Cartan-Karlhede algorithm and Cartan invariants for spacetimes

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Outline

Preliminaries: The metric

Spacetime is a Lorentzian manifold, $(M, \mathbf{g})$.

A coordinate system, $\mathbf{x}$, for the manifold M,

$$x^\alpha, \ \alpha \in [1, 4]. \tag{1}$$

The coordinate (or *holonomic*) basis for $T_p M$,

$$\mathbf{V} = V^\alpha \partial_{x^\alpha}. \tag{2}$$

The dual basis for the cotangent space of one-forms, $T_p^* M$,

$$\mathbf{w} = w_\alpha dx^\alpha. \tag{3}$$

The metric is:

$$\mathbf{g} = g_{(\alpha\beta)}(x^\gamma) dx^\alpha dx^\beta, \ \alpha, \beta, \gamma \in [1, 4] \ \text{ with } det(\mathbf{g}) \neq 0. \tag{4}$$

Preliminaries: A frame basis Perspective

Let's pick a new basis for $T^*_p M$, called a *coframe basis*, $\{\mathbf{h}^a\}$ such that

$$\mathbf{h}^a = \mathbf{h}^a{}_\alpha dx^\alpha. \tag{5}$$

We want a *frame basis* for the tangent space, so we will assume $h^a{}_\alpha h_b{}^\beta = \delta^a{}_b$, then

$$\mathbf{h}_a = h_a{}^\alpha \partial_{x^\alpha}. \tag{6}$$

The coframe basis must satisfy:

$$g_{\alpha\beta} = \eta_{ab} h^a{}_\alpha h^b{}_\beta, \;\; \eta = diag(-1,1,1,1), \; a,b \in [1,4]. \tag{7}$$

This will be an *anholonomic* frame since

$$[\mathbf{h}_a, \mathbf{h}_b] = C^c{}_{ab} \mathbf{h}_c \neq \mathbf{0}. \tag{8}$$

The components $C^a{}_{bc}$ are *the coefficients of anholonomy*.

The connection

To differentiate vectors and one-forms covariantly, pick a connection ($\nabla_a = \nabla_{\mathbf{h}_a}$) so that

$$\nabla_a \mathbf{h}^b = \Gamma^b{}_{ca}\mathbf{h}^c, \;\; \nabla_a \mathbf{h}_b = -\Gamma^c{}_{ba}\mathbf{h}_c. \tag{9}$$

For any vector or one-form:

$$\mathbf{v} \in T_pM, \quad \nabla_a v^b = v^b{}_{;a} = \mathbf{h}_a(v^b) + \Gamma^a{}_{cb}v^c, \tag{10}$$

$$\mathbf{w} \in T_p^*M \quad \nabla_a w_b = w_{b;a} = \mathbf{h}_a(w_b) - \Gamma^c{}_{ba}w_c. \tag{11}$$

General Relativity (GR) uses the *Levi-Civita connection*, Γ.

$$\begin{aligned} \textit{metric compatible} : \nabla_a g_{bc} = g_{bc;a} = 0, \\ \textit{torsion-free} : \mathbf{T}(\mathbf{v},\mathbf{w}) = w^b v_{a;b} - v^b w_{a;b} - [\mathbf{v},\mathbf{w}] = \mathbf{0}, \mathbf{v},\mathbf{w} \in T_pM. \end{aligned} \tag{12}$$

The tensor, $\mathbf{T}$, is known as the torsion tensor.

