# On ODEs satisfied by modular forms

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# Plan:

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- Jacobi forms
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*S. Opanasenko, E.V. Ferapontov, On ODEs satisfied by modular forms; arXiv:2212.01413.*

# General picture

Let $f$ be a modular form (classical, Jacobi, Siegel, Picard, ...) on a discrete subgroup $\Gamma$ of a Lie group $G$. Then $f$ solves a nonlinear ODE/PDE system $\Sigma$ such that:

- System $\Sigma$ is involutive ($\sim$ compatible)
- System $\Sigma$ is of finite type ($\sim$ finite-dimensional solution space)
- System $\Sigma$ is $G$-invariant, furthermore, the Lie group $G$ acts on the solution space of $\Sigma$ locally transitively and with an open orbit ($\sim$ dimension of the solution space of $\Sigma$ equals $\dim G$); system $\Sigma$ is expressible via differential invariants of $G$
- The modular form $f$ is a generic solution of system $\Sigma$ ($\sim f$ belongs to the open orbit), with discrete stabilizer $G$

Classical modular forms $f(\tau),\ \tau \in \mathcal{H}$: $\Gamma = \mathrm{SL}(2, \mathbb{Z}),\ G = \mathrm{SL}(2, \mathbb{R})$, system $\Sigma$ is a third-order nonlinear $\mathrm{SL}(2, \mathbb{R})$-invariant ODE for $f(\tau)$.

# Modular forms

A modular form on a discrete subgroup $\Gamma$ of SL$(2, \mathbb{R})$ is a holomorphic function $f(\tau)$ on the upper half-plane $\mathcal{H}$ that satisfies the modular transformation property,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where $k$ is the weight of a modular form. An example of $\Gamma$ is the Hecke congruence subgroup of SL$(2, \mathbb{Z})$ of level $N$, denoted $\Gamma_0(N)$, characterised by the condition $a, b, c, d \in \mathbb{Z}$, $c \equiv 0 \pmod{N}$; note that $\Gamma_0(1)$ coincides with the full modular group SL$(2, \mathbb{Z})$.

# Eisenstein series

The Eisenstein series are defined as

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad q = e^{2\pi i \tau},$$

where $B_k$ are the Bernoulli numbers and $\sigma_{k-1}(n)$ denotes the sum of the $(k-1)$st powers of the positive divisors of $n$. Explicitly,

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24q - 72q^2 - \ldots,$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + \ldots,$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n = 1 - 504q - 16632q^2 - \ldots.$$

For every even $k > 2$, $E_k(\tau)$ is a modular form of weight $k$ on SL$(2, \mathbb{Z})$.

## Ramanujan equations

The Eisenstein series $E_2$, $E_4$ and $E_6$ satisfy the Ramanujan equations,

$$E_2' = \frac{E_2^2 - E_4}{12}, \qquad E_4' = \frac{E_2 E_4 - E_6}{3}, \qquad E_6' = \frac{E_2 E_6 - E_4^2}{2},$$

here prime denotes the operator $q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. These equations are invariant under the following action of the Lie group $\mathrm{SL}(2,\mathbb{R})$:

$$\tilde{\tau} = \tfrac{a\tau+b}{c\tau+d}, \quad \tilde{E}_2 = (c\tau+d)^2 E_2 + 12c(c\tau+d),$$
$$\tilde{E}_4 = (c\tau+d)^4 E_4, \quad \tilde{E}_6 = (c\tau+d)^6 E_6.$$

Every modular form $f$ on SL$(2,\mathbb{Z})$ is a homogeneous polynomial in $E_4$ and $E_6$. Differentiation of $f$ with the help of Ramanujan equations gives four polynomial expressions for $f$, $f'$, $f''$, $f'''$ in terms of $E_2$, $E_4$ and $E_6$. The elimination of $E_2$, $E_4$, $E_6$ leads to a third-order nonlinear ODE for $f$ which inherits $\mathrm{SL}(2,\mathbb{R})$ symmetry from the Ramanujan equations.

