Symmetric trilinear forms and Einstein-like equations: from affine spheres to Griess algebras

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Structure of talks

- Overview and motivation.
- Curvature equations coupling a metric to auxiliary tensor.
- Examples and properties of solutions.
- Affine differential geometry of hypersurfaces.
- Generalized affine hypersurface (AH) structures.
- Einstein equations for AH structures.
- Examples and properties of solutions.
- Metrized algebras. Killing/Ricci metrized algebras.

Review: conformal and Weyl structures

- Metrics g and $\bar{g}$ are conformally equivalent if $\bar{g} = e^f g$ for $f \in C^\infty(M)$.
- A conformal structure is a conformal equivalence class $[g]$ of metrics.
- $[g]$ conformally flat $\iff$ charts in which represented by flat metric.
- Conformal Schouten tensor: $\mathcal{P}_{ij} = -\frac{1}{n-2}(Ric_{ij} - \frac{1}{2(n-1)} s g_{ij})$.
- Conformal Weyl tensor: $\mathcal{W}_{ijkl} = \mathcal{R}_{ijkl} + 2g_{l[i}\mathcal{P}_{j]k} - 2g_{k[i}\mathcal{P}_{j]l}$.
- Conformal Cotton tensor: $2\nabla_{[i}\mathcal{P}_{j]k}$.
- If $n > 3$, $[g]$ conformally flat $\iff$ conformal Weyl tensor vanishes.
- If $n = 3$, $[g]$ conformally flat $\iff$ conformal Cotton tensor vanishes.
- If $n = 2$, $[g]$ always conformally flat.
- Weyl structure: Torsion-free ∇ and conformal structure $[h]$ such that for every $h \in [h]$ there is a one-form γ such that $\nabla h = 2\gamma \otimes h$.

Review: projective structures

- Torsion-free connections ∇ and $\tilde{\nabla}$ projectively equivalent if

$$\tilde{\nabla} = \nabla + \gamma \otimes \mathrm{Id} + \mathrm{Id} \otimes \gamma.$$

- Equivalently: traces of geodesics of ∇ and of $\tilde{\nabla}$ are the same.
- Equivalence class $[\nabla]$ is projective structure.
- $[\nabla]$ projectively flat $\iff$ charts in which geodesics are straight lines.
- Projective Schouten tensor: $P_{ij} = -\frac{1}{n-1}\left(Ric_{ij} - \frac{2}{n+1}Ric_{[ij]}\right)$.
- Projective Weyl tensor: $B_{ijk}{}^{l} = R_{ijk}{}^{l} + \delta_i{}^{l}P_{jk} - \delta_j{}^{l}P_{ik} - 2\delta_k{}^{l}P_{[ij]}$.
- Projective Cotton tensor: $2\nabla_{[i}P_{j]k}$.
- If $n > 2$, ∇ projectively flat $\iff$ projective Weyl tensor vanishes.
- If $n = 2$, ∇ projectively flat $\iff$ projective Cotton tensor vanishes.

Classical hierarchy of curvature conditions

(pseudo)-Riemannian metric h. Curvature conventions (signs!):

$$(h \owedge h)_{ijkl} = 2h_{k[i}h_{j]l}, \qquad Ric_{ij} = Riem_{pij}{}^{p}, \qquad s = Ric_{p}{}^{p}.$$

- Constant sectional curvature: $Riem = -\frac{\kappa}{n(n-1)}(h \owedge h)$.
- Einstein equation: ($G = Ric - \frac{1}{2}sh$ is the Einstein tensor)

$$Ric - \frac{1}{n}sh = 0 \qquad \Longleftrightarrow \qquad G + \frac{n-2}{2n}\kappa h = 0.$$

- Constant scalar curvature: $s = \mathrm{tr}_h\, Ric = \kappa$ is constant.

Beltrami Theorem

A metric has constant sectional curvature if and only if it is projectively flat. In this case it is conformally flat and Einstein, and conversely if $n > 2$.

Goal: study similar hierarchy for more general class of AH structures.

Background contexts

Common element

- Linear algebra of cubic forms on quadratic spaces.
- Equivalently: symmetric bilinear form and a symmetric trilinear form.