The curvature

In GR, gravity is encoded in the non-zero Riemann curvature tensor:

$$\mathbf{R}(\mathbf{v},\mathbf{w})\mathbf{x} = \nabla_{\mathbf{v}}\nabla_{\mathbf{w}}\mathbf{x} - \nabla_{\mathbf{w}}\nabla_{\mathbf{v}}\mathbf{x} - \nabla_{[\mathbf{v},\mathbf{w}]}\mathbf{x}. \tag{13}$$

The components relative to the frame,

$$R^a{}_{bcd} = e_c(\Gamma^a{}_{bd}) - e_d(\Gamma^a{}_{bc}) + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed}. \tag{14}$$

The Ricci tensor and Ricci scalar, respectively,

$$Ric_{ab} = R^c{}_{acb}, \text{ and } R = Ric^a{}_b. \tag{15}$$

Einstein tensor:

$$G_{ab} = Ric_{ab} - \frac{1}{2}Rg_{ab}, \quad G^a{}_{b;a} = 0. \tag{16}$$

A frame approach

Define the *connection one-form* relative to the frame $\mathbf{h}^a$,

$$\omega^a{}_b = \Gamma^a{}_{bc}\mathbf{h}^c. \tag{17}$$

Cartan's Structure equations:

$$d\mathbf{h}^a + \omega^a{}_c \wedge \mathbf{h}^c = \frac{1}{2} T^a{}_{bc}\mathbf{h}^b \wedge \mathbf{h}^c, \tag{18}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd}\mathbf{h}^c \wedge \mathbf{h}^d. \tag{19}$$

The metric compatibility condition can be restated as the vanishing of the *non-metricity* tensor:

$$Q_{abc} = g_{ab;d} = 0 \leftrightarrow \omega_{ab} = -\omega_{ba}. \tag{20}$$

Field equations of $f(R)$ theories

For a reasonably motivated stress-energy tensor, T_{ab}, the Einstein field equations are,

$$G_{ab} = \kappa T_{ab}, \quad \kappa = \frac{8\pi G}{c^4}. \tag{21}$$

The field equations of a gravitational theory can be derived by varying an action with a particular function $f(R)$

$$S = \int \left[\frac{1}{\kappa} f(R) + \mathcal{L}_m \right] \sqrt{-g} d^4 x. \tag{22}$$

The $f(R)$ field equations are:

$$\frac{df}{dR} R_{ab} - \frac{1}{2} f(R) g_{ab} + (g_{ab} g^{cd} \nabla_c \nabla_d - \nabla_a \nabla_b)[f(R)] = \kappa T_{ab}. \tag{23}$$

If $f(R) = R$ we recover the Einstein-Hilbert action and the standard field equations.

The equivalence problem for Riemannian manifolds

Any diffeomorphism between two smooth manifolds, $\Phi : M \to \overline{M}$, induces two related mappings

$$\Phi_* : T_pM \to T_{\bar{p}}\overline{M} \text{ and } \Phi^* T^*_{\bar{p}}\overline{M} \to T^*_p M. \tag{24}$$

For any two solutions, $(M, \mathbf{g})$ and $(M, \bar{\mathbf{g}})$, it is worthwhile to know if a diffeomorphism, Φ, exists such that

$$\Phi^* \bar{\mathbf{g}} = \mathbf{g}.$$

We call this an isometry, and we say $(M, \mathbf{g})$ and $(\overline{M}, \bar{\mathbf{g}})$ are equivalent.

Scalar polynomial curvature invariants

Definition

A *scalar polynomial curvature invariant (SPI)* is constructed from full contractions of tensors constructed from copies of the Riemann tensor, its Hodge dual R^*_{abcd}, and its covariant derivatives, $R_{abcd;e_1,\dots e_p}$.

We will call the set of all SPIs formed from all of the curvature tensors, $\mathcal{I}$.

As an example, consider the Kretschmann scalar, K_1, and the Karlhede invariant, K_2:

$$K_1 = R_{abcd}R^{abcd} \text{ and } K_2 = R_{abcd;e}R^{abcd;e}. \tag{25}$$

- Gravitational theories can be generated by varying an action involving SPIs
- Most spacetimes (but not all), can be locally characterized by $\mathcal{I}$.

Minkwowski spacetime vs a plane wave spacetime

Minkowski spacetime

$$\mathbf{g_M} = -dt^2 + dx^2 + dy^2 + dz^2 \tag{26}$$

or equivalently,

$$\mathbf{g_M} = 2dudv + dy^2 + dz^2. \tag{27}$$

homogeneous plane wave

$$\begin{aligned} \mathbf{g_P} &= 2dudv + dy^2 + dz^2 + 2H(u,x,y)du^2, \\ H &= \frac{1}{2}(y^2 - z^2)\cos(2\epsilon u) - yz\sin(2\epsilon u). \end{aligned} \tag{28}$$

where ϵ is some real-valued constant.