## Example: ODEs for the modular discriminant

The modular discriminant is a cusp form of weight $k = 12$ on $\mathrm{SL}(2,\mathbb{Z})$:

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \frac{1}{1728}(E_4^3 - E_6^2).$$

It satisfies an $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODE (Resnikoff),

$$36\Delta^4\Delta'''^2 - 14(18\Delta\Delta'' - 13\Delta'^2)\Delta^2\Delta'\Delta''' + 48\Delta^3\Delta''^3$$
$$+285\Delta^2\Delta'^2\Delta''^2 - 468\Delta\Delta'^4\Delta'' + 169\Delta'^6 + 48\Delta^7 = 0,$$

as well as the $\mathrm{GL}(2,\mathbb{R})$-invariant fourth-order ODE (van der Pol – Rankin),

$$2\Delta^3\Delta'''' - 10\Delta^2\Delta'\Delta''' - 3\Delta^2\Delta''^2 + 24\Delta\Delta'^2\Delta'' - 13\Delta'^4 = 0,$$

possessing an extra scaling symmetry $\Delta \to \lambda\Delta$ that is not present in the third-order ODE. Note that the fourth-order ODE is a differential consequence of the third-order ODE.

# Modularity theorem

Every elliptic curve $\gamma$ over $\mathbb{Q}$ possesses a modular parametrisation, that is, for some $N \geq 11$ there is a rational map

$$\Gamma_0(N)\backslash\mathcal{H} \longrightarrow \gamma$$

such that the pull-back of the holomorphic differential from $\gamma$ equals $\pi i f(\tau) d\tau$ where $f(\tau)$ is a cusp form of weight two on $\Gamma_0(N)$ with integer coefficients. This is the statement of the Taniyama-Shimura-Weyl conjecture proved by Wiles and Taylor for semistable elliptic curves (1995), and by Breuli, Conrad, Diamond, and Taylor in full generality (2001).

For instance, the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$ (with $j$-invariant $j = -2^{12}31^3 11^{-5}$) corresponds to the cusp form of weight two on $\Gamma_0(11)$,

$$f(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2.$$

We will see that the ODE for $f(\tau)$ 'knows' the corresponding elliptic curve $\gamma$.

## Fermat's last theorem

The most spectacular application of the modularity theorem is the proof of Fermat's Last Theorem. Suppose

$$a^p + b^p = c^p$$

is a counter-example to Fermat's Last Theorem. Then the (Hellegouarch-Frey) elliptic curve,

$$y^2 = x(x - a^p)(x + b^p),$$

cannot be modular (Ribet, 1990). Thus, the modularity theorem implies Fermat's Last Theorem.

## Third-order ODEs for modular forms

Every modular form $f(\tau)$ of weight $k$ satisfies a third-order ODE which can be represented by an algebraic relation

$$F(I_k, J_k) = 0$$

where $I_k$ and $J_k$ are differential invariants of a certain $\mathrm{SL}(2, \mathbb{R})$-action:

$$I_k = \frac{kff'' - (k+1)f'^2}{f^{2+\frac{4}{k}}} = \frac{[f,f]_2}{(k+1)f^{2+\frac{4}{k}}},$$

$$J_k = \frac{k^2 f^2 f''' - 3k(k+2)ff'f'' + 2(k+1)(k+2)f'^3}{f^{3+\frac{6}{k}}} = \frac{[f,[f,f]_2]_1}{(k+1)f^{3+\frac{6}{k}}}.$$

Here $[\cdot,\cdot]_i$ is the $i$th Rankin–Cohen bracket. Thus, there is a plane algebraic curve $C\colon\ F(I_k, J_k) = 0$ naturally associated with every modular form (note that relation $F$ depends on the modular form $f$).

Generic solution of this ODE: $\frac{1}{(c\tau+d)^k} f(\frac{a\tau+b}{c\tau+d}),\ a, b, c, d \in \mathbb{R},\ ad - bc = 1.$

## Third-order ODEs for modular forms of weight $k = 2$

Every modular form $f(\tau)$ of weight $k = 2$ satisfies a third-order ODE which can be represented by an algebraic relation

$$F(I_2, J_2) = 0$$

where $I_2$ and $J_2$ are the following modular functions:

$$I_2 = \frac{2ff'' - 3f'^2}{f^4}, \quad J_2 = \frac{4f^2f''' - 24ff'f'' + 24f'^3}{f^6},$$

note a relation

$$\frac{\mathrm{d}I_2}{J_2} = \pi i f(\tau)\mathrm{d}\tau.$$

We will see that for newforms of weight $k = 2$ on congruence subgroups $\Gamma_0(N)$ of genus $\geq 1$, the corresponding curves $C : F(I_2, J_2) = 0$ have genus $1$ and holomorphic differential $dI_2/J_2$. Furthermore, equations for $I_2, J_2$ provide a modular parametrisation of $C$.