- Hypersurface in (flat) affine manifold: Blaschke metric and Pick form.
- Statistical structure (∇, h) = torsion-free ∇, metric h such that $\nabla_{[i}h_{j]k} = 0 \implies \omega_{ijk} = \nabla_i h_{jk}$ is symmetric.
- Lagrangian submanifolds of (para/pseudo)-Kähler manifolds: Induced metric and modified second fundamental form.
- Metrized commutative algebra $(\mathbb{A}, \circ, h)$: Bilinear multiplication $\circ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and nondegenerate symmetric bilinear h (a metric) that is invariant, meaning $h(x \circ y, z) = h(x, y \circ z)$.

Main theme: Einstein equations in these contexts.

Preview: affine hypersurface (AH) structures

AH (affine hypersurface) structures are a class of geometric structures that includes all those mentioned as special cases.

Definition: An AH structure is a pair $([\nabla], [h])$ comprising a projective structure, $[\nabla]$, and a conformal structure, $[h]$, such that for each $\nabla \in [\nabla]$ and each $h \in [h]$ there is a one-form γ_i such that $\nabla_{[i}h_{j]k} = 2\gamma_{[i}h_{j]k}$. (Equivalently: the trace-free part of $\nabla_i h_{jk}$ is completely symmetric.)

The condition does not depend on the choices involved.

- There is an involutive notion of conjugacy of AH structures.
- Self-conjugate AH structures = Weyl structures.
- AH structures are locally statistical structures.
- Exact AH structure = homothety class of special statistical structures.
- Locally Kähler affine structures = projectively flat AH structures.

Einstein-like equations

Einstein-like equations

Some Ricci tensor is a multiple of the metric and some closed/coclosed conditions on some auxiliary tensor from which Ricci is constructed.

- Einstein-Weyl structures.
- Affine spheres. Cheng-Yau metric for convex flat projective structures.
- Mean curvature zero Lagrangian submanifolds of Kähler space forms.
- Einstein equations for statistical manifolds.
- Commutative algebras metrized by Killing form: $\tau(x, y) = \operatorname{tr} L(x)L(y)$ or Ricci form: $\rho(x, y) = \operatorname{tr} L(x \circ y) - \operatorname{tr} L(x)L(y)$.

Einstein-Maxwell equations

Metric h and 2-form F satisfying

$$dF = 0 = d \star F, \qquad Ric - F \circ F = \frac{1}{n}(s - |F|^2)h.$$

Idea: replace F, d, and $d\star$ ...

General context: linear algebra of symmetric k-forms

- k-linear form: $\omega \in \otimes^k \mathbb{V}^*$.
- Symmetric group S_k acts on $\otimes^k \mathbb{V}^*$ by $(\sigma \cdot \omega)_{i_1 \dots i_k} = \omega_{i_{\sigma^{-1}(1)} \dots i_{\sigma^{-1}(k)}}$.
- ω symmetric if $\sigma \cdot \omega = \omega$. That is, $\omega_{i_1 \dots i_k} = \omega_{(i_1 \dots i_k)}$.
- k-form (quadratic, cubic, ...) $=$ degree k homogeneous polynomial P.
- k-linear forms and symmetric k-forms correspond via polarization:

$$
\begin{aligned}
&P_\omega(x) = \omega(x, \dots, x),\\
&k!\omega(x_1, \dots, x_k) = \sum_{s=1}^{k} \sum_{|I|=s} (-1)^{k-s} P(x_{i_1} + \dots + x_{i_s}).
\end{aligned}
$$

- Kulkarni-Nomizu product and its Ricci trace:

$$
\begin{aligned}
&(\omega \owedge \omega)_{ijkl} = 2\omega_{k[i}{}^{A}\omega_{j]lA},\\
&\rho(\omega \owedge \omega)_{ij} = (\omega \owedge \omega)_{pijq} h^{pq} = \omega_i{}^{A}\omega_{jA} - w_{ij}{}^{A} h^{pq} \omega_{pqA}.
\end{aligned}
$$

Geometric interpretations: symmetric k-linear forms

Linear algebra of pair (h, ω), where h is nondegenerate symmetric bilinear form, ω is symmetric k-linear form.