Failure of SPI characterization

In the Riemannian signature, $(++++)$, we could (in principle) compute all SPIs and characterize two metrics.

In the Lorentzian signature things are not as simple.

Denoting I_M and I_P as the set of all SPIs for the spacetimes with the metric $\mathbf{g_M}$ and $\mathbf{g_P}$, we have

$$I_M = I_P = \{0\}. \tag{29}$$

As far as SPIs are concerned, Minkowski and the homogeneous plane-wave solution are the same!

Symmetries

These are different solutions.

Compute the *Killing vector fields*, **X** for each metric, i.e,

$$\mathcal{L}_{\mathbf{X}}\mathbf{g} = \mathbf{0} \leftrightarrow \frac{1}{2}(X_{a;b} + X_{b;a}) = 0, \tag{30}$$

where $\mathcal{L}$ is the Lie derivative.

The set of Killing vectors forms a n-dimensional Lie algebra, with an associated n-dimensional Lie group, G_n.

Comparing the two spacetimes we see they are inequivalent:

Minkowski spacetime	G_{10}
homogeneous plane wave spacetime	G_6

Could we compare two plane wave solutions with the same G_6?

A more general problem

We will consider the classification problem for a manifold, M, equipped with a (co)frame and a structure group, G [Olver, 1995].

Definition

On a smooth m-dimensional manifold, a *frame* on M is an ordered set of vector fields $\{\mathbf{h}_a\}_{a=1}^m$ which form a basis for T_pM at each point $p \in M$.

A coframe on M is an ordered set of one-forms $\{\mathbf{h}^a\}_{a=1}^m$ which form a basis for T_p^*M at each $p \in M$.

We will assume that $\{\mathbf{h}_a\}_{a=1}^m$ and $\{\mathbf{h}^a\}_{a=1}^m$ are dual to each other.

Notation: **(Abuse of notation)** I will denote an individual (co)frame element with arbitrary index and the (co)frame basis by $\mathbf{h}_a$ ($\mathbf{h}^a$).

The equivalence problem

We need a matrix Lie Group, $G \subset GL(m)$ for the structure group.

Essentially, the structure group will act on the frame basis to produce a new frame basis.

Definition

Let $G \subset GL(m)$ be a Lie group. Let $\mathbf{h}^a$ and $\bar{\mathbf{h}}^a$ be coframes defined on the m-dimensional manifolds, M and $\overline{M}$, respectively.

The *G-valued equivalence problem* for these coframes is to determine whether there exists a (local) diffeomorphism $\Phi : M \to \overline{M}$ and a G-valued function $g : M \to G$ such that

$$\Phi^* \bar{\mathbf{h}}^a = g^a{}_b \mathbf{h}^b \tag{31}$$

where $g(\mathbf{x})$ is required to belong to the structure group G at each point $\mathbf{x} \in M$.

The $\{e\}$-valued equivalence problem

Suppose that $\Phi : M \to \overline{M}$ or $\overline{x}^\alpha = \Phi^\alpha(x^\beta)$ satisfies

$$\Phi^* \bar{\mathbf{h}}^a = \mathbf{h}^a. \tag{32}$$

If this holds, we can take an exterior derivative

$$\Phi^* d\bar{\mathbf{h}}^a = d\mathbf{h}^a. \tag{33}$$

We have a relationship to the $C^a_{\ bc}$, which we will now call *structure functions*

$$d\mathbf{h}^a = -C^a_{\ bc}\mathbf{h}^b \wedge \mathbf{h}^c \tag{34}$$

Using (32) and (33) we find

$$\bar{C}^a_{\ bc}(\Phi(\mathbf{x})) = C^a_{\ bc}(\mathbf{x}) \tag{35}$$

In addition, we have derived invariants, since if $I(\mathbf{x})$ is an invariant then

$$\Phi^* d\bar{I} = dI \leftrightarrow [\bar{\mathbf{h}}_a(\bar{I})](\Phi(\mathbf{x})) = [\mathbf{h}_a(I)](\mathbf{x}). \tag{36}$$