# Fourth-order ODEs for modular forms

Every modular form $f(\tau)$ of weight $k$ satisfies a fourth-order ODE which can be represented by an algebraic relation

$$\mathcal{F}(P_k, Q_k) = 0$$

where $P_k$ and $Q_k$ are differential invariants of a certain $\mathrm{GL}(2, \mathbb{R})$-action,

$$P_k = \frac{(k^2 f^2 f''' - 3k(k+2) f f' f'' + 2(k+1)(k+2) f'^3)^2}{(k f f'' - (k+1) f'^2)^3} = \frac{(k+1)[f, [f,f]_2]_1^2}{[f,f]_2^3},$$

$$Q_k = \frac{f^2 \big(k(k+1) f f'''' - 4(k+1)(k+3) f' f''' + 3(k+2)(k+3) f''^2\big)}{(k f f'' - (k+1) f'^2)^2} = \frac{12(k+1)^2 f^2 [f,f]_4}{(k+2)(k+3)[f,f]_2^2}.$$

Thus, there is another plane algebraic curve $\mathcal{C}\colon\ \mathcal{F}(P_k, Q_k) = 0$ naturally associated with every modular form (turns out to be rational in all examples).

Generic solution of this ODE: $\frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right),\ a, b, c, d \in \mathbb{R}$.

Fourth-order ODE is a differential consequence of the third-order ODE.

## Eisenstein series $E_4$

Third-order ODE:

$$80E_4^2E_4'''^2 + 120(5E_4'^3 - 6E_4E_4'E_4'')E_4''' + 576E_4E_4''^3$$
$$-20(27E_4'^2 + 4E_4^3)E_4''^2 + 200E_4^2E_4'^2E_4'' - 125E_4E_4'^4 = 0.$$

Invariant form:

$$5J_4^2 + 144I_4^3 - 80I_4^2 = 0.$$

Fourth-order ODE:

$$16Q_4 - 5P_4 - 144 = 0.$$

## Eisenstein series $E_6$

Third-order ODE:

$$343(J_6^3 - 216 I_6^3) + 2(256 I_6^3 + 7 J_6^2)^2 = 0.$$

Fourth-order ODE:

$$(6Q_6 - 32)^2 - 7(Q_6 - 4)P_6 = 0.$$

## Modular discriminant

Third-order ODE:

$$36\Delta^4\Delta'''^2 - 14(18\Delta\Delta'' - 13\Delta'^2)\Delta^2\Delta'\Delta''' + 48\Delta^3\Delta''^3$$
$$+285\Delta^2\Delta'^2\Delta''^2 - 468\Delta\Delta'^4\Delta'' + 169\Delta'^6 + 48\Delta^7 = 0.$$

Invariant form:

$$J_{12}^2 + 16I_{12}^3 + 27648 = 0.$$

Fourth-order ODE:

$$2\Delta^3\Delta'''' - 10\Delta^2\Delta'\Delta''' - 3\Delta^2\Delta''^2 + 24\Delta\Delta'^2\Delta'' - 13\Delta'^4 = 0.$$

Invariant form:

$$Q_{12} = 6.$$

# Jacobi theta constants

Jacobi theta constants (thetanulls) are modular forms of weight $1/2$ defined as

$$\theta_2 = \sum_{n=-\infty}^{\infty} \mathrm{e}^{(n-1/2)^2\pi i\tau}, \quad \theta_3 = \sum_{n=-\infty}^{\infty} \mathrm{e}^{n^2\pi i\tau}, \quad \theta_4 = \sum_{n=-\infty}^{\infty} (-1)^n \mathrm{e}^{n^2\pi i\tau}.$$

Third-order ODE:

$$(\theta^2\theta_{\tau\tau\tau} - 15\theta\theta_\tau\theta_{\tau\tau} + 30\theta_\tau^3)^2 + 32(\theta\theta_{\tau\tau} - 3\theta_\tau^2)^3 + \pi^2\theta^{10}(\theta\theta_{\tau\tau} - 3\theta_\tau^2)^2 = 0.$$