- $k = 2$: Endomorphism $\omega_i{}^j = h^{jp}\omega_{ip}$ diagonalizable with respect to h.
  - Geometric application: induced metric and second fundamental form.
- $k = 3$: $\omega_{ij}{}^k = h^{kp}\omega_{ijp}$ can be viewed as structure tensor of commutative product invariant with respect to h.
  - $\nabla_i h_{jk}$ for statistical structure (∇, h).
  - Equiaffine metric and cubic Pick form on hypersurface in affine space.
  - Induced metric and modified second fundamental form of a Lagrangian immersion in a (para/pseudo)-Kähler manifold.
- $k \geqslant 4$: $\omega_{i_1 \dots i_{k-1}}{}^j = h^{jp}\omega_{i_1 \dots i_{k-1} p}$ gives $(k-1)$-ary product ...
  - Similarly nice geometric interpretations lacking (?).

What is special about $k = 3$

Trilinear forms $\iff$ metrized commutative algebras

- Commutative multiplication $\circ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and metric h determine

$$\omega(x, y, z) = h(x \circ y, z).$$

- h metrizes $(\mathbb{A}, \circ) \iff \omega(x, y, z) = h(x \circ y, z)$ is symmetric.
- $\circ$ and h determine the associated cubic form $6P(x) = h(x \circ x, x)$.
- Conversely: metric h and cubic form P determine $\circ$ by polarization:

$$\begin{aligned} h(x,y \circ z) &= P(x + y + z) \\ &\quad - P(x + y) - P(y + z) - P(x + z) + P(x) + P(y) + P(z). \end{aligned}$$

Metrized commutative algebras as statistical structures

- Metrized commutative algebra $(\mathbb{A}, \circ, h)$, $\omega(x, y, z) = h(x \circ y, z)$.
- $L : \mathbb{A} \to \mathrm{End}(\mathbb{A})$ defined by $L(x)y = x \circ y$. $L(x)_i{}^j = x^p \omega_{pi}{}^j$.
- D Levi-Civita connection of h.
- $\nabla = D + \omega_{ij}{}^k$ satisfies $\nabla_i h_{jk} = -2\omega_{ijk} \implies (\nabla, h)$ is statistical.
- ω_{ijk} is D-parallel, so curvature of ∇ is algebraic in ω_{ijk}.

Curvature $R(x, y)$ of ∇ measures nonassociativity of $\circ$:

$$R(x, y)_k{}^l = -x^i y^j (L \owedge L)_{ijk}{}^l = [L(x), L(y)]_k{}^l,$$

$$\rho(R)(x, y) = x^i y^j (\omega_{ip}{}^q \omega_{jq}{}^p - \omega_{ij}{}^a \omega_{ap}{}^p) = \operatorname{tr} L(x)L(y) - \operatorname{tr} L(x \circ y).$$

Associator: $[x, z, y] = (x \circ z) \circ y - x \circ (z \circ y) = -[L(x), L(y)]z$.

Affine normal line bundle

$\mathcal{A}^{n+1}$ = flat $(n+1)$-dimensional affine space.

If $M^n \subset \mathcal{A}^{n+1}$ is locally convex, affine normal at p is the tangent to the curve formed by barycenters of slices of M by parallel translates of T_pM.

A different definition is needed in nonconvex case.

A cooriented nondegenerate hypersurface in flat affine space acquires a pair of conjugate AH structures, one of which is projectively flat:

- Second fundamental form induces a conformal structure.
- Affine normal induces an affine connection.
- Codazzi equations give compatibility between the two.
- Conormal Gauss map induces flat projective structure.

Affine spheres: motivation for Einstein AH equations

A nondegenerate $M^n \subset \mathcal{A}^{n+1}$ is an affine sphere if its affine normals all meet in a point (its center) or are parallel (center at infinity).

Motivating observation:
There are Einstein equations for AH structures such that the conjugate pair of AH structures induced on a cooriented nondegenerate hypersurface in flat affine space are Einstein if and only if the hypersurface is an affine sphere.

- Affine spheres are the umbilics in affine geometry.
- Examples: ellipsoids, hyperboloids, paraboloids.
- There are others! Unlike in Euclidean geometry, in affine geometry umbilics abound.

Convex affine spheres

Let M be a convex affine sphere with complete Blaschke metric h.

- (Blaschke-Deicke-Calabi) If M is elliptic, then h has positive constant curvature and M is an ellipsoid.
- (Jörgens-Calabi-Cheng-Yau) If M is a parabolic affine sphere, then h is flat and M is an elliptic paraboloid.