Classifying manifolds

In this context, the invariants come from the structure functions,

$$C_\sigma = \mathbf{h}_{d_s}\mathbf{h}_{d_{s-1}} \ldots, \mathbf{h}_{d_1}[C^a_{\ bc}], \text{where } \sigma = (a, b, c, d_1, \ldots d_{s-1}, d_s). \tag{37}$$

Here, σ is a non-decreasing multi-index. We say $s =$ order σ is the *order* of the derived invariant.

Definition

The s^{th} order classifying space $K_m^{(s)}$ associated with M, is the Euclidean space of dimension

$$\frac{1}{2}m^2(m-1)\binom{m+s}{m}$$

with coordinates $z^{(s)} = (\ldots, z_\sigma, \ldots)$.

The s^{th} structure map associated a coframe θ on M is $\mathbf{T}^{(s)} : M \to K_m^{(s)}$ whose components are $z_\sigma = C_\sigma$ for $\sigma \leq s$.

Classifying manifolds

If the structure map is regular, then the coframe $\mathbf{h}^a$ is *regular*.

If $\rho_s = \text{rank}\ \mathbf{T}^{(s)}$ this corresponds to the number of functionally independent structure invariants up to order s.

Definition

The s^{th} order classifying set $\mathcal{C}^{(s)}(\mathbf{h}^a)$ associated with the coframe $\{\mathbf{h}^a\}$ is defined as

$$\mathcal{C}^{(s)}(\mathbf{h}^a) = \{\mathbf{T}^{(s)}(\mathbf{x})|\mathbf{x} \in M\} \subset K^{(s)}. \tag{38}$$

We say the *order* of a coframe is the smallest integer, s, where

$$dim\, \mathcal{C}^{(s)}(\mathbf{h}^a) = dim\, \mathcal{C}^{(s+1)}(\mathbf{h}^a).$$

Proposition

Let $\{\mathbf{h}^a\}$ and $\{\bar{\mathbf{h}}^a\}$ be smooth, fully regular coframes defined on M and $\overline{M}$, respectively. There exists a local diffeomorphism $\Phi : M \to \overline{M}$ so that $\Phi^* \bar{\mathbf{h}}^a = \mathbf{h}^a$ if and only if they have the same order $\bar{s} = s$ and the manifolds $\mathcal{C}^{(s+1)}(\mathbf{h}^a)$ and $\bar{\mathcal{C}}^{(s+1)}(\bar{\mathbf{h}}^a)$ overlap.

Framing the problem: equivalence of metrics

For $(M, \mathbf{g})$ and $(\overline{M}, \bar{\mathbf{g}})$, we can rephrase the problem of finding an isometry $\Phi : M \to \overline{M}$ with

$$\Phi^* \bar{\mathbf{g}} = \mathbf{g}. \tag{39}$$

We employ the coframes for M and $\overline{M}$

$$\mathbf{g} = \eta_{ab}\mathbf{h}^a\mathbf{h}^b, \text{ and } \bar{\mathbf{g}} = \eta_{ab}\bar{\mathbf{h}}^a\bar{\mathbf{h}}^b, \quad \eta = diag(-1, 1, 1, 1). \tag{40}$$

Then, the two metrics are equivalent if and only if the coframes satisfy

$$\Phi^* \bar{\mathbf{h}}^a = \Lambda^a{}_b \mathbf{h}^b \tag{41}$$

where $\Lambda^a{}_b$ is required to take values in the group $SO(1, 3)$.

