Symmetric trilinear forms and Einstein-like equations: from affine spheres to Griess algebras

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General scheme: hierarchy of coupled equations

- $SO(n)$-irreducible $\mathbb{W}$. Associated module of trace-free tensors.
- $\mathcal{MC}$ = curvature tensors.
- Symmetric $SO(n)$-map $\phi : \mathbb{W} \times \mathbb{W} \to \mathcal{MC}$, $\mathrm{tr}_h \, \rho(\phi(\omega, \omega)) = |\omega|^2$.
- Generalized gradients $\mathcal{A}_1, \dots, \mathcal{A}_N$, symbols $\sigma_{\mathcal{A}_i}(X)(\omega)$.
- Stress-energy tensor: $\mathcal{T}(\omega) = \rho(\phi(\omega, \omega)) + \frac{1}{2r}|\omega|^2 h$, some $r \in \mathbb{R}$.

$$h(X, \mathrm{div}\,(\mathcal{T})) = \sum_{i=1}^{N} \beta_i h(\sigma_{\mathcal{A}_i}(X)(\omega), \mathcal{A}_i(\omega)).$$

- Coupled projective flatness:

$$\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N, \quad Riem - \epsilon\phi(\omega, \omega) = -\tfrac{\kappa}{n(n-1)} h \owedge h, \quad \epsilon \in \{\pm 1\}.$$

- Coupled Einstein equations:

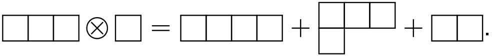
$$\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N, \quad G + \Lambda h = \epsilon \mathcal{T} \iff Ric - \epsilon\, \rho(\phi(\omega, \omega)) = \tfrac{\kappa}{n} h.$$

- $\mathcal{A}_i(\omega) = 0, 1 \leqslant i \leqslant N$, and constancy of $\Lambda = \frac{n-2}{2n}\left(s + \epsilon \frac{n+2r}{r(n-2)}|\omega|^2\right)$.

Coupled equations for trace-free Codazzi tensors

Suppose $\omega_{i_1 \dots i_k} = \omega_{(i_1 \dots i_k)}$ is trace-free. $\operatorname{div}(\omega)_{i_1 \dots i_{k-1}} = D^p \omega_{p i_1 \dots i_{k-1}} = 0$, $\mathcal{C}(\omega)_{i j i_1 \dots i_{k-1}} = D_{[i}\omega_{j] i_1 \dots i_{k-1}}$.

$$\mathcal{K}(\omega)_{i j i_1 \dots i_{k-1}} = \mathcal{C}(\omega)_{i j i_1 \dots i_{k-1}} - \tfrac{1}{n+k-3} \sum_{s=1}^{k-1} h_{i_s [i} \operatorname{div}(\omega)_{j] i_1 \dots \hat{i}_s \dots i_{k-1}}.$$



- Coupled projective flatness:

$$\mathcal{K}(\omega) = 0, \quad \operatorname{div}(\omega) = 0, \quad Riem - \epsilon(\omega \owedge \omega) = -\tfrac{\kappa}{n(n-1)}(h \owedge h) \quad \epsilon \in \{\pm 1$$

- Coupled Einstein equations:

$$\mathcal{K}(\omega) = 0, \quad \operatorname{div}(\omega) = 0, \quad G + \Lambda h = \epsilon \mathcal{T}(\omega) \iff Ric - \epsilon\, \rho(\omega \owedge \omega) = \tfrac{\kappa}{n}$$

- $\mathcal{K}(\omega) = 0, \operatorname{div}(\omega) = 0$ and constancy of $\Lambda = \tfrac{n-2}{2n}(s - \epsilon|\omega|^2) = \tfrac{n-2}{2n}\kappa$.

Submanifold examples, $k \in \{2, 3\}$

($k = 2$) For a mean curvature zero nondegenerate hypersurface in a pseudo-Riemannian space form having induced metric h and second fundamental form Π, (h, Π) is coupled projectively flat.

($k = 3$) For a mean curvature zero nondegenerate Lagrangian immersion in a (para/pseudo)-Kähler manifold of constant (para)-holomorphic sectional curvature having induced metric h and twisted second fundamental form Π, (h, Π) is coupled projectively flat.

- General properties of solutions: results in Riemannian signature
- General existence theorems? Affine spheres.
- Examples of coupled Einstein not coupled projectively flat.