Theorem (Cheng-Yau '86): The interior of a nonempty pointed closed convex cone is foliated in a unique way by hyperbolic affine spheres asymptotic to its boundary and center at its vertex.

- Together with duality of cones yields duality of affine spheres.

No systematic study of nonconvex affine spheres. Examples from level sets of real forms of relative invariants of irreducible prehomogeneous vector spaces and from nonassociative algebras (Fox, R. Hildebrand).

Cheng-Yau metric

Cheng-Yau: a pointed convex cone is foliated by complete hyperbolic affine spheres asymptotic to its boundary and centered on its vertex.

Together with duality of cones yields duality of affine spheres.

Corollary: a properly convex flat projective manifold carries a canonical homothety class of metrics.

Proof (J. Loftin 2001): descend the Blaschke metric of an affine sphere asymptotic to the cone over its universal cover.

- This is like Kähler-Einstein metric in negative Chern class case.
- Corollary yields a canonical duality of properly convex flat real projective manifolds that extends the duality of affine spheres.

Klartag: there is an (incomplete) elliptic affine sphere over a bounded convex domain with center at the Santaló point of the domain.

Overview: Einstein AH structures

There are Einstein equations for AH structures such that:

- An AH structure is Einstein $\iff$ conjugate AH structure is Einstein.
- For Weyl structures specialize to the usual Einstein-Weyl equations.
- The conjugate AH structures on a hypersurface in flat affine space are Einstein if and only if the hypersurface is an affine sphere.
- A properly convex flat real projective manifold has a pair of conjugate Einstein AH structures.
- There is AH structure induced on a mean curvature zero nondegenerate Lagrangian submanifold of a para-Kähler space form and it is Einstein.
- Examples can be constructed by solving usual Einstein field equations with a stress energy tensor built from a trace-free symmetric 3-tensor (like Einstein-Maxwell equations).

Overview: Einstein AH structures

- There are Einstein AH structures which are not Weyl and which are not locally equivalent to ones induced on affine spheres.
- The simplest example is defined on $SU(n)$.
- Algebraic avatar: Killing metrized exact commutative algebras - commutative algebras metrized by Killing form $\tau(x, y) = \operatorname{tr} L(x)L(y)$.
- Many examples, including:
  - Tensor products of semisimple Lie algebras.
  - Trace-free Hermitian matrices with the trace-free Jordan product.
  - Metric curvature tensors with the Hamilton-Huisken product.
  - Griess algebra of the Monster finite simple group.
  - Combinatorial constructions.

Hierarchy of curvature conditions

h a (pseudo)-Riemannian metric. Notations (Sign conventions!):

$$(h \owedge h)_{ijkl} = 2h_{k[i}h_{j]l}, \qquad Ric_{ij} = \rho(Riem)_{ij} = Riem_{pijq}h^{pq}.$$

- Constant sectional curvature: $Riem = -\frac{\kappa}{n(n-1)}(h \owedge h)$.
- Einstein equation: ($G = Ric - \frac{1}{2}sh$ is the Einstein tensor)

$$Ric - \frac{1}{n}sh = 0 \qquad \Longleftrightarrow \qquad G + \frac{n-2}{2n}\kappa h = 0. \qquad .$$

- Constant scalar curvature: $s = \mathrm{tr}_h\, Ric = \kappa$ is constant.

Beltrami Theorem

A metric has constant sectional curvature if and only if it is projectively flat. In this case it is conformally flat and Einstein, and conversely if $n > 2$.

Goal: explain similar hierarchy for the geometric structure abstracting that induced on a hypersurface in flat affine space.

Example: Einstein-Maxwell equations

- Electromagnetic 2-form F. $(F \circ F)_{ij} = F_i{}^p F_{jp}$. $|F|^2 = h^{ik}h^{jl}F_{ij}F_{kl}$.
- Einstein-Maxwell equations:

$$dF = 0 = d \star F, \qquad Ric - F \circ F = \tfrac{1}{n}(s - |F|^2)h.$$

Equivalently, stress-energy tensor: $\mathcal{T} = F \circ F - \frac{1}{4}|F|^2 h$,

$$dF = 0 = d \star F, \quad G + \Lambda h = \mathcal{T}, \quad \Lambda = \tfrac{2-n}{2n}\left(s - \tfrac{n-4}{2(n-2)}|F|^2\right).$$

- $\operatorname{div}(G) = 0$ and $\operatorname{div}(\mathcal{T}) = 0 \implies$ constancy of Λ.