Normalization

Instead consider the problem

$$\bar{g}^a{}_b(\bar{\mathbf{x}})\bar{\mathbf{h}}^b = g^a{}_c(\mathbf{x})\mathbf{h}^c. \tag{42}$$

GOAL: Find $\bar{g}$ and g such that on both manifolds:

$$\bar{\mathbf{h}}'^a = \bar{g}^a{}_b\mathbf{h}^b, \text{ and } \mathbf{h}'^a = g^a{}_b\mathbf{h}^b \tag{43}$$

satisfying,

$$\Phi^*\bar{\mathbf{h}}'^a = \mathbf{h}'^a. \tag{44}$$

Use scalar invariant combinations of g and h^a such that

$$H(\bar{g}(\bar{\mathbf{x}}), \bar{\mathbf{h}}^a|\bar{\mathbf{x}}) = H(g(\mathbf{x}), \mathbf{h}^a|_{\mathbf{x}}) \tag{45}$$

whenever $\bar{\mathbf{x}} = \Phi(\mathbf{x})$ and $\Phi^*\bar{g}^a{}_b\mathbf{h}^b = g^a{}_b\mathbf{h}^b$

Normalization

Let N denote the range of H, due to the equivalence condition $\bar{g}^a_{\ b}(\bar{\mathbf{x}})\bar{\mathbf{h}}^b = g^a_{\ c}(\mathbf{x})\mathbf{h}^c$:

$$g^*(\mathbf{x}) = \bar{g}^*(\Phi(\mathbf{x})) = \bar{g}^*(\bar{\mathbf{x}}), \tag{46}$$

and so

$$\bar{g}^{*a}_{\ b}\bar{g}^b_{\ c}\bar{\mathbf{h}}^c = g^{*a}_{b}g^b_{\ c}\mathbf{h}^c. \tag{47}$$

The invariant of H then implies

$$H(\bar{g}^* g, \bar{\mathbf{h}}^a) = H(g^* g, \mathbf{h}^a) \tag{48}$$

There is an induced global action of G on the range of H!

Definition

An invariant function $H(g, \mathbf{h}^a)$ is said to define a normalization of *constant type* for the coframe $\mathbf{h}^a$ if for each $\mathbf{x} \in M$ the values $H(e, \mathbf{h}^a|_{\mathbf{x}})$ lie in a single orbit $\mathcal{O}_{\mathbf{h}} \subset N$ of the induced action of G on the range of H.

A useful theorem

Theorem

Assume that $\mathbf{h}^a$ *and* $\bar{\mathbf{h}}^a$ *are both of constant type for the invariant function* $H(g, \mathbf{h}^a)$*. If* $\mathbf{h}^a$ *and* $\bar{\mathbf{h}}^a$ *are equivalent, then on their associated orbits in N are necessarily the same* $\mathcal{O}_{\bar{\mathbf{h}}} = \mathcal{O}_{\mathbf{h}}$.

Fixing a point z in the orbit, let $\tilde{G} = G_z$ *the associated isotropy subgroup. We can construct smooth G-valued functions* $\bar{g}_0(\bar{\mathbf{x}})$ *and* $g_0(\mathbf{x})$ *so that*

$$H(\bar{g}_0, \bar{\mathbf{h}}^a) = z = H(g_0, \mathbf{h}^a), \tag{49}$$

which gives the following modified coframes

$$\mathbf{h}'^a = g_0{}^a{}_b \mathbf{h}^a, \text{ and } \bar{\mathbf{h}}'^a = \bar{g}_0{}^a{}_b \bar{\mathbf{h}}^a. \tag{50}$$

The original two coframes are equivalent if $g^a{}_b \mathbf{h}^b = \bar{g}^a{}_b \bar{\mathbf{h}}^b$ *if and only if*

$$\tilde{g}^a{}_b \mathbf{h}'^a = \bar{\tilde{g}}^a{}_b \bar{\mathbf{h}}'^a, \tag{51}$$

for some $\tilde{g}, \bar{\tilde{g}} \in \tilde{G}$.