Invariant form:

$$J_{1/2}^2 + 16\, I_{1/2}^3 - \frac{1}{16} I_{1/2}^2 = 0.$$

Fourth-order ODE:

$$\theta^3(\theta\theta_{\tau\tau} - 3\theta_\tau^2)\theta_{\tau\tau\tau\tau} - \theta^4\theta_{\tau\tau\tau}^2 + 2\theta^2\theta_\tau(\theta\theta_{\tau\tau} + 12\theta_\tau^2)\theta_{\tau\tau\tau}$$
$$+\theta^3\theta_{\tau\tau}^3 - 24\theta^2\theta_\tau^2\theta_{\tau\tau}^2 - 18\theta\theta_\tau^4\theta_{\tau\tau} + 18\theta_\tau^6 = 0.$$

Invariant form:

$$Q_{1/2} - 6P_{1/2} - 102 = 0.$$

# Eisenstein series $E_{1,3}$

This is a modular form of weight $1$ (with character) on $\Gamma_0(3)$ defined as

$$E_{1,3}(\tau) = \sum_{(\alpha,\beta)\in\mathbb{Z}^2} q^{(\alpha^2-\alpha\beta+\beta^2)} = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + .....$$

Third-order ODE:

$$f^2 f'''^2 - 6f'(3ff'' - 4f'^2)f''' + 18ff''^3 - (f^6 + 27f'^2)f''^2 + 4f^5 f'^2 f'' - 4f^4 f'^4 = 0.$$

Invariant form:

$$J_1^2 + 18I_1^3 - I_1^2 = 0.$$

Fourth-order ODE:

$$ff''''(ff'' - 2f'^2) - f^2 f'''^2 + 2f'(ff'' + 4f'^2)f''' - 9f'^2 f''^2 = 0.$$

Invariant form:

$$Q_1 - 2P_1 - 36 = 0.$$

## Modular form of weight 2 on $\Gamma_0(11)$

There is a unique cusp form of weight $2$ on the congruence subgroup $\Gamma_0(11)$,

$$f(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2,$$

labelled as case 11.2.a.a in the LMFDB database. It satisfies a third-order ODE,

$$\begin{gathered} J_2^4 + 32(I_2-8)(I_2^2+72I_2-944)J_2^2 \\ +256(I_2+8)(I_2^3+152I_2^2-704I_2+1168)(I_2-8)^2 = 0, \end{gathered}$$

which defines a singular curve $C$ of genus $1$ with $j = -2^{12}31^3 11^{-5}$. This value coincides with the $j$-invariant of the curve $y^2+y = x^3-x^2-10x-20$, which corresponds to the same cusp form $f$ via the modularity theorem. Thus, both curves are birationally equivalent:

$$I_2 = -\frac{8x^2+8x-119}{(x-5)^2}, \quad J_2 = -\frac{44(4x-9)(2y+1)}{(x-5)^3}.$$

The inverse transformation is

$$x = \frac{-11J_2^2 + 16(I_2 - 8)(9I_2^2 - 2468I_2 + 11712)}{64(I_2 - 83)(I_2^2 - 64)},$$

$$y = \frac{(11I_2 - 264)J_2^3 + 176(I_2 - 8)(I_2^3 + 80I_2^2 - 5584I_2 + 43904)J_2}{512(I_2^2 - 64)^2(I_2 - 83)} + \frac{1}{2}.$$

Note that $\mathrm{d}I_2/J_2 = \pi i f(\tau)\mathrm{d}\tau$ is the holomorphic differential on $C$. The form $f(\tau)$ also satisfies the fourth-order ODE,

$$\begin{aligned}&5616022359375P_2^4 - 2^4 3^5 5^3(34618195Q_2 - 763426383)P_2^3\\ &-2^8 3^3(173368000Q_2^3 - 8479136175Q_2^2 + 183916606320Q_2 - 1561600055241)P_2^2\\ &+2^{12}3^2(64349800Q_2^4 - 3828348951Q_2^3 + 88775864253Q_2^2 - 1000262056761Q_2 + 4759648412715)P_2\\ &+131072(4Q_2 - 63)(5329Q_2^3 - 204861Q_2^2 + 2745099Q_2 - 14039703)(2Q_2 - 105)^2 = 0,\end{aligned}$$

which defines a singular rational curve $\mathcal{C}$.