Properties of solutions in Riemannian signature

For general scheme, Weitzenböck/Kato inequality machinery applies:

- A priori restrictions on the coupled tensors: vanishing theorems.
  - k-forms: Hodge theory.
  - Codazzi tensors: (Berger-Ebin '69, Stepanov '92).
  - (Conformal) Killing tensors: (Dairbekov-Sharafutdinov '11, Heil-Semmelmann-Moroianu '16).
  - General machinery (Hitchin '15).
- Weitzenböck formulas for $\mathcal{A}^*\mathcal{A}$. (Semmelmann-Weingart '10).
- Refined Kato inequalities. (Calderbank-Gauduchon-Herzlich/Branson).
- Together these yield differential inequality for the Laplacian of the coupled tensor and its norm.
- Depending on signs, can be integrated (Simons style gap theorem) or can apply maximum principles (Cheng-Yau style growth estimates).

Model result (Calabi 1971): If Blaschke metric of a hyperbolic affine sphere is complete, then it has nonpositive Ricci curvature.

Weitzenböck machinery for Codazzi tensors

- h a Riemannian metric on M^n, $n > 2$, and $\omega \in \Gamma(S^k_0(T^*M))$.
- $\mathcal{L}$ the trace-free part of the symmetrized covariant derivative.

General formulas:

$$\widehat{\mathcal{R}}(\omega)_{i_1\dots i_k} = k\,\rho(\mathcal{R})_{p(i_1}\omega_{i_2\dots i_k)}{}^{p} - k(k-1)\mathcal{R}_{p(i_1 i_2}{}^{q}\omega_{i_3\dots i_k)q}{}^{p},$$

$$\mathcal{Q}_{\mathcal{R}}(\omega) = \langle \omega, \widehat{\mathcal{R}}(\omega)\rangle = k\langle \rho(\omega \owedge \omega), \rho(\mathcal{R})\rangle + \binom{k}{2}\langle \omega \owedge \omega, \mathcal{R}\rangle,$$

$$\Delta\omega = \tfrac{1}{k}\widehat{\mathcal{R}}(\omega) + \tfrac{n+2(k-2)}{n+k-3}\mathcal{L}\mathrm{div}\,(\omega) + 2\mathcal{K}^*\mathcal{K}(\omega).$$

Weitzenböck formula:

$$\tfrac{1}{2}\Delta|\omega|^2 = |D\omega|^2 + \tfrac{n+2(k-2)}{n+k-3}\langle \omega, \mathcal{L}\mathrm{div}\,(\omega)\rangle + 2\langle \omega, \mathcal{K}^*\mathcal{K}(\omega)\rangle + \tfrac{1}{k}\mathcal{Q}_{\mathcal{R}}(\omega).$$

With refined Kato $\implies$ if $\omega \in \ker \mathcal{C} \cap \ker \mathcal{C}^*$, wherever $\omega \neq 0$,

$$|\omega|^{(n+2(k-1))/(n+k-2)}\Delta_h|\omega|^{(n-2)/(n+k-2)} \geqslant \tfrac{n-2}{k(n+k-2)}\mathcal{Q}_{\mathcal{R}}(\omega).$$

Properties of solutions

Calabi-Cheng-Yau style growth theorem for $\boxed{\epsilon > 0}$:

M a manifold of dimension $n \geqslant 3$. Suppose h is a complete Riemannian metric which with a trace-free symmetric k-tensor ω solves

$$\mathcal{C}(\omega) = 0 = \mathcal{C}^*(\omega), \quad Riem \boxed{-} (\omega \owedge \omega) = -\tfrac{\kappa}{n(n-1)}(h \owedge h), \quad \text{for } \kappa \in \mathbb{R}.$$

If $\kappa \geqslant 0$ then $\omega \equiv 0$, and h has constant curvature. If $\kappa < 0$ then:

- if $k \geqslant 3$: $\sup_M |\omega|^2 \leqslant -\kappa$, so $s_h = |\omega|^2 + \kappa \leqslant 0$;
- if $k = 2$: $\sup_M |\omega|^2 \leqslant -\kappa/(n-1)$, so $s_h = |\omega|^2 + \kappa \leqslant \frac{n-2}{n-1}\kappa$.

Applies to maximal spacelike hypersurfaces in AdS space (Ishihara '87).

$k = 3$: a complete solution with $\kappa < 0$ has nonpositive Ricci curvature.

