Is 4-dimensional special case of ambient obstruction tensor.
- On a 4-manifold there is a Bach tensor for AH structures. When the curvature is normal it has the properties:
    - Symmetric, divergence free, and self-conjugate.
    - For a closed Einstein AH structure with normal curvature, $\mathcal{O}_{ij} = 0$.
    - For a projectively flat AH structure with normal curvature, $\mathcal{O}_{ij} = 0$.
- Definition is formally the same as for Weyl structures, but checking the properties is more involved.

Convex affine spheres

Let M be a convex affine sphere with complete Blaschke metric h.

- (Blaschke-Deicke-Calabi) If M is elliptic, then h has positive constant curvature and M is an ellipsoid.
- (Jörgens-Calabi-Cheng-Yau) If M is a parabolic affine sphere, then h is flat and M is an elliptic paraboloid.
- (Calabi) If M is a hyperbolic, then h has nonpositive Ricci curvature.

Theorem (Cheng-Yau '86): The interior of a nonempty pointed closed convex cone is foliated in a unique way by hyperbolic affine spheres asymptotic to its boundary and center at its vertex.

Many examples of nonconvex affine spheres from level sets of real forms of relative invariants of irreducible prehomogeneous vector spaces and from nonassociative algebras (Fox, R. Hildebrand).

Cheng-Yau metric

Corollary: a properly convex flat projective manifold carries a a canonical homothety class of metrics that constitute with its projective structure an exact Einstein AH structure.

Proof (Idea due to J. Loftin 2001): descend the Blaschke metric of an affine sphere asymptotic to the cone over its universal cover.

- Yields abundance of nontrivial compact Riemannian Einstein AH.
- This is like Kähler-Einstein metric in negative Chern class case.
- Corollary yields a canonical duality of properly convex flat real projective manifolds that extends the duality of affine spheres.

Klartag: there is an (incomplete) elliptic affine sphere over a bounded convex domain with center at the Santaló point of the domain.

Are there Einstein special statistical structures with self-conjugate curvature that are not locally equivalent to affine spheres?

Link with algebra

Theorem (Fox, 2009)

For a metric h with Levi-Civita D and curvature $\mathcal{R}_{ijk}{}^{l}$ on M^n, $n > 2$, the following are equivalent:

- ω_{ijk} is trace-free, Codazzi and (h, ω) solves coupled Einstein equations for $\kappa \in \mathbb{R}$ and $c = 1$.
- $\nabla = D + \omega_{ij}{}^{k}$ (where $\omega_{ijk} = \omega_{ij}{}^{p} h_{pk}$) constitutes with h a special statistical structure (∇, h) with self-conjugate curvature satisfying

$$R_{ij} = R_{pij}{}^{p} = \mathcal{R}_{ij} - \omega_{ip}{}^{q}\omega_{jq}{}^{p} = \tfrac{\kappa}{n} h_{ij},$$

  and $h^{ij} R_{ij} = \mathcal{R}_h - |\omega|^2 = \kappa$ is constant.

Theorem applies with h flat and ω_{ijk} parallel $\iff$ ω is trilinear form of metrized commutative algebra that is exact Killing metrized with $\tau = \kappa h$.

Metrized commutative algebras

- Commutative $\Bbbk$-algebra $(\mathbb{A}, \circ)$: symmetric $\Bbbk$-bilinear $\circ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$.
- metric: a nondegenerate symmetric bilinear form $h \in S^2\mathbb{A}^*$.
- $L : \mathbb{A} \to \mathrm{End}(\mathbb{A})$ defined by $L(x)y = x \circ y$. (So $(L(x)y)^k = x^i y^j L_{ij}{}^k$.)

If h is flat and $\mathcal{L}_{ijk}$ is parallel, can view $\mathcal{L}_{ij}{}^k$ as the structure tensor of a commutative (generally nonassociative) multiplication.

- exact if $\operatorname{tr} L(x) = 0$ for all $x \in \mathbb{A}$. (So $L_{ip}{}^p = 0$.)
- metrized if it admits a metric h that is invariant, meaning:

$$h(x \circ y, z) = h(x, y \circ z), \qquad \text{(Equivalently: } L_{ijk} = L_{(ijk)}.\text{)}$$

Killing metrized commutative algebras

$(\mathbb{A}, \circ)$ a commutative $\Bbbk$-algebra ($\operatorname{char} \Bbbk = 0$ for simplicity).

- Killing metrized: metrized by its Killing form $\tau(x, y) = \operatorname{tr} L(x)L(y)$.
- Associator: $(x \circ z) \circ y - x \circ (z \circ y) = -[L(x), L(y)]z$.
- Projectively associative: there is symmetric bilinear form c such that

$$-[L(x), L(y)] = [x, \cdot\,, y] = c(y, \cdot\,) \otimes x - c(x, \cdot\,) \otimes y.$$

In this case $(1 - n)c(x, y) = \operatorname{tr} L(x \circ y) - \operatorname{tr} L(x)L(y)$ is invariant.