Structure equations again

Consider the lifted coframe, $\mathbf{H}^a = g^a_{\ b}\mathbf{h}^b$ then

$$d\mathbf{H}^a = dg^a_{\ b} \wedge \mathbf{h}^b + g^a_{\ b} d\mathbf{h}^b. \tag{52}$$

So that

$$d\mathbf{H}^a = \gamma^a_{\ b} \wedge \mathbf{H}^b - c^a_{\ bc}(\mathbf{x}, g)\mathbf{h}^b \wedge \mathbf{H}^c \tag{53}$$

where $\gamma^a_{\ b} = dg^a_{\ c}(g^{-1})^c_{\ b}$ are the Maurer-Cartan forms on the structure group G or relative to the basis α^A:

$$d\mathbf{H}^a = A^a_{\ bA}\alpha^A \wedge \mathbf{h}^b - c^a_{\ bc}(\mathbf{x}, g)\mathbf{H}^b \wedge \mathbf{H}^c. \tag{54}$$

Equivalently, if we have $\bar{\mathbf{H}}^a = \bar{g}^a_{\ b}\mathbf{h}^b$ then we find

$$d\bar{\mathbf{H}}^a = A^a_{\ bA}\bar{\alpha}^A \wedge \bar{\mathbf{h}}^b - \bar{c}^a_{\ bc}(\bar{\mathbf{x}}, \bar{g})\bar{\mathbf{H}}^b \wedge \bar{\mathbf{H}}^c. \tag{55}$$

Note: $c^a_{\ bc}(\mathbf{x}, g)$ are called torsion coefficients.

Invariant quantities

Assume a consistent definition for the group parameters is given, $g(\mathbf{x}), \mathbf{x} \in M$

The Maurer Cartan forms, α^A, drop down to M

$$\alpha^A = z^A_{\ a}\mathbf{H}^a \tag{56}$$

then

$$d\theta^a = [B^a_{\ bc}[\mathbf{z}] - c^a_{\ bc}(\mathbf{x}, g)]\theta^b \wedge \theta^c, \text{ where } B^a_{\ bc} = 2A^a_{\ [b|A|}z^A_{\ c]}. \tag{57}$$

If the two original coframes can be mapped to each other for some choice of (unknown) $z^A(\mathbf{x})$ and $\bar{z}^A(\bar{\mathbf{x}})$ so that

$$\bar{\mathbf{H}}^a = \mathbf{H}^a$$

then

$$[B^a_{\ bc}[\bar{\mathbf{z}}] - \bar{c}^a_{\ bc}(\bar{\mathbf{x}})] = [B^a_{\ bc}[\mathbf{z}] - c^a_{\ bc}(\mathbf{x})], \text{ when } \bar{\mathbf{x}} = \Phi(\mathbf{x}). \tag{58}$$

Normalization and Absorption

If, for some fixed a, b, and c: $B^a{}_{bc} = \bar{B}^a{}_{bc} = 0$ then

$$c^a_{bc}(\mathbf{x}, g) = \bar{c}^a_{bc}(\bar{\mathbf{x}}, \bar{g})$$

is *essential torsion* for $d\mathbf{H}^a$.

- If $c^a{}_{bc}(\mathbf{x})$ then it is an invariant.
- Else $c^a{}_{bc}(\mathbf{x}, g) = H(g, \mathbf{h}^a) = 0$ or ± 1 by choosing g.

To remove inessential torsion, use *absorption of torsion*:

$$\pi^A = \alpha^A - z^A_a \mathbf{H}^a \tag{59}$$

where z^A_a are solutions to the absorption equations, giving

$$d\mathbf{H}^a = A^a{}_{bA}\pi^A \wedge \mathbf{H}^b - U^a{}_{bc}\mathbf{H}^b \wedge \mathbf{H}^c. \tag{60}$$

Outline of the Cartan method

0) Compute structure equations

1) Check if there is essential torsion, normalize if possible

2) Eliminate inessential torsion and recompute structure equations. Go back to 1.

This process ends when

i) All group parameters have been specified, or

ii) A subset of group parameters cannot be specified [Prolongation].