Properties of solutions

Simons style gap theorem for $\boxed{\epsilon < 0}$:

M a compact oriented manifold of dimension $n \geqslant 3$. If a Riemannian metric h and trace-free symmetric k-tensor ω solve

$$\mathcal{C}(\omega) = 0 = \mathcal{C}^*(\omega), \quad Riem \boxed{+} (\omega \owedge \omega) = -\tfrac{\kappa}{n(n-1)}(h \owedge h), \quad \text{for } \kappa \in \mathbb{R},$$

then there is a constant $c_{n,k} > 0$ such that

$$\begin{aligned} 0 &\geqslant \int_M |\omega|^2 \left(\tfrac{n+k-2}{n(n-1)}\kappa - c_{n,k}|\omega|^2\right) d\mathrm{vol}_h, \\ &= c_{n,k} \int_M (\kappa - s)\left(\left(\tfrac{n+k-2}{n(n-1)c_{n,k}} - 1\right)\kappa + s\right) d\mathrm{vol}_h, \end{aligned}$$

where

$$\kappa = s + |\omega|^2,$$

$c_{n,2} = 1$ (Simons '68), $\quad c_{n,3} = \frac{2n-1}{n}$ (Chen-Ogiue '74),

$c_{n,k} \leqslant (1 + \frac{(2n+1)(k-1)}{n}) \quad k > 3$ (Fox 2021) (Not sharp!).

Algebraic solutions on flat background

Flat h with Levi-Civita D. $\omega_{i_1\dots i_k} = \omega_{(i_1\dots i_k)}$ is trace-free and parallel.

- Coupled projective flatness: $\omega \owedge \omega = \frac{\kappa}{n(n-1)}(h \owedge h)$.
- Coupled Einstein equations: $\rho(\omega \owedge \omega) = -\frac{\kappa}{n} h$.

- Purely algebraic equations for a tensor.
- The coupled Einstein equations admit many such solutions.
- Determining whether these are coupled projectively flat requires some additional argument.
- When $k = 3$, real coupled projectively flat solutions are classified. Unique up to isomorphism in each dimension.

Algebraic solutions from isoparametric polynomials

n-dimensional Euclidean vector space $(\mathbb{V}, h)$.

- F an h-harmonic polynomial homogeneous of degree $g \geqslant 2$ on $\mathbb{V}$.
- $\omega = D(g)F$ is trace-free, Codazzi, and divergence-free.

(h, ω) solves coupled Einstein equations $\iff$ there is $c \in \mathbb{R}$ such that:

$$(\star) \qquad \begin{aligned} 0 &= F_{ip_1 \dots p_{g-1}} F_j{}^{p_1 \dots p_{g-1}} - ch_{ij}, \\ 0 &= D_i D_j \left(|D^{(g-1)} F|^2 - c|x|^2 \right), \\ 0 &= |D^{(g-1)} F|^2 - c|x|^2. \end{aligned}$$

In this case $nc = |D^{(g)} F|^2$.

Next are given some examples for which it can be shown the resulting solutions are not coupled projectively flat.

Algebraic solutions from isoparametric polynomials

$\Sigma^{n-2} \subset \mathbb{S}^{n-1}$ is isoparametric if its principal curvatures are constant.
(Münzner 1980-81) showed:

- Number g of distinct principal curvatures λ_i satisfies $g \in \{1, 2, 3, 4, 6\}$;
- Multiplicities of $\lambda_1 > \lambda_2 > \cdots > \lambda_g$ satisfy $m_i = m_{i+2}$ (indices mod g) $\implies$ at most two distinct m_1 and m_2 (if $g = 3$ then $m_1 = m_2$);
- Σ part of a level set in $\mathbb{S}^{n-1}$ of a degree g homogeneous polynomial $P : \mathbb{R}^n \to \mathbb{R}$ (a Cartan-Münzner polynomial) solving

$$|dP|^2 = g^2|x|^{2g-2}, \qquad \Delta P = \tfrac{m_2 - m_1}{2} g^2 |x|^{g-2}.$$

Corollary (Fox): For a Cartan-Münzner polynomial P, the trace-free part $\omega_{i_1 \dots i_g}$ of $D^{(g)}P$ solves $(\star)$, so (h, ω) solves the coupled Einstein equations. When $g = 3$, (h, ω) is not coupled projectively flat.

Proof: algebra with harmonic polynomials.

Combinatorial algebraic solutions

- $E = \{e_1, \dots, e_n\}$ edge set of k-regular graph with finite vertex set V.
- Partial Steiner system $\mathcal{B}$ = collection of k-element subsets (blocks) of $\{1, \dots, n\}$ such that edges $e_{i_1}, \dots, e_{i_k}$ incident at some vertex in V.
- $\mathbb{V} = \mathbb{R}\langle E \rangle$ with metric h for which E is ordered orthonormal basis.
- $(x_1, \dots, x_n)$ coordinates of $x \in \mathbb{V}$. $I \in \mathcal{B} \to$ monomial $x_I = x_{i_1} \dots x_{i_k}$.
- Choose $\epsilon \in \{\pm 1\}^{\mathcal{B}}$, so $\epsilon_I \in \{\pm 1\}$ for each $I \in \mathcal{B}$.

