Dispersionless integrable equations and modular forms

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Plan:

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Coefficients of dispersionless integrable PDEs in 3D can be written in terms of generalised hypergeometric functions/modular forms.

Dispersionless integrability

Hydrodynamic reductions: A PDE in 3D is said to be integrable if it possesses infinitely many reductions to a collection of commuting 2D systems of hydrodynamic type.

Dispersionless Lax pairs: A PDE in 3D is said to be integrable if it possesses a dispersionless Lax pair, that is, if it can be represented as the commutativity condition of two vector fields depending on a spectral parameter.

Integrability 'on solutions': A PDE in 3D is said to be integrable if its characteristic variety defines a conformal structure which is Einstein-Weyl on every solution.

All three approaches are equivalent!

3D dispersionless Hirota type equations

Dispersionless Hirota type equation is a second-order PDE of the form

$$F(u_{ij}) = 0$$

where $u(x_1, x_2, x_3)$ is a function of three independent variables, $u_{ij} = u_{x_i x_j}$.

Example 1. Dispersionless Kadomtsev-Petviashvili equation

$$u_{xt} - \frac{1}{2}u_{xx}^2 - u_{yy} = 0.$$

Example 2. Boyer-Finley equation

$$u_{xx} + u_{yy} - e^{u_{tt}} = 0.$$

Modular example

Example 3. Equation of the form

$$u_{tt} - \frac{u_{xy}}{u_{xt}} - \frac{1}{6} h(u_{xx}) u_{xt}^2 = 0$$

is integrable if and only if the coefficient h satisfies the Chazy equation

$$h''' + 2hh'' - 3(h')^2 = 0$$

(Pavlov, 2003). Its general solution can be expressed in terms of the Eisenstein series of weight 2 on the modular group $SL(2, \mathbb{Z})$: $h(s) = E_2(is/\pi)$ where $(q = e^{2\pi i \tau})$

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n \tau} = 1 - 24\, q - 72\, q^2 - 96\, q^3 + \ldots .$$

3D Hirota type equations: summary of known results

- The class of Hirota equations is invariant under the symplectic group $Sp(6, \mathbb{R})$:

$$U \mapsto (AU + B)(CU + D)^{-1}.$$

  Here $U = \text{Hess}(u) = u_{ij}$ is the Hessian matrix of the function u.

- The parameter space of integrable Hirota type equations is 21-dimensional. Furthermore, the action of the equivalence group $Sp(6, \mathbb{R})$ on the parameter space is locally free. Since $\dim Sp(6, \mathbb{R}) = 21$, there exists a generic Hirota master-equation generating an open 21-dimensional $Sp(6, \mathbb{R})$-orbit.

- Geometrically, Hirota type equation $F(u_{ij}) = 0$ can be viewed as the defining equation of a hypersurface M^5 in the Lagrangian Grassmannian Λ^6.

E.V. Ferapontov, L. Hadjikos and K.R. Khusnutdinova, Integrable equations of the dispersionless Hirota type and hypersurfaces in the Lagrangian Grassmannian, IMRN (2010) 496-535.

3D Hirota master-equation

Theorem. *A 3D Hirota master-equation is given by the formula*

$$\vartheta_m(u_{ij}) = 0$$

where ϑ_m is any genus 3 theta constant with an even characteristic m.

F. Cléry, E.V. Ferapontov, Dispersionless Hirota equations and the genus 3 hyperelliptic divisor, Comm. Math. Phys. **376***, no. 2 (2020) 1397-1412.*

Note that θ-constants are Siegel modular forms of weight $1/2$.

The corresponding hypersurface $M^5 \subset \Lambda^6$ is the genus 3 hyperelliptic divisor.

This theorem was proved by uncovering geometry behind the Odesskii-Sokolov construction that parametrises dispersionless integrable systems via generalised hypergeometric functions.