Physics and Computation

“Physical”

In GR, and other gravity theories, the Riemann tensor is seen as physical.

The frame, the coefficients of anholonomy and their derivatives are not considered physical.

In some instances, the Cartan algorithm will not be able to fix all parameters (without prolongation) this should say something about the symmetry group of the spacetime.

Computational

Normalization and Absorption of torsion are both problems based in linear algebra.

It would be nice to combine these steps into one and reduce the number of calculations.

The frame problem for spacetimes

Taking the frame basis for the metric, $\mathbf{h}^a$, we lift the coframe to

$$\mathbf{H}^a = \Lambda^a{}_b \mathbf{h}^b, \ \Lambda \in SO(1,3).$$

The structure equations become

$$d\mathbf{H}^a = d(\Lambda^a{}_c)(\Lambda^{-1})^c{}_b \wedge \mathbf{H}^b - c^a{}_{bc}\mathbf{H}^b \wedge \mathbf{H}^c. \tag{61}$$

Notice that $d(\Lambda^a{}_c)(\Lambda^{-1})^c{}_b$ is the skew-symmetric Maurer-Cartan form on $SO(1,3)$.

Absorb all of the torsion:

$$d(\Lambda^a{}_c)(\Lambda^{-1})^c{}_b = z^a{}_{bc}\mathbf{H}^c, \text{ with } z_{abc} = -z_{bac}. \tag{62}$$

The absorption equation (Cartan's test) is

$$z^a{}_{bc} - z^a{}_{cb} = -c^a{}_{bc}, \tag{63}$$

where the solution is

$$2z^a{}_{bc} = c^a{}_{bc} + c^b{}_{ac} + c^c{}_{ba}. \tag{64}$$

A connection between methods

Since

$$c^a_{\ bc} = \Lambda^a_{\ d}(\Lambda^{-1})^e_{\ b}(\Lambda^{-1})^f_{\ c}C^d_{\ ef},$$

this becomes

$$\begin{aligned} z^a_{\ bc} &= \frac{1}{2}\Lambda^a_{\ d}(\Lambda^{-1})^e_{\ b}(\Lambda^{-1})^f_{\ c}[C^d_{\ ef} + C^e_{\ df} + C^f_{\ ad}] \\ &= \Lambda^a_{\ d}(\Lambda^{-1})^e_{\ b}(\Lambda^{-1})^f_{\ c}\Gamma^d_{\ ef}. \end{aligned} \tag{65}$$

Consider a new frame problem on $M^1 = M \times O(M)$ with the forms:

$$-\boldsymbol{\omega}^a_{\ b} = d(\Lambda^a_{\ c})(\Lambda^{-1})^c_{\ b} - z^a_{\ bc}\mathbf{H}^c. \tag{66}$$

We find,

$$d\mathbf{H}^a = -\boldsymbol{\omega}^a_{\ b} \wedge \mathbf{H}^b, \;\; d\boldsymbol{\omega}^a_{\ b} = -\boldsymbol{\omega}^a_{\ c} \wedge \boldsymbol{\omega}^c_{\ b} + S^a_{\ bcd}\mathbf{H}^c \wedge \mathbf{H}^d. \tag{67}$$

Use curvature instead

On M^1 we have

$$d\mathbf{H}^a = -\omega^a{}_b \wedge \mathbf{H}^b, \quad d\omega^a{}_b = -\omega^a{}_c \wedge \omega^c{}_b + S^a{}_{bcd}\mathbf{H}^c \wedge \mathbf{H}^d. \tag{68}$$

where

$$\begin{aligned} \omega^a{}_b &= -d(\Lambda^a{}_c)(\Lambda^{-1})^c{}_b + [\Lambda^a{}_d \Gamma^d{}_{ef}(\Lambda^{-1})^e{}_b(\Lambda^{-1})^f{}_c]\mathbf{H}^c, \\ S^a{}_{bcd} &= \Lambda^a{}_{a'}(\Lambda^{-1})^{b'}{}_b(\Lambda^{-1})^{c'}{}_c(\Lambda^{-1})^{d'}{}_d R^{a'}{}_{b'c'd'}. \end{aligned} \tag{69}$$

Theorem

A complete set of structure invariants for a Lorentzian manifold are provided by the invariant components of the higher orer curvature tensors $\nabla^q \mathbf{R}$, $p = 0, 1, 2 \ldots, s+1$.