Is there a coupled projectively flat condition that traces to give Einstein-Maxwell equations?

Coupled equations for p-forms

- For a p-form $F_{ija_1\dots a_{p-2}} = F_{ijA}$:

$$(F\cdot F)_{ijkl} = \tfrac{2}{3}(F_{k[i}{}^{A}F_{j]lA} - F_{ij}{}^{A}F_{klA}), \qquad \rho(F\cdot F)_{ij} = F_{i}{}^{qA}F_{jqA}.$$

- Stress-energy tensor: $\mathcal{T} = \rho(F\cdot F) - \frac{1}{2p}|F|^2 h.$

$$h(X, \operatorname{div}(\mathcal{T})) = h(\iota(X)F, \operatorname{div}(F)) - \tfrac{1}{p}h(\iota(X)dF, F) \quad \text{(Baird '08)}.$$

- Coupled projective flatness:

$$dF = 0 = d\star F, \quad Riem - \epsilon(F\cdot F) = -\tfrac{\kappa}{n(n-1)} h \owedge h, \quad \epsilon\in\{\pm 1\}$$

- Coupled Einstein equations:

$$dF = 0 = d\star F, \qquad G + \Lambda h = \epsilon\mathcal{T}, \iff Ric - \epsilon\,\rho(F\cdot F) = \tfrac{\kappa}{n}h,$$

- $dF = 0 = d\star F$ and constancy of $\Lambda = \frac{n-2}{2n}\left(s - \epsilon\frac{n-2p}{p(n-2)}|F|^2\right)$.

Einstein-Maxwell couplings: examples

Let (h, J) be a Kähler structure with Kähler form $\omega(\cdot\,,\cdot\,) = h(J\cdot\,,\cdot\,)$.

- (E. Flaherty '78, C. Lebrun '10) If $n = 4$ and (h, J) has constant scalar curvature, taking the primitive part of the Ricci form,

$$F = \omega + \tfrac{1}{2}(\rho - \tfrac{1}{4}s\omega), \qquad \rho(\cdot\,,\cdot\,) = Ric(J\cdot\,,\cdot\,),$$

  (h, F) solves the Einstein-Maxwell equations.

- (h, J) has constant holomorphic sectional curvature 4κ if and only if

$$Riem - 3\kappa(\omega \cdot \omega) = -\kappa(h \owedge h) \qquad (\kappa = s/24).$$

  If $\kappa > 0$, $F = \sqrt{3\kappa}\omega$, (h, F) is coupled projectively flat with $\epsilon = 1$.

- If $\kappa < 0$, $F = \sqrt{-3\kappa}\omega$, (h, F) is coupled projectively flat with $\epsilon = -1$.

- Are there any other solutions?

General scheme

- $SO(n)$-irreducible $\mathbb{W}$. $\mathcal{MC}$ = curvature tensors. Symmetric $SO(n)$-map $\phi : \mathbb{W} \times \mathbb{W} \to \mathcal{MC}$ normalized so $\operatorname{tr}_h \rho(\phi(\omega, \omega)) = |\omega|^2$.
- Stress-energy tensor: $\mathcal{T}(\omega) = \rho(\phi(\omega, \omega)) + \frac{1}{2r}|\omega|^2 h$, some $r \in \mathbb{R}$.
- Generalized gradient $\mathcal{A}$ so that, for some $\beta, \gamma \in \mathbb{R}$,

$$h(X, \operatorname{div}(\mathcal{T})) = \beta h(\iota(X)\omega, \mathcal{A}^*(\omega)) + \gamma h(\iota(X)\mathcal{A}(\omega), \omega).$$

- Coupled projective flatness:

$$\mathcal{A}(\omega) = 0 = \mathcal{A}^*(\omega), \quad Riem - \epsilon\phi(\omega, \omega) = -\tfrac{\kappa}{n(n-1)} h \owedge h, \quad \epsilon \in \{\pm 1\}.$$

- Coupled Einstein equations:

$$\mathcal{A}(\omega) = 0 = \mathcal{A}^*(\omega), \quad G + \Lambda h = \epsilon\mathcal{T} \iff Ric - \epsilon\,\rho(\phi(\omega, \omega)) = \tfrac{\kappa}{n} h.$$

- $\mathcal{A}(\omega) = 0 = \mathcal{A}^*(\omega)$ and constancy of $\Lambda = \frac{n-2}{2n}\left(s + \epsilon\frac{n+2r}{r(n-2)}|\omega|^2\right)$.