ϑ-constants

Theta constants (of genus 3) with characteristics are defined by

$$\vartheta_m(\tau) = \vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau) = \sum_{n\in\mathbb{Z}^3} e^{i\pi(n+\mu/2)\left(\tau(n+\mu/2)^t+\nu^t\right)}$$

where $\mu, \nu \in \{0,1\}^3$ and τ is a 3×3 symmetric matrix. The characteristic $m = \left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]$ is called even if $\mu\nu^t$ is even. In genus 3, there are 36 even characteristics and they give rise to 36 theta constants. Recall that there is an action of the group $Sp(6,\mathbb{Z})$ on the set of even characteristics which is transitive, so that all equations

$$\vartheta_m(u_{ij}) = 0$$

for different characteristics m are equivalent. It is known that in genus 3 the vanishing of even theta constants characterises the hyperelliptic divisor (Schottky).

Open problems

- Find a purely computational proof that even theta constants satisfy the 3D integrability conditions by deriving $Sp(6,\mathbb{R})$-invariant differential equations that characterise theta constants.
- Classify 3D integrable Hirota type equations corresponding to singular $Sp(6,\mathbb{R})$-orbits of lower dimension (degenerations of theta constants).

3D Integrable Lagrangians $\int f(v_{x_1}, v_{x_2}, v_{x_3})\, dx$

Euler-Lagrange equation:

$$(f_{v_{x_1}})_{x_1} + (f_{v_{x_2}})_{x_2} + (f_{v_{x_3}})_{x_3} = 0.$$

Example 1. Dispersionless Kadomtsev-Petviashvili equation

$$v_{x_1x_3} - v_{x_1}v_{x_1x_1} - v_{x_2x_2} = 0, \qquad f = v_{x_1}v_{x_2} - \frac{1}{3}v_{x_1}^3 - v_{x_2}^2.$$

Example 2. Boyer-Finley equation

$$v_{x_1x_1} + v_{x_2x_2} - e^{v_{x_3}}v_{x_3x_3} = 0, \qquad f = v_{x_1}^2 + v_{x_2}^2 - 2e^{v_{x_3}}.$$

E.V. Ferapontov, K.R. Khusnutdinova and S.P. Tsarev, On a class of three-dimensional integrable Lagrangians, Comm. Math. Phys. **261***, N1 (2006) 225-243.*

E.V. Ferapontov and A.V. Odesskii, Integrable Lagrangians and modular forms, J. Geom. Phys. **60***, no. 6-8 (2010) 896-906.*

Modular example

Example 3. Lagrangian density $f = v_{x_1} v_{x_2} g(v_{x_3})$ gives the Euler-Lagrange equation

$$(v_{x_2} g(v_{x_3}))_{x_1} + (v_{x_1} g(v_{x_3}))_{x_2} + (v_{x_1} v_{x_2} g'(v_{x_3}))_{x_3} = 0.$$

Integrability condition for $g(z)$:

$$g''''(g^2 g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''g''' + 8(g')^3 g''' - g^2(g''')^2 = 0.$$

The generic solution $g(z)$ can be represented in the form ($q = e^{2\pi i z}$)

$$g(z) = \sum_{(k,l)\in\mathbb{Z}^2} e^{2\pi i (k^2+kl+l^2)z} = 1 + 6q + 6q^3 + 6q^4 + 12q^4 + \ldots .$$

Note that g coincides with the Eisenstein series $E_{1,3}(z)$ which is a modular form of weight one and level three.

Summary of known results

- The parameter space of integrable Lagrangian densities f is 20-dimensional.
- Integrability conditions for f are invariant under a 20-dimensional symmetry group that acts on the parameter space with an open orbit.

Problem: construct master-Lagrangian corresponding to the open orbit.

Remarkably, this leads to the theory of Picard modular forms.

F. Cléry, E.V. Ferapontov, A. Odesskii, D. Zagier, Integrable Lagrangians and modular forms, https://people.mpim-bonn.mpg.de/zagier/files/preprints/cfoz.pdf, work in progress.