# Jacobi forms

A Jacobi form of weight $k$ and index $m$ is a holomorphic function $f(\tau, z) \colon \mathcal{H} \times \mathbb{C} \mapsto \mathbb{C}$ with the transformation property

$$f\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right) = (c\tau{+}d)^k \mathrm{e}^{2\pi i m\left(\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} - \lambda^2\tau - 2\lambda z - \lambda\mu\right)} f(\tau, z),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$.

Let us introduce the associated $G_{k,m}$-group action,

$$\tilde{\tau} = \frac{a\tau+b}{c\tau+d}, \quad \tilde{z} = \frac{z+\lambda\tau+\mu}{c\tau+d}, \quad \tilde{f} = (c\tau{+}d)^k \mathrm{e}^{2\pi i m\left(\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} - \lambda^2\tau - 2\lambda z - \lambda\mu\right)} f,$$

where all parameters are arbitrary real numbers.

# Differential invariants

**Second-order differential invariants:**

$$L_{k,m} = \frac{1}{(2kff_{zz} - 8m\pi i f f_\tau - (2k-1)f_z^2)^2} \Big( 64m^2\pi^2 f^3 f_{\tau\tau} + 32m\pi i f^2 f_z f_{\tau z} - 4k(k+1)f^2 f_{zz}^2 + 4(8m(k+1)\pi i f f_\tau + (2k^2+k-2)f_z^2) f f_{zz} - (16(2k+3)m\pi i f f_\tau + (4k^2-7)f_z^2) f_z^2 \Big),$$

$$M_{k,m} = \frac{4m\pi i f^2 f_{\tau z} - f f_z f_{zz} - (4m\pi i f f_\tau - f_z^2) f_z}{(2kff_{zz} - 8m\pi i f f_\tau - (2k-1)f_z^2)^{3/2}}.$$

**Third-order differential invariants:** $N_{k,m}$, $P_{k,m}$, $Q_{k,m}$, $R_{k,m}$ (similar formulas).

Any generic third-order $G_{k,m}$-invariant involutive PDE system (governing Jacobi forms) can be obtained by expressing all third-order invariants $N_{k,m}$, $P_{k,m}$, $Q_{k,m}$, $R_{k,m}$ as functions of the second-order invariants $L_{k,m}$, $M_{k,m}$.

Non-generic case: the denominator $2kff_{zz} - 8m\pi i f f_\tau - (2k-1)f_z^2 = 0$.

## Jacobi form $\varphi_{-1,2}(\tau, z)$

The weak Jacobi form $f(\tau, z) = \varphi_{-1,2}(\tau, z)$ of weight $-1$ and index $2$ is defined as $\varphi_{-1,2}(\tau, z) = \Delta^{-1/8}(\tau)\vartheta_1(\tau, 2z)$ where $\Delta$ is the modular discriminant.

Third-order involutive PDE system:

$$\begin{aligned} 2\pi i f f_{\tau\tau\tau} &= 2\pi i f_\tau f_{\tau\tau} + f_{\tau\tau} f_{zz} - f_{\tau z}^2, \\ f f_{\tau\tau z} &= 3 f_z f_{\tau\tau} - 2 f_\tau f_{\tau z}, \\ f f_{\tau zz} &= 8\pi i (f f_{\tau\tau} - f_\tau^2) + 2 f_z f_{\tau z} - f_\tau f_{zz}, \\ f f_{zzz} &= 16\pi i (f f_{\tau z} - f_\tau f_z) + f_z f_{zz}. \end{aligned}$$

Invariant form:

$$N_{-1,2} = 32 M^2_{-1,2} - L_{-1,2}, \qquad P_{-1,2} = -M_{-1,2},$$

$$Q_{-1,2} = -\frac{1}{4}(L_{-1,2} + 1), \qquad R_{-1,2} = 2M_{-1,2}.$$

## Jacobi theta functions

The Jacobi theta function,

$$\vartheta_3(\tau, z) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi i n z + \pi i n^2 \tau},$$

is a Jacobi form of index $\frac{1}{2}$ and weight $\frac{1}{2}$. It satisfies the heat equation

$$4\pi i (\vartheta_3)_\tau = (\vartheta_3)_{zz},$$

as well as a sixth-order equation involving $z$-derivatives of $\vartheta_3$ only,

$$\left( \frac{(\ln \vartheta_3)_{zzzzz}}{(\ln \vartheta_3)_{zzz}} + 12 (\ln \vartheta_3)_{zz} \right)_z = 0.$$

The same system holds for other Jacobi theta functions.