Theorem (Fox): Associate with (E, V) and $\epsilon \in \{\pm 1\}^{\mathcal{B}}$ the k-form

$$P(x) = \sum_{I \in \mathcal{B}} \epsilon_I x_I.$$

The pair $(h, D^{(k)}P)$ solves the coupled Einstein equations.

When $k = 3$ there are one-parameter variants of this construction.

Conformal structure from second fundamental form

A cooriented nondegenerate immersed hypersurface in a manifold with projective structure acquires a conformal structure.

- $i : M^n \to (N^{n+1}, [\hat{\nabla}])$ immersed hypersurface. Projective structure $[\hat{\nabla}]$.
- Second fundamental form Π: normal bundle valued symmetric two tensor defined by projecting $\hat{\nabla}_X Ti(Y)$ onto normal bundle.
- Π depends only on the projective equivalence class $[\hat{\nabla}]$ of $\hat{\nabla}$.
- Immersion is nondegenerate $\iff$ Π is nondegenerate.
- Local transversal W determines h representing Π and a connection ∇:

$$\hat{\nabla}_X Ti(Y) = Ti(\nabla_X Y) + h(X, Y)W.$$

- $W \to \tilde{W} = f(W + Z) \implies h \to \tilde{h} = f^{-1}h$ and $\tilde{\nabla} - \nabla = -h \otimes Z$.
- M cooriented $\implies$ conformal structure $[h]$.
- $[h]$ invariant with respect to automorphisms of $[\hat{\nabla}]$.

Blaschke metric

Nothing so far depends on choice of transversal W.

- Ψ a volume density on N.
- Determines a unique representative $\hat{\nabla} \in [\hat{\nabla}]$ such that $\hat{\nabla}\Psi = 0$.
- Local transversal W determines two volume densities:

$$|\iota(W)\Psi| \qquad \text{and} \qquad |\det h|^{1/2}.$$

- For cooriented W such that these coincide, h is the equiaffine metric.
- Also called the Blaschke metric.
- Depends only on trivialization of normal bundle, not on splitting.
- Preserved by automorphisms of $([\hat{\nabla}], \Psi)$.

Affine normal in flat affine space

Nothing so far depends on direction of W.

If $M^n \subset \mathcal{A}^{n+1}$ is locally convex, affine normal at p is the tangent to the curve formed by barycenters of slices of M by parallel translates of T_pM.

A nondegenerate immersed hypersurface, M^n, in $(\mathcal{A}^{n+1}, \widehat{\nabla})$ acquires a connection, ∇, via projection along the affine normal line bundle.

The affine normal line bundle is defined more generally:

- With nondegeneracy in place of convexity.
- Without assuming flatness of $\widehat{\nabla}$.

For indefinite signature $[h]$ or nonflat $\widehat{\nabla}$ need a different definition.

Alignment

Lemma: Given a projective structure $[\nabla]$ and a conformal structure $[h]$ on M^n there is a unique $\nabla \in [\nabla]$ that is aligned with respect to $[h]$, meaning:

$$h^{pq}\nabla_i h_{pq} = nh^{pq}\nabla_p h_{qi} \qquad \text{(alignment condition)}.$$

Equivalently $H^{ij} = |\det h|^{1/n} h^{ij}$ is divergence free: $\nabla_p H^{ip} = 0$.

An AH structure on M^n is a pair $([\nabla], [h])$ comprising a projective structure, $[\nabla]$, and a conformal structure, $[h]$, such that for each $\nabla \in [\nabla]$ and each $h \in [h]$ there is a one-form γ_i such that $\nabla_{[i}h_{j]k} = 2\gamma_{[i}h_{j]k}$.

Identify AH structure with the pair $(\nabla, [h])$ where $\nabla \in [\nabla]$ is aligned.

The affine normal

Lemma: The affine normal line bundle of a cooriented nondegenerate $M^n \subset (N^{n+1}, \hat{\nabla})$ is determined uniquely by the requirement that the induced connection be aligned with respect to the conformal structure determined by the second fundamental form and coorientation.

- A transversal W determines a connection ∇ on M.
- If $\tilde{W} = f(W + Z)$, then $\tilde{\nabla} - \nabla = -h_{ij}Z^k$.
- ∇ depends on line bundle spanned by W, not on W.
- ∇ varies projectively if $\hat{\nabla}$ is varied projectively with W fixed.

Lemma: There is a unique choice of transverse line bundle so that the induced connection ∇ is aligned with respect to $[h]$.