Integrability conditions

Integrability conditions form a PDE system for the density f:

$$d^4 f = d^3 f \frac{dH}{H} + \frac{3}{H}\det(dM).$$

Here $d^3 f$ and $d^4 f$ are the symmetric differentials of f while the Hessian H and the 4×4 matrix M are defined as

$$H = \det \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & f_x & f_y & f_z \\ f_x & f_{xx} & f_{xy} & f_{xz} \\ f_y & f_{xy} & f_{yy} & f_{yz} \\ f_z & f_{xz} & f_{yz} & f_{zz} \end{pmatrix}.$$

Here $(x, y, z) = (v_{x_1}, v_{x_2}, v_{x_3})$. The system for f is in involution and its solution space is 20-dimensional.

Weierstrass sigma function σ and integers C_k

Let σ be the Weierstrass sigma function (equianharmonic case $g_2 = 0$). It solves the ODE

$$\sigma\sigma'''' - 4\sigma'\sigma''' + 3\sigma''^2 = 0$$

and possesses a power series expansion

$$\sigma(z) = \sum_{k\geq 0} C_k \frac{z^{6k+1}}{(6k+1)!}$$

where C_k are certain integers:

$$1,\ 1,\ -6,\ -552,\ 18600,\ -9831240,\ \ldots.$$

These integers will feature in the formulas for the density f.

Lagrangian densities $f = v_{x_1} v_{x_2} g(v_{x_3})$

Euler-Lagrange equation:

$$(v_{x_2} g(v_{x_3}))_{x_1} + (v_{x_1} g(v_{x_3}))_{x_2} + (v_{x_1} v_{x_2} g'(v_{x_3}))_{x_3} = 0.$$

Integrability condition:

$$g''''(g^2 g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''g''' + 8(g')^3 g''' - g^2(g''')^2 = 0.$$

Below we give 3 equivalent representations of the generic solution g:

Theta representation;

Power series representation;

Parametric representation.

Auxiliary hypergeometric equation

Consider the auxiliary hypergeometric equation

$$u(1-u)h_{uu} + (1-2u)h_u - \frac{2}{9}h = 0.$$

The geometry behind this equation is a one-parameter family of genus 2 trigonal curves,

$$r^3 = t(t-1)(t-u)^2,$$

supplied with the holomorphic differential $\omega = dt/r$. The corresponding periods, $h = \int_a^b \omega$ where $a, b \in \{0, 1, \infty, u\}$, form a 2-dimensional vector space and satisfy the above (Picard-Fuchs) hypergeometric equation.

Generic solution $g(z)$

1. **Theta representation:**

$$g(z) = \sum_{(k,l)\in\mathbb{Z}^2} e^{2\pi i(k^2+kl+l^2)z} = 1 + 6q + 6q^3 + 6q^4 + 12q^4 + ...$$

2. **Power series:**

$$g(z) = \sum_{k\geq 0} C_k^2 \frac{z^{6k+1}}{(6k+1)!}.$$

3. **Parametric form:**

$$z = \frac{h_1(u)}{h_2(u)}, \quad g = h_2(u),$$

here h_i are two linearly independent solutions of the above hypergeometric equation.

Lagrangian densities $f = v_{x_1} g(v_{x_2}, v_{x_3})$

The corresponding Euler-Lagrange equation is

$$(g)_{x_1} + \left(v_{x_1} g_{v_{x_2}}\right)_{x_2} + \left(v_{x_1} g_{v_{x_3}}\right)_{x_3} = 0.$$

The integrability conditions lead to an involutive PDE system for $g(y, z)$ which is invariant under a ten-dimensional symmetry group. This invariance allows one to linearise the integrability conditions for g.