Aside: relaxing $\mathcal{A}^*(\omega) = 0$

In some cases the condition $\mathcal{A}^*(\omega) = 0$ can be relaxed.

- Suppose $\mathcal{A}^* : \mathbb{W} \to \mathbb{U}$.
- Symmetric bilinear map $\Psi : \mathbb{W} \times \mathbb{W} \to \mathbb{U}$.
- Consider $\mathcal{A}^*(\omega) = \Psi(\omega, \omega)$ in place of $\mathcal{A}^*(\omega) = 0$:

$$\mathcal{A}(\omega) = 0, \qquad \mathcal{A}^*(\omega) = \Psi(\omega, \omega), \qquad G + \Lambda h = \epsilon \mathcal{T}.$$

- To have $\operatorname{div}(\mathcal{T}) = 0$, and corresponding constancy of Λ, need $h(\iota(X)\omega, \Psi(\omega, \omega)) = 0$ when $\mathcal{A}(\omega) = 0$.

Might be interesting to explore systematically other similar variations of the basic scheme.

Example: $(3p-1)$-dimensional supergravity

- Data: pseudo-Riemannian (M^n, h), p-form F.
- Suppose $n+1 = 3p$ (e.g. $(p,n) = (2,5), (3,8), (4,11)$, etc.)
- $dF = 0$ and $d \star F = cF \wedge F$, constant $c \implies \operatorname{div} \mathcal{T} = 0$.

- Coupled projective flatness:

$$dF = 0, \quad d \star F = cF \wedge F, \quad Riem - \epsilon(F \cdot F) = -\tfrac{\kappa}{n(n-1)} h \owedge h.$$

- Variant of supergravity equations:

$$\begin{aligned} dF &= 0, \qquad d \star F = cF \wedge F, \\ G + \Lambda h &= \epsilon \mathcal{T} \iff Ric - \epsilon\, \rho(F \cdot F) = \Lambda h. \end{aligned}$$

(Up to normalizations, generalizes usual 11-D supergravity equations.)

- $dF = 0$, $d \star F = cF \wedge F$, constancy of $\Lambda = \tfrac{n-2}{2n}\left(s - \epsilon \tfrac{n-2p}{p(n-2)} |F|^2\right)$.

Codazzi and Killing tensors, stress-energy tensors

$\omega_{i_1\dots i_k}$ completely symmetric, $\omega_{i_1\dots i_k} = \omega_{(i_1\dots i_k)}$, trace-free.

- Kulkarni-Nomizu product and its Ricci trace:

$$(\omega \owedge \omega)_{ijkl} = 2\omega_{k[i}{}^{A}\omega_{j]lA}, \quad \rho(\omega \owedge \omega)_{ij} = (\omega \owedge \omega)_{pij}{}^{p} = \omega_{i}{}^{A}\omega_{jA}.$$

- Divergence: $\mathrm{div}\,(\omega)_{i_1\dots i_{k-1}} = D^p\omega_{pi_1\dots i_{k-1}}$.
- Codazzi tensor: $\mathcal{C}(\omega) = 0$ where $\mathcal{C}(\omega)_{ijp_1\dots p_{k-1}} = D_{[i}\omega_{j]p_1\dots p_{k-1}}$.
- Killing tensor: $\mathcal{L}(\omega) = 0$ where $\mathcal{L}(\omega)_{i_1\dots i_{k+1}} = D_{(i}\omega_{i_2\dots i_{k+1})}$.
- Formally $\mathcal{C} = -\mathrm{div}^*$ and $\mathcal{L} = -\mathrm{div}^*$.
- Note: when $\mathrm{tr}_h\,\omega = 0$, $\mathcal{C}(\omega) = 0$ or $\mathcal{L}(\omega) = 0 \implies \mathrm{div}\,(\omega) = 0$.