- This unique choice is the affine normal line bundle.
- A $\hat{\nabla}$-parallel volume density determines an affine normal vector field.

Flat ambient space: conormal Gauss map

A nondegenerate immersed hypersurface, M^n, in flat affine space, $\mathcal{A}^{n+1}$, acquires a flat projective structure.

- Conormal Gauss map: $\nu : M \to \mathbb{P}(\mathcal{A}^*)$, $\nu(p) = \text{Ann } T_p M$.
- M nondegenerate $\iff$ ν is an immersion.
- $[\bar{\nabla}] =$ pullback via ν of flat projective structure on $\mathbb{P}(\mathcal{A}^*)$.

If the immersion is cooriented, the projective structure $[\bar{\nabla}]$ generates an AH structure with the conformal structure determined by the second fundamental form and coorientation.

AH structure induced on an affine hypersurface

An AH structure on M^n is a pair $([\nabla], [h])$ comprising a projective structure, $[\nabla]$, and a conformal structure, $[h]$, such that for each $\nabla \in [\nabla]$ and each $h \in [h]$ there is a one-form γ_i such that $\nabla_{[i}h_{j]k} = 2\gamma_{[i}h_{j]k}$.

A cooriented nondegenerate immersed hypersurface M in flat affine space $(\mathcal{A}, \hat{\nabla})$ acquires two conjugate exact AH structures:

- The AH structure $([\nabla], [h])$ induced via the affine normal.
- The pullback $[\bar{\nabla}]$ via the conormal Gauss map $M \to \mathbb{P}^+(\mathcal{A}^*)$ of the flat projective structure on $\mathbb{P}^+(\mathbb{V})$ constitutes with $[h]$ the AH structure conjugate to $([\nabla], [h])$.

- Definition: $([\nabla], [h])$ is conjugate projectively flat.
- Characterizes AH structures locally equivalent to affine hypersurfaces.

Special statistical structures on an affine hypersurface

Alternative formulation.

A cooriented nondegenerate hypersurface in flat affine space acquires a pair of conjugate special statistical structures, one of which is projectively flat.

- Special statisical $\iff$ cubic form $\mathcal{L}_{ijk} = \nabla_i h_{jk}$ symmetric, trace-free.
- $\bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^k$ generates with h the conjugate statistical structure $(\bar{\nabla}, h)$ having trilinear form $\bar{\mathcal{L}}_{ijk} = -\mathcal{L}_{ijk}$ and also special.

$$\nabla - \tfrac{1}{2}\mathcal{L}_{ij}{}^k = D = \bar{\nabla} + \tfrac{1}{2}\mathcal{L}_{ij}{}^k, \quad \bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^k.$$

D is Levi-Civita of h. $\mathcal{L}_{ij}{}^k = h^{kp}\nabla_p h_{ij}$. $\mathcal{L}_{ip}{}^p = 0$.

- Conjugate projectively flat $\implies$ locally an affine hypersurface.

Einstein special statistical structures and affine spheres

A nondegenerate $M^n \subset \mathcal{A}^{n+1}$ is an affine sphere if its affine normals meet in a point (its center) or are parallel (center at infinity).

Theorem (Fox 2009): For a nondegenerate cooriented $M^n \subset \mathcal{A}^{n+1}$ with induced conjugate special statistical structures (∇, h) and $(\bar{\nabla}, h)$ the following are equivalent:

- M is an affine sphere.
- (∇, h) and $(\bar{\nabla}, h)$ have self-conjugate curvature.
- (∇, h) and $(\bar{\nabla}, h)$ are Einstein.

These conditions imply and, if $n > 2$, are implied by

- (∇, h) and $(\bar{\nabla}, h)$ are projectively flat.

Yields lots of solutions of the coupled projectively flat equations.

Convex affine spheres

Let M be a convex affine sphere with complete Blaschke metric h.

- (Blaschke-Deicke-Calabi) If M is elliptic, then h has positive constant curvature and M is an ellipsoid.
- (Jörgens-Calabi-Cheng-Yau) If M is a parabolic affine sphere, then h is flat and M is an elliptic paraboloid.
- (Calabi) If M is a hyperbolic, then h has nonpositive Ricci curvature.

Theorem (Cheng-Yau '86): The interior of a nonempty pointed closed convex cone is foliated in a unique way by hyperbolic affine spheres asymptotic to its boundary and center at its vertex.

Many examples of nonconvex affine spheres from level sets of real forms of relative invariants of irreducible prehomogeneous vector spaces and from nonassociative algebras (Fox, R. Hildebrand).

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