Auxiliary hypergeometric system

Consider the auxiliary (Appell) hypergeometric system

$$h_{u_1u_2} = \frac{1}{3}\frac{h_{u_1} - h_{u_2}}{u_1 - u_2},$$

$$h_{u_1u_1} = -\frac{h}{9u_1(u_1-1)} + \frac{h_{u_2}}{3(u_1-u_2)}\frac{u_2(u_2-1)}{u_1(u_1-1)} - \frac{h_{u_1}}{3}\left(\frac{1}{u_1-u_2} + \frac{2}{u_1} + \frac{2}{u_1-1}\right),$$

$$h_{u_2u_2} = -\frac{h}{9u_2(u_2-1)} + \frac{h_{u_1}}{3(u_2-u_1)}\frac{u_1(u_1-1)}{u_2(u_2-1)} - \frac{h_{u_2}}{3}\left(\frac{1}{u_2-u_1} + \frac{2}{u_2} + \frac{2}{u_2-1}\right).$$

Its solutions are periods of the holomorphic differential $\omega = dq/r$ of the genus three Picard trigonal curves (Picard, 1883):

$$r^3 = q(q-1)(q-u_1)(q-u_2).$$

Generic solution $g(y,z)$

1. **Theta representation:**

$$g(y,z) = y + \sum_{(k,l)\in\mathbb{Z}^2\setminus 0} \frac{\sigma((k+\epsilon l)y)}{k+\epsilon l} e^{2\pi i(k^2+kl+l^2)z}, \quad \epsilon = e^{\pi i/3}.$$

2. **Power series:**

$$g(y,z) = \sum_{j,k\geq 0} C_j C_k C_{j+k} \frac{y^{6j+1}}{(6j+1)!} \frac{z^{6k+1}}{(6k+1)!}.$$

3. **Parametric form:**

$$y = \frac{h_1(u_1,u_2)}{h_3(u_1,u_2)}, \quad z = \frac{h_2(u_1,u_2)}{h_3(u_1,u_2)}, \quad g = F(s),\ s = \frac{u_1(u_2-1)}{u_2(u_1-1)}.$$

where $F' = [s(s-1)]^{-2/3}$.

Relation to Picard modular forms

The period map,

$$y = \frac{h_1(u_1, u_2)}{h_3(u_1, u_2)}, \quad z = \frac{h_2(u_1, u_2)}{h_3(u_1, u_2)},$$

was inverted by Picard (1883):

$$u_1 = \frac{\varphi_1(y, z)}{\varphi_0(y, z)}, \;\; u_2 = \frac{\varphi_2(y, z)}{\varphi_0(y, z)},$$

where φ_ν are modular forms on a 2-dimensional complex ball $2\mathrm{Re}y + |z|^2 < 0$ with respect to the Picard modular group $\Gamma[\sqrt{-3}] = \{g \in U(2, 1; \mathbb{Z}[\rho]) : g \equiv 1(\mathrm{mod}\sqrt{-3})\}, \; \rho = e^{2\pi i/3}$. Picard modular forms were extensively studied by Holzapfel, Feustel, Finis, Shiga, Cléry and van der Geer.

Differential dg via Picard modular forms

There exists a simple expression of the differential dg is terms of φ_ν:

$$dg = \frac{\varphi_1\varphi_2(\varphi_2 - \varphi_1)d\varphi_0 + \varphi_0\varphi_2(\varphi_0 - \varphi_2)d\varphi_1 + \varphi_0\varphi_1(\varphi_1 - \varphi_0)d\varphi_2}{\zeta^2}$$

where ζ is a modular form defined as

$$\zeta^3 = \varphi_0\varphi_1\varphi_2(\varphi_1 - \varphi_0)(\varphi_2 - \varphi_0)(\varphi_2 - \varphi_1).$$

Up to a constant factor, the differential dg has appeared in the theory of vector-valued Picard modular forms.