$[L(x), L(y)]_k{}^l = -x^i y^j (L \owedge L)_{ijk}{}^l$, so, for $(h, \mathcal{L})$ and $\nabla = D + \frac{1}{2}\mathcal{L}_{ij}{}^k$:

$$R(x, y) = \tfrac{1}{4}[L(x), L(y)], \qquad \rho(R) = \tfrac{1}{4}\operatorname{tr} L(x)L(y) = \tfrac{1}{4}\tau(x, y).$$

- $(h, \mathcal{L})$ coupled projectively flat $\iff$ $(\mathbb{A}, \circ)$ projectively associative.
- $(h, \mathcal{L})$ coupled Einstein $\iff$ $(\mathbb{A}, \circ)$ Killing metrized with $\tau = \kappa h$.

Equivalent problems

Partial differential equations

Classify $CO(n)$-orbits of orthogonally indecomposable cubic homogeneous polynomials $P(x)$ solving

$$(*) \qquad \Delta P = 0, \qquad |\operatorname{Hess} P(x)|^2 = \kappa |x|^2, \qquad 0 \neq \kappa \in \mathbb{R}.$$

Equivalent algebraic problem

Classify isomophism classes of simple exact Killing metrized commutative algebras $(\mathbb{A}, \circ, h)$ such that Killing form $\tau(x, y) = \operatorname{tr} L(x)L(y)$ satisfies $\tau = \kappa h$.

Characterization of Griess-Harada algebras

$\mathbb{E}^n(\Bbbk)$ is the n-dimensional commutative algebra generated by $n+1$ idempotents $\{e_i : 0 \leqslant i \leqslant n\}$ satisfying

$$\sum_{i=0}^{n} e_i = 0, \qquad e_i \circ e_j = -\tfrac{1}{n-1}(e_i + e_j), \qquad i \neq j \in \{0, \dots, n\}.$$

Example: $\mathbb{E}^2(\mathbb{R})$ is isomorphic to $\mathbb{C}$ with para-Hurwitz product $x \diamond y = \bar{x}\bar{y}$.

Theorem (Fox '20): If $\Bbbk$ is algebraically closed or $\Bbbk = \mathbb{R}$, an n-dimensional commutative $\Bbbk$-algebra is isomorphic to $\mathbb{E}^n(\Bbbk)$ if and only if it is exact, Killing metrized (and τ definite if $\Bbbk = \mathbb{R}$), and projectively associative.

Corollary: an exact Killing metrized commutative algebra not isomorphic to $\mathbb{E}^n(\mathbb{R})$ yields a solution of the coupled Einstein equations that is not coupled projectively flat.

Griess algebras

Most spectacular example.

Griess constructed a simple, exact 196883-dimensional exact commutative $\mathbb{R}$-algebra whose group of automorphisms is the Monster finite simple group $\mathbb{M}$. That it is Killing metrized follows from calculations made by Tits and Conway.

Frenkel-Lepowsky-Meurman constructed a vertex operator algebra (VOA) with automorphism group $\mathbb{M}$. Its 2-graded piece is Griess's algebra.

Matsuo made calculations like those of Tits and Conway for certain other VOAs satisfying a highly technical condition. Their 2-graded pieces are also exact, Killing metrized commutative algebras.

Probably many of the algebras on the next slide fit into the same setup.

Classification problem

Problem: classify simple exact Killing metrized commutative algebras.

Incomplete list of known examples over $\mathbb{R}$:

- $\mathbb{E}^n(\Bbbk)$ for $n \geqslant 2$. (Projectively associative.)
- Deunitalizations of simple Euclidean Jordan algebras.
- Finitely many Hsiang algebras (Tkachev).
- Nonunital Griess algebras of certain VOAs, e.g. Monster VOA.
- Algebra of Weyl curvature tensors (Fox '21).
- Certain Norton algebras of association schemes.
- Certain algebras associated with Steiner triple systems (Fox '22).
- Tensor products of simple Lie algebras. ($\mathfrak{so}(3) \otimes \mathfrak{so}(3) \leftrightarrow \det$)
- Tensor products of simple EKMCs with $\mathbb{E}^3(\Bbbk)$. (3 is special).

Classification of EKMCS for $n \leqslant 4$ when $\Bbbk = \mathbb{R}$, $\tau > 0$:

- $n = 2 \implies \mathbb{E}^2(\Bbbk)$.
- $n = 3 \implies \mathbb{E}^3(\Bbbk)$.
- $n = 4 \implies \mathbb{E}^2(\Bbbk) \oplus \mathbb{E}^2(\Bbbk) \simeq \mathbb{E}^2(\Bbbk) \otimes \mathbb{E}^2(\Bbbk)$ or $\mathbb{E}^4(\Bbbk)$.

Example: Algebraic solutions on nonflat background

Lemma (Fox): G a connected compact simple Lie group.

- Killing form $B < 0$. Biinvariant metric $h = -B$ on G.
- $\omega_{i_1 \dots i_k} \in S^k(\mathfrak{g}^*)$ the complete polarization of a Δ_h-harmonic homogeneous G-invariant polyonomial P of degree $k \geqslant 3$.