In the regular case, two Riemannian manifolds are locally isometric if and only if their curvature tensors parameterize overlapping classifying manifolds.

Note: The order of a Riemannian metric is the order of the coframe

The Cartan-Karlhede Algorithm

Denote the set of components $\{R_{abcd}, R_{abcd;e_1}, \ldots e_q\}$ as R^q.

The algorithm is then:

1. Let $q = 0$.
2. Compute R^q.
3. Fix the frame as much as possible using Lorentz frame transformations.
4. Find the invariance group H^q of the frame which leaves R^q invariant.
5. Find the number of functionally independent components t^q amongst the set R^q.
6. If $t^q \neq t^{q-1}$ or $dim(H^q) \neq dim(H^{q-1})$ then set $q = q + 1$ and go to step 2. Otherwise, the algorithm stops and set $q = p + 1$.

The set $\{H^r, t^r, R^r\}$, $r = 1, ..., p + 1$ classifies the solution, locally.

Definition

The set R^p relative to the frame basis determined by the Cartan-Karlhede algorithm are called *Cartan invariants*.

Lorentz frame transformations

Consider a null frame basis, $\{\ell, n, m, \bar{m}\}$ such that

$$g_{ab} = -2\ell_{(a}n_{b)} + m_{(a}\bar{m}_{b)}. \tag{70}$$

For a null frame, the Lorentz frame transformation group is then:

- Boosts and Spins:

$$\ell' = \lambda\ell, \quad n' = \lambda^{-1}n, m' = e^{i\theta}m. \tag{71}$$

- Null rotations about ℓ:

$$\ell' = \ell, \quad n' = n + Bm + \bar{B}\bar{m} + |B|^2\ell, m' = m + \bar{B}\ell. \tag{72}$$

- Null rotations about n:

$$n' = n, \quad \ell' = \ell + Cm + \bar{C}\bar{m} + |C|^2 n, m' = m + \bar{C}n. \tag{73}$$

In addition to a simpler form for $SO(1,3)$ we can use this basis and shift the problem to spinors. This is the Newman-Penrose (NP) formalism.

Alignment classification

To determine the canonical form of R_{abcd} and $R_{abcd;e_1\dots e_p}$, consider a boost for an arbitrary tensor [Milson et al., 2005],

$$T'_{a_1 a_2 \dots a_n} = \lambda^{b_{a_1 a_2 \dots a_n}} T_{a_1 a_2 \dots a_n}, \quad b_{a_1 a_2 \dots a_n} = \sum_{i=1}^{n} (\delta_{a_i 0} - \delta_{a_i 1}). \tag{74}$$

The quantity, $b_{a_1 a_2 \dots a_n}$, is called the *boost weight* (b.w) of the frame component $T_{a_1 a_2 \dots a_p}$.

Boost order, $\mathcal{B}_{\mathbf{T}}(\ell)$, is the maximum b.w. of a tensor, **T**, for a null direction ℓ.

Alignment types of the Weyl tensor, C_{abcd}, and Ricci tensor, Ric_{ab}, are

$$\begin{array}{cccccc} \textit{Type} & \mathbf{G} & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{N} \\ \mathcal{B}_{\mathbf{T}}(\ell) & 2 & 1 & 0 & -1 & -2. \end{array} \tag{75}$$

If **T** vanishes, then it belongs to alignment type **O**.

References

Milson, R., Coley, A., Pravda, V., and Pravdova, A. (2005).
Alignment and algebraically special tensors in Lorentzian geometry.
International Journal of Geometric Methods in Modern Physics, 2(01):41–61.

Olver, P. J. (1995).
Equivalence, invariants and symmetry.
Cambridge University Press.

Thank you for your attention!