$$\mathcal{T}^{Cod}(\omega) = \rho(\omega \owedge \omega) - \tfrac{1}{2}|\omega|^2 h, \qquad \mathcal{T}^{Kil}(\omega) = \rho(\omega \owedge \omega) + \tfrac{1}{2k}|\omega|^2 h.$$

$$h(X, \mathrm{div}\,(\mathcal{T}^{Cod}(\omega))) = h(\iota(X)\omega, \mathrm{div}\,(\omega)) - 2h(\omega, \iota(X)\mathcal{C}(\omega)),$$
$$h(X, \mathrm{div}\,(\mathcal{T}^{Kil}(\omega))) = h(\iota(X)\omega, \mathrm{div}\,(\omega)) + \tfrac{k+1}{k}h(\omega, \iota(X)\mathcal{L}(\omega)).$$

Coupled equations for trace-free Codazzi/Killing tensors

Suppose $\omega_{i_1\dots i_k} = \omega_{(i_1\dots i_k)}$ is trace-free.

- In both cases, coupled projective flatness:

$$\mathcal{C}(\omega) = 0 = \mathcal{C}^*(\omega), \quad Riem - \epsilon(\omega \owedge \omega) = -\tfrac{\kappa}{n(n-1)}(h \owedge h) \quad \epsilon \in \{\pm 1\}.$$

- In both cases, coupled Einstein equations:

$$\mathcal{C}(\omega) = \mathcal{C}^*(\omega), \quad G + \Lambda h = \epsilon \mathcal{T}^{Cod/Kil}(\omega) \iff Ric - \epsilon\, \rho(\omega \owedge \omega) = \tfrac{\kappa}{n} h.$$

- Codazzi case: constancy of $\Lambda = \frac{n-2}{2n}(s - \epsilon|\omega|^2) = \frac{n-2}{2n}\kappa$.
- Killing case: constancy of $\Lambda = \frac{n-2}{2n}(s + \epsilon\frac{n+2k}{k(n-2)}|\omega|^2)$.

- Comment: in the Killing case, κ need not be constant.

$k = 2$ Example: minimal hypersurfaces in space forms

- $i : M^n \to (N^{n+1}, g)$ in a space form with curvature $-\frac{s_g}{n(n+1)}(g \owedge g)$.
- Suppose $h = i^*(g)$ is nondegenerate.
- Π the second fundamental form with respect to a unimodular orthogonal transversal T satisfying $\epsilon = -g(T, T) \in \{\pm 1\}$.

Gauss-Codazzi equations:

$$\mathcal{C}(\Pi) = 0, \quad \operatorname{div}(\Pi) = D \operatorname{tr}_h \Pi, \quad Riem - \epsilon(\Pi \owedge \Pi) = -\frac{\kappa}{n(n-1)}(h \owedge h),$$

with $\kappa = s_h + \epsilon(|\Pi|_h^2 - (\operatorname{tr}_h \Pi)^2) = \frac{n-1}{n+1} s_g$.

For a mean curvature zero nondegenerate hypersurface in a pseudo-Riemannian space form, (h, Π) is coupled projectively flat.

$k = 3$ Example: minimal Lagrangian submanifolds

- (N^{2n}, g, J) (para/pseudo-)Kähler. $J \circ J = \epsilon \,\mathrm{Id}$, $\omega(\cdot\,,\,\cdot) = g(J\cdot\,,\,\cdot)$.
- Constant (para)-holomorphic curvature $4\hat{c}$:

$$\widehat{Riem} = -3\hat{c}\epsilon(\omega \cdot \omega) - \hat{c}(g \owedge g).$$

- $i : M^n \to N^{2n}$ nondegenerate ($h = i^*(g)$ nondegenerate) Lagrangian.
- $\Pi \in \Gamma(S^3 T^* M)$ defined by $\Pi(X, Y, Z) = \omega(\hat{\nabla}_X Ti(Y), Ti(Z))$.
- Mean curvature one-form: $\mathcal{H}_i = h^{pq}\Pi_{ipq}$.

Gauss-Codazzi equations:

$$\mathcal{C}(\Pi) = 0, \quad \mathrm{div}\,(\Pi) = D\mathcal{H}, \quad Riem - \epsilon(\Pi \owedge \Pi) = -\hat{c}(h \owedge h).$$

For a mean curvature zero nondegenerate Lagrangian immersion in a (para/pseudo)-Kähler manifold of constant (para)-holomorphic sectional curvature, (h, Π) is coupled projectively flat.

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