The pair (h, ω) solves the coupled Einstein equations on G.

Proof: G invariance decouples the equations.

- G-invariance implies $\omega_{i_1 \dots i_k}$ is parallel, so Codazzi.
- G-invariance implies $\rho(\omega \owedge \omega)$ is G-invariant, so is a multiple of h.
- For same reason h is Einstein in usual sense.

In some cases it can be shown that the resulting solutions are not coupled projectively flat. This will be shown for $G = SU(n)$.

Example: $SU(n)$

- $G = SU(n)$, $n \geqslant 3$.
- $\mathfrak{g} = \mathfrak{su}(n) =$ Lie algebra of skew-Hermitian $n \times n$ matrices.
- $h = -B$ for Killing form $B(X, Y) = 2n\,\mathrm{tr}(XY)$.
- $t \in \mathbb{R}$, commutative multiplication $\circ$ on $\mathfrak{g}$ defined by:

$$(\star) \qquad X \circ Y = t\mathrm{i}\left(XY + YX - \tfrac{2}{n}\,\mathrm{tr}(XY) I_n\right).$$

- Trilinear form: $\chi(X, Y, Z) = h(X \circ Y, Z) = -2nt\mathrm{i}\,\mathrm{tr}(XYZ + YXZ)$.
- Cubic form: $P(X) = -\frac{2nt\mathrm{i}}{3}\,\mathrm{tr}\, X^3$.
- $\chi \in S^3(\mathfrak{g}^*)$ is invariant under G-action induced by adjoint action.
- Since G acts orthogonally on $\mathfrak{g}$, the h-Laplacian is G-equivariant, and so $\Delta_h P$ is G-invariant. An invariant homogeneous linear form on a simple Lie algebra must be zero, so P is h-harmonic.

Example: $SU(n)$

- Define $L : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by $L(X)Y = X \circ Y$.
- $\Delta P = 0 \iff \operatorname{tr} L(X) = 0$.
- Invariance of $\chi \iff [\mathrm{ad}(X), L(Y)] = L([X, Y]) = [L(X), \mathrm{ad}(Y)]$.
- Killing form $\tau(X, Y) = \operatorname{tr} L(X)L(Y)$ is G-invariant:

$$\begin{aligned}&\operatorname{tr} L([X, Y])L(Z) - \operatorname{tr} L(X)L([Y, Z])\\ &\qquad = \operatorname{tr}\left([L(X), \mathrm{ad}(Y)]L(Z) - L(Z)[\mathrm{ad}(Y), L(Z)]\right) = 0.\end{aligned}$$

- Einstein-like condition: Because h is G-invariant, $\tau = \frac{1}{\dim \mathfrak{g}}|\chi|^2 h$.
- Associator $[X, Y, Z] = (X \circ Y) \circ Z - X \circ (Y \circ Z)$:

$$[X, Y, Z] = 4t^2 \left(\tfrac{1}{4}[Y, [Z, X]] + h(Z, X)Y - h(X, Y)Z\right).$$

Example: $SU(n)$

- Levi-Civita connection D of h satisfies $D_X Y = \frac{1}{2}[X, Y]$.
- χ is D-parallel by the invariance of χ.
- Biinvariant torsion-free connection $\nabla^t = D - t\chi_{ij}{}^k$.

$$\begin{aligned}
\nabla^t_X Y &= \tfrac{1}{2}[X, Y] - \mathrm{i}\tfrac{t}{2}\left(XY + YX - \tfrac{2}{n}\operatorname{tr}(XY)I_n\right),\\
R^t(X, Y)Z &= -\tfrac{1+t^2}{4}[[X, Y], Z] + 2t^2\left(h(Y, Z)X - h(X, Z)Y\right),\\
\rho^t(X, Y) &= \left(\tfrac{1}{4} + (\tfrac{1}{n^2} - \tfrac{1}{4})t^2\right)h(X, Y),\\
\operatorname{tr}_h \rho^t &= (n^2 - 1)\left(\tfrac{1}{4} + (\tfrac{1}{n^2} - \tfrac{1}{4})t^2\right).
\end{aligned}$$

- ∇^t is not projectively flat for any $t \in \mathbb{R}$.

(∇, h) is Einstein special statistical structure with self-conjugate curvature, so (h, χ) solve coupled Einstein equations, but ∇^t is not projectively flat for any $t \in \mathbb{R}$, so (h, χ) not coupled projectively flat.

No other examples on compact Lie groups

H. T. Laquer (also H. Naitoh)

A compact simple Lie group G admits a unique biinvariant torsion-free affine connection (the Levi-Civita connection of the Killing form) unless G has type A_n with $n \geqslant 2$, in which case there is a unique one-parameter family of biinvariant torsion-free affine connections.

There is a similar theorem for Riemannian symmetric spaces, with exceptional examples on the symmetric spaces:

$SU(n)/SO(n)$	$(SU(n) \times SU(n))/SU(n)$	$SU(2n)/Sp(n)$	E_6/F_4
$n \geqslant 3$	$n \geqslant 3$	$n \geqslant 3$	
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$

Some relevant references (incomplete)

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