H. Shiga, On the representation of the Picard modular function by θ constants, I, II. Publ. Res. Inst. Math. Sci. **24***, no. 3 (1988) 311-360.*

F. Cléry, G. van der Geer, Generators for modules of vector-valued Picard modular forms, Nagoya Math. J. **212** *(2013) 19-57.*

Generic Lagrangian densities $f(v_{x_1}, v_{x_2}, v_{x_3})$

Euler-Lagrange equation:

$$(f_{v_{x_1}})_{x_1} + (f_{v_{x_2}})_{x_2} + (f_{v_{x_3}})_{x_3} = 0.$$

Integrability conditions lead to a PDE system for $f(x, y, z)$ which is invariant under 20-dimensional symmetry group:

$$\tilde{x} = \frac{l_1(x,y,z)}{l(x,y,z)}, \quad \tilde{y} = \frac{l_2(x,y,z)}{l(x,y,z)}, \quad \tilde{z} = \frac{l_3(x,y,z)}{l(x,y,z)}, \quad \tilde{f} = \frac{f}{l(x,y,z)},$$

as well as obvious symmetries of the form

$$\tilde{f} = \epsilon f + \alpha x + \beta y + \gamma z + \delta,$$

This symmetry allows one to linearise the integrability conditions for f.

Auxiliary hypergeometric system

Consider the auxiliary (Appell) hypergeometric system

$$h_{u_i u_j} = \frac{1}{3}\frac{h_{u_i} - h_{u_j}}{u_i - u_j},$$

$$h_{u_i u_i} = -\frac{2}{9}\frac{h}{u_i(u_i-1)} - \frac{1}{3u_i(u_i-1)}\sum_{j\neq i}^{3}\frac{u_j(u_j-1)}{u_j-u_i}h_{u_j} + \\ -\frac{1}{3}\Big(\sum_{j\neq i}^{3}\frac{1}{u_i-u_j} + \frac{2}{u_i} + \frac{2}{u_i-1}\Big)h_{u_i}.$$

Its solutions are periods of the holomorphic differential $\omega = dq/r$ of the genus four Picard trigonal curves,

$$r^3 = q(q-1)(q-u_1)(q-u_2)(q-u_3).$$

Inhomogeneous hypergeometric extension

We will also need the inhomogeneous hypergeometric system

$$F_{u_i u_j} = \frac{1}{3}\frac{F_{u_i} - F_{u_j}}{u_i - u_j} + \epsilon_{ijk}\frac{u_k(u_k-1)(u_i-u_j)}{U^{2/3}},$$

$$F_{u_i u_i} = -\frac{2}{9}\frac{F}{u_i(u_i-1)} - \frac{1}{3u_i(u_i-1)}\sum_{j\neq i}^{3}\frac{u_j(u_j-1)}{u_j-u_i}F_{u_j} + \\ -\frac{1}{3}\Big(\sum_{j\neq i}^{3}\frac{1}{u_i-u_j} + \frac{2}{u_i} + \frac{2}{u_i-1}\Big)F_{u_i},$$

where ϵ_{ijk} is the totally antisymmetric tensor and

$$U = u_1u_2u_3(u_1-1)(u_2-1)(u_3-1)(u_1-u_2)(u_2-u_3)(u_3-u_1).$$

This system for F is in involution.

Generic solution $f(x, y, z)$

1. **Theta representation:**

$$f(x,y,z) = xy + \sum_{(k,l)\in\mathbb{Z}^2\setminus 0} \frac{\sigma((k+\epsilon l)x)\sigma((k+\epsilon l)y)}{(k+\epsilon l)^2} e^{2\pi i(k^2+kl+l^2)z}.$$

2. **Power series:**

$$f(x,y,z) = \sum_{i,j,k\geq 0} C_i C_j C_k C_{i+j+k} \frac{x^{6i+1}}{(6i+1)!}\frac{y^{6j+1}}{(6j+1)!}\frac{z^{6k+1}}{(6k+1)!}.$$

3. **Parametric form:**

$$x = \frac{h_1}{h_4}, \quad y = \frac{h_2}{h_4}, \quad z = \frac{h_3}{h_4}, \quad f = \frac{F}{h_4}$$

   where F is a solution of the inhomogeneous system.

Open problem

- Classify integrable Lagrangian densities corresponding to singular orbits of lower dimension (degenerations of Picard modular forms).

What do dispersionless integrable systems give to the theory of modular forms?

- Differential equations for modular forms (integrability conditions);
- Potentials for vector-valued